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Shock-Waves in General Relativity

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ABSTRACT. In this paper we set out a general theory of shock-waves and we apply this to explicitly construct a class of spherically symmetric shock-wave solutions of the Einstein gravitational equations for a perfect fluid. Our construction entails matching a Friedmann-Robertson-Walker metric to an Oppenheimer-Tolman metric at a shock-wave interface across which the gravitational metric is Lipschitz continuous. In our dynamically matched solution, we imagine the F-R-W solution as an exploding inner core, and the boundary of this inner core is a shock surface that is driven by the expansion behind the shock out into the static O-S solution. Our solution solves the problem first posed by Oppenheimer and Snyder of extending their solution to the case of non-zero pressure. In the terminology of the classical theory of shock-waves, the interface of the O-S solution is a contact discontinuity, and this means that their solution is time-reversible. In contrast, our shock-wave solution, in which the pressure is non-zero, is an irreversible solution of the Einstein equations in which the irreversibility, the loss of information, and increase of entropy in the fluid puts time irreversibility into the dynamics of the gravitational field.

1. The Einstein gravitational field equations can be written in the form

$$G = \kappa T, \tag{1}$$

where G is the Einstein tensor, T is the stress-energy tensor (the source of the gravitational field), and $\kappa = 8\pi/c^2$, where c denotes the speed of light. In Einstein's theory, the gravitational field is identified with a Lorentzian metric g, a symmetric $(0,2)$ tensor which at any point can be diagonalized as $\mathrm{diag}(-1,1,1,1)$, and equation (1) describes, simultaneously, the evolution of g and sources in T.

In a given coordinate system, $x = (x^0, x^1, x^2, x^3)$, on 4-dimensional space-time, the components of G are given by $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$, where the Ricci tensor R_{ij} and the scalar curvature R are obtained from the Riemann curvature tensor $R^i_{jk\ell}$, computed from the metric g_{ij}. In this paper, we shall be concerned with a perfect fluid, whereby the stress energy tensor T takes the form

$$T_{ij} = (p + \rho c^2)u_i u_j + p g_{ij}, \tag{2}$$

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where ρ is the mass-energy density of the fluid (as measured in a frame moving with the fluid), p is the pressure, and (u^i) denotes the normalized unit 4-velocity vector of the fluid. When an equation of state $p = p(\rho)$ is specified, system (1) imposes 10 equations in 14 unknowns (10 metric components g_{ij}, the density ρ, and 3 velocity components). Four remaining equations can then be imposed in order to specify a coordinate system.

Note that (1) shows that the Einstein equations couple the gravitational field to the *undifferentiated* fluid variables. Now it is well known that, as a consequence of the fundamental geometric identities called the Bianchi identities, div $G \equiv 0$. (Here div denotes the covariant divergence computed with respect to the unique symmetric connection compatible with the metric g, [11].) The Einstein theory thus has the remarkable property that on solutions of (1), the equation

$$\operatorname{div} T = 0 \tag{3}$$

must hold, and (3) is equivalent to the conservation of energy and momentum. If g is the flat Minkowski metric of special relativity, "div" reduces to the ordinary divergence, and hence (2) and (3) reduce to the relativistic Euler equations, [8], which in turn reduce to the classical Euler equation for an inviscid fluid, in the limit of small velocities [11]. Thus in the setting of General Relativity, the Euler equations follow from (1) and (2), as a consequence of geometrical identities.

In the papers [8,9,10], we study shock-wave solutions of the Einstein equations (1), (2). This is done by solving a problem first posed by Oppenheimer and Snyder on black holes in 1939, [5], namely, how does one remove the assumption that the pressure be identically zero in the Oppenheimer-Snyder model? That is, the Oppenheimer-Snyder model, which is in virtually every textbook on the subject of general relativity, is an exact solution of the Einstein equations, that models the collapse of a star to a black hole. The simplifying assumptions that they make is that the pressure inside the star must be identically zero and the space outside the boundary of the star must be devoid of matter. In [8], we give a general technique for extending the Oppenheimer-Snyder model to the case of non-zero pressure. We then generalize these ideas and apply them to some interesting astrophysical problems, [9].

Our technique to obtain solutions of (1) consists of the matching of two well-known spherically symmetric solutions of (1) across a shock-wave interface. The inner metric is a standard model of an expanding universe, which we match to a well-known metric which models the interior of a star. What we obtain in [9] is a solution which models a shock-wave explosion into a singular, static isothermal sphere. This solution is a General Relativistic version of how stars are formed. In fact, this solution can model explosions on any scale: supernova explosions, or even the "big-bang." If we allow the sound speed $\sigma \to 0$, we then get models for spherical explosions in classical Newtonian gravity. In the paper [10], we demonstrate that our theory generates a large class of physically meaningful outgoing shock-waves that model blast waves in a general relativistic setting. We also obtain explicit formulas for the physical quantities that evolve according to the equations; these formulas are important for the numerical simulation of these solutions.

2. An important issue in general relativity is the problem of how time irreversibility should enter the theory. Indeed, all of the exact solutions of the Einstein equations (1) for classical sources assume the stress tensor for a perfect fluid, or for electro-magnetism, and it is well known that smooth solutions of the Einstein equations with these sources are time reversible. However, *it is our thesis here that time-irreversibility has been built into the theory of general relativity from the very start by way of the stress tensor for a perfect fluid, and that the irreversibility, loss of information and increase of entropy in the solutions of* (1) *is encoded in the structure of the shock-waves.*

The idea is that, in the case of a perfect fluid (2), the relativistic compressible Euler equations appear as a subsystem of the Einstein gravitational field equations (1) through the identity (3), and the relativistic Euler equations form a system of nonlinear hyperbolic conservation laws, the setting for the modern theory of shock-waves. Therefore, like the classical compressible Euler equations, solutions of the Einstein equations should be time reversible only up until the formation of shock-waves. For the classical Euler equations, the dissipation and increase of entropy at shock-waves can be interpreted physically as the non-negligible effects of viscosity that must appear because the inviscid Euler equations represent a *singular limit* of the Navier-Stokes equations as the viscosity is taken to zero: however small the viscosity parameters, the dissipative effects of viscosity are not negligible in the surface layer of a shock, and this dissipation is fully accounted for in the shock-waves that appear in the zero viscosity limit. One of the complications in the theory of relativity is that there is no simple model like the classical Navier-Stokes equations that plays a similar role, because the introduction of second order parabolic terms to model dissipation introduces an infinite speed of propagation similar to that observed in the linear heat equation. However, it is important to emphasize that the irreversibility, loss of information and increase of entropy at shock-waves in a generic system of conservation laws div $T = 0$ is explicitly determined at the level of the zero viscosity limit, and this applies even when the precise form of the viscosity is not known. The conclusion one must draw from the theory of shock-waves is that shock-waves will always form in solutions of a system of conservation laws except in very special cases when the fluid is everywhere expansive [1,3,6]. We believe that this result carries over to the Einstein equations for a perfect fluid as well. Indeed, in locally Lorentzian coordinates at a point, (coordinates in which $g_{ij} = \eta_{ij} = \text{diag}(-1,1,1,1)$, and $g_{ij,k} = 0$ for all $i,j,k = 0,\dots,3$), the covariant divergence equals the classical divergence; and thus, to within higher order errors, the perfect fluid in general relativity satisfies the relativistic Euler equations in special coordinates at each point. Thus if shock-waves never form in T, then in essence, the gravitational field has the effect of removing all of the compressive action of the Euler equations, and thereby has the effect of eliminating dissipation in fluids! This we find physically implausible, and thus we maintain that shock-waves are as fundamental to solutions (1), (2) as they are to the classical Euler equations. This suggests that when viscosity and heat conduction are neglected, (as they are in the case of a perfect fluid), it is the shock-waves in the sources T that fundamentally introduce irreversibility into the dynamics of the gravitational field. This point of view seems to have been essentially overlooked [2,4]. In [8], we organize the general theory of matching two metric solutions of (1) across an interface from the point of view of the theory of shock-waves, and we apply this theory to explicitly construct spherically symmetric shock-waves solutions of the Einstein equations. This construction solves the problem first posed by Oppenheimer and Snyder in 1939 of extending their model to the case of non-zero pressure.

If the fluid variables shock, the entries in the stress tensor T become discontinuous, and thus by the Einstein equation, the metric loses continuity in the second derivative. In Section 3 of [8], we present a self-contained treatment of the general theory of shock-waves for the Einstein equations by considering the general problem of allowing the metric to be only Lipschitz continuous across a hypersurface Σ in space-time. This treatment reorganizes and extends previous results of Israel. (See also [4] and references therein where this topic is referred to by the heading *Junction Conditions*). The result is the following theorem:

THEOREM 1. *Let Σ denote a smooth, 3-dimensional shock surface in space-time with space-like normal vector* $\mathbf{n}$. *Assume that the components g_{ij} of the gravitational metric g are smooth on either side of Σ, (continuous up to the boundary on either side separately), and Lipschitz continuous across Σ in some fixed coordinate system. Then the following statements are equivalent:*

(i) *$[K]=0$ at each point of Σ.*
(ii) *The curvature tensors $R^i_{jk\ell}$ and G_{ij}, viewed as second order operators on the metric components g_{ij}, produce no delta function sources on Σ.*
(iii) *For each point $P\in\Sigma$ there exists a $C^{1,1}$ coordinate transformation defined in a neighborhood of P, such that, in the new coordinates, (which can be taken to be the Gaussian normal coordinates for the surface), the metric components are $C^{1,1}$ functions of there coordinates.*
(iv) *For each $P\in\Sigma$, there exists a coordinate frame that is locally Lorentzian at P, and can be reached within the class of $C^{1,1}$ coordinate transformations.*

Moreover, if any one of these equivalencies hold, then the Rankine-Hugoniot jump conditions

$$[G]^\sigma_i n_\sigma = 0,$$

(which express the weak form of conservation of energy and momentum across Σ when $G=\kappa T$), hold at each point on Σ.

Here $[K]$ denotes the jump in the second fundamental form (extrinsic curvature) K across Σ, (this being determined by the metric separately on each side of Σ because g_{ij} is only Lipschitz continuous across Σ), and by $C^{1,1}$ we mean that the first derivatives are Lipschitz continuous. Theorem 1 should be credited mostly to Israel, who obtained results (i)–(iii) in Gaussian normal coordinates. Our contribution was to identify the covariance class of $C^{1,1}$ transformations, and to thereby obtain precise coordinate independent statements for (ii) and (iii), as well as the equivalence with (iv). As a consequence of this, we obtain the result that the Ricci scalar curvature R *never* has delta function sources at a Lipschitz continuous matching of the metrics, as well as the following theorem that partially validates the statement that shock-wave singularities in the source free Einstein equations $R_{ij}=0$ or $G_{ij}=0$ can only appear as coordinate anomalies, and can be transformed away by coordinate transformation:

THEOREM 2. *If a smooth shock surface Σ forms in weak solutions of $R_{\alpha\beta}=0$ or $G_{\alpha\beta}=0$ posed in some given coordinate system y, such that the y-components*

$g_{\alpha\beta}$ of the metric tensor g are Lipschitz continuous across Σ, and C^k functions of y on either side of Σ (continuous up to the boundary on either side separately), then there exists a regular $C^{1,1}$ coordinate transformation taking $y \to x$, such that, the components g_{ij} of g in the x-coordinates are actually C^k functions of x in a neighborhood of each point on the surface Σ.

Note that when the metric g is only Lipschitz continuous across a surface Σ, the Einstein tensor G, which contains second derivatives of g, can in general contain delta function sources, and then Σ should be interpreted as a surface layer and not as a shock-wave. In the presence of spherical symmetry, however, the weak form of conservation of energy and momentum, (4), implies the absence of surface layers, so long as the areas of the spheres of symmetry match up on Σ. This is implied in the following theorem, proved in [8].

THEOREM 3. *Let g and $\bar{g}$ be 2 spherically symmetric metrics:*

$$g\colon\ ds^2 = -a(t,r)\,dt^2 + b(t,r)\,dr^2 + c(t,r)d\Omega^2,$$

$$\bar{g}\colon\ d\bar{s}^2 = -\bar{a}(\bar{t},\bar{r})\,d\bar{t}^2 + \bar{b}(\bar{t},\bar{r})\,d\bar{r}^2 + \bar{c}(\bar{t},\bar{r})\,d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\theta$. Assume that there exists a smooth map: $\psi(t,r) \to (\bar{t},\bar{r})$ defined in a neighborhood of a shock interface Σ, such that the metrics agree on Σ. Assume too that $c(t,r) = \bar{c}(\psi(t,r))$, the normal $\bar{n}$ to Σ is non-null, and that the derivative of c in the direction $\bar{n}$ is non-zero. Then the following are equivalent:

(a) *$g \cup \bar{g}$ is $C^{1,1}$ in some neighborhood of Σ.*
(b) $[G^i_j]n_i = 0$, $j = 1,2$.
(c) $[G^{ij}]n_i n_j = 0$.
(d) $[K] = 0$.

In the above theorem and in what follows, $[f]$ denotes the jump in the quantity f across the shock interface and $C^{1,1}$ denotes the class of functions with Lipschitz continuous first derivatives. Note that the condition $c(t,r) = \bar{c}(\psi(t,r))$ implies that the areas of the spheres of symmetry in the barred and unbarred coordinates agree on Σ. The condition $\bar{n}(c) \neq 0$ implies that the areas of the spheres of symmetry change monotonically in the normal direction if $c(t,r) = r^2$, e.g., and if this condition failed, it would imply that the shock speed would exceed the speed of light.

We conclude that in the presence of spherical symmetry one condition, (c), is equivalent to the jump conditions (4); i.e., to the fact that the weak form of conservation of energy and momentum holds. This is an important fact in our extension of the (OS) model to non-zero pressure.

3. We now discuss how we explicitly construct spherically symmetric shock-wave solutions of the Einstein equations for a perfect fluid. To do this, we shall match different metrics, Lipschitz-continuously across a surface of discontinuity for the fluid variables (a shock-wave). In what follows, we shall always assume that the fluid is comoving relative to the metric; thus for diagonal metrics, $u^i = 0$, $i = 1,2,3$, and $u^0 = (-g_{00})^{-1/2}$.

In our shock-wave model, the *inner* metric is the Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + R(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2\, d\Omega^2 \right], \tag{FRW}$$

where k is a constant, $k = 0, 1$, or -1, and

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

This metric is spherically symmetric and describes a homogeneous, isotropic universe (no preferred point and no preferred direction), [11]. The space expands or contracts in time according to the function $R(t)$, the cosmological scale factor. The (FRW) metric is used to model an expanding universe, and there is, generally speaking (e.g. if $R(t) > 0$ for some t), a singularity ($R(t) = 0$) in "backwards" time. The function $R(t)$ determines the "red-shift" factor for distant astronomical objects. If we are given an equation of state of the form $p = p(\rho)$, $R(t)$ is determined by the Einstein equations (1), (2), under the assumption that $\rho = \rho(t)$.

The *outer* metric in our model is the Oppenheimer-Tolman (OT) metric:

$$d\bar{s}^2 = -B(\bar{r})\, d\bar{t}^2 + \left(1 - \frac{2\mathcal{G}M(\bar{r})}{\bar{r}}\right)^{-1} d\bar{r}^2 + \bar{r}^2\, d\Omega^2. \tag{OT}$$

(We use barred coordinates here to distinguish from the unbarred coordinates in the FRW metric.) If we are given an equation of state of the form $\bar{p} = \bar{p}(\bar{\rho})$, then using Einstein's equations, we get the following differential equations for the unknown quantities $\bar{\rho}$, B, and M (see [11]):

$$\frac{B'}{B} = \frac{-2\bar{p}'}{\bar{p} + \bar{\rho}}, \quad M'(\bar{r}) = 4\pi\bar{r}^2\bar{\rho}(\bar{r}), \tag{4}$$

and (the Oppenheimer-Volkoff equation),

$$-\bar{r}^2\bar{p}' = \mathcal{G}M(\bar{r})\bar{\rho}(\bar{r}) \left(1 + \frac{\bar{p}}{\bar{\rho}}\right) \left(1 + \frac{4\pi\bar{r}^3\bar{p}}{M}\right) \left(1 - \frac{2\mathcal{G}M(\bar{r})}{\bar{r}}\right)^{-1}. \tag{5}$$

Here $M(\bar{r})$ denotes the "total mass" inside a ball of radius $\bar{r}$. The (OT) metric is a time independent spherically symmetric solution of the Einstein equations (1), (2); it is used to model the interior of a star, [11].

We now discuss the Oppenheimer-Snyder (OS) model, [5]. In the (OS) model, the *outer* metric is the empty space Schwarzschild metric ([11]):

$$d\bar{s}^2 = -\left(1 - \frac{2\mathcal{G}M}{\bar{r}}\right) d\bar{t}^2 + \left(1 - \frac{2\mathcal{G}M}{\bar{r}}\right)^{-1} d\bar{r}^2 + \bar{r}^2\, d\Omega^2,$$

where M is a positive constant. This metric describes the gravitational field outside of a ball in $\mathbb{R}^3$, assuming that there is no matter in the exterior of the ball; i.e., $T_{ij} \equiv 0$. The OS result is the following: Take $p \equiv 0$ in the FRW metric. Then there exists a coordinate transformation $(t, r) \to (\bar{t}, \bar{r})$, where $Rr = \bar{r}$, such that under this identification of coordinates the (FRW) and Schwarzschild metrics agree and are Lipschitz continuous across a 3-dimensional interface, the surface of a star. In (r, t) coordinates, the surface of the star is $r = a$ (fixed in the FRW metric). Then given $\rho(0)$ and M, a is determined by $M = \frac{4\pi}{3}\rho(0)$ and $k = \frac{2M\mathcal{G}}{a^3}$. The functions $\rho(t)$ and $R(t)$ are determined by the Einstein equations. On the other hand, in $\bar{r}$-$\bar{t}$ coordinates the surface collapses to a black hole: $\bar{r}(\bar{t}) \to \bar{r} = 2\mathcal{G}M$, and $\rho(\bar{t}) \to \infty$ as $\bar{t} \to \infty$. This (OS) result was the first example of gravitational collapse.

The total mass inside the interface is fixed since mass does not cross the interface. The (OS) solution is thus a contact discontinuity, [10] (as opposed to a true fluid-dynamical shock-wave). The function $R(t)$ vanishes at some finite T; thus, (cf. [11]) "a fluid sphere of initial density $\rho(0)$ and zero pressure will collapse from rest to a state of infinite density in the finite time T."

In the shock-wave model in [8,9,10], we also take the inner metric to be the (FRW) metric but with a non-zero pressure function $p = p(\rho)$, $\rho = \rho(t)$, and we take the outer metric to be the (OT) metric, with non-zero pressure function $\bar{p} = \bar{p}(\bar{\rho})$, $\bar{\rho} = \bar{\rho}(\bar{r})$. We show that we can define a coordinate mapping $(t, r) \to (\bar{t}, \bar{r})$, where $Rr = \bar{r}$, such that under this identification of coordinates, the (FRW) and (OT) metric match Lipschitz continuously along a 3-dimensional interface: the shock surface. That is, the following theorem holds [8]:

THEOREM 4. *Under the above procedure, there exists a shock-wave solution for arbitrary equation of state $p = p(\rho)$ in the* (FRW) *space, and arbitrary equation of state $\bar{p} = \bar{p}(\bar{\rho})$ in the* (OT) *space.*

It is instructive to outline the matching procedure. Thus, we first take $\bar{r} = Rr$, so that the spheres of symmetry have the same area. In order to match the $d\bar{r}^2$ term in the (OT), namely $\left(1 - \frac{2\mathcal{G}M(\bar{r})}{\bar{r}}\right)$, to the corresponding term in (the coordinate transformation of) the (FRW) metric, we find the explicit relation

$$M(\bar{r}) = \frac{4\pi}{3}\rho(t)\bar{r}^3; \tag{6}$$

this defines the shock surface. In order to match B, the $d\bar{t}^2$ term in (OT), to the corresponding term in the (FRW) metric we find that $\bar{t} = \bar{t}(t, r)$ solves a linear partial differential equation with initial conditions on the shock surface (6) chosen to match B on (6). The result is the following theorem, [8].

THEOREM 5. *The shock surface* (6) *is non-characteristic for this partial differential equation away from the Schwarzschild radius.*

Thus the transformation $\bar{t} = \bar{t}(t, r)$ exists, but is not explicit. So the coordinate transformation is defined; we shall not need any explicit information about $\bar{t}$.

Now there arises the following problem: If the metrics match Lipschitz-continuously across a shock surface Σ, does the weak form of the conservation of energy and momentum hold across the shock surface; i.e., does the following (jump condition) hold:

$$\left[T^{ij}\right] n_i = 0 \tag{7}$$

where $\bar{n} = \left(n^i\right)$ is the normal vector to Σ, cf. Figure 1.
Now in view of (1), (7) holds if and only if

$$\left[G^{ij}\right] n_i = 0 \tag{8}$$

That is, assume div $G = 0$ on each side of Σ, and g is Lipschitz continuous across Σ. Does this imply that (8) holds? The answer is no, in general. To understand this, we consider the extrinsic curvature (second fundamental form) K

$$K : T_p\Sigma \to T_p\Sigma$$
$$\bar{x} \to \nabla_{\bar{x}}\bar{n},$$

where $T_p\Sigma$ is the tangent space to Σ at p. Then Israel, [2], has proved the following theorem.

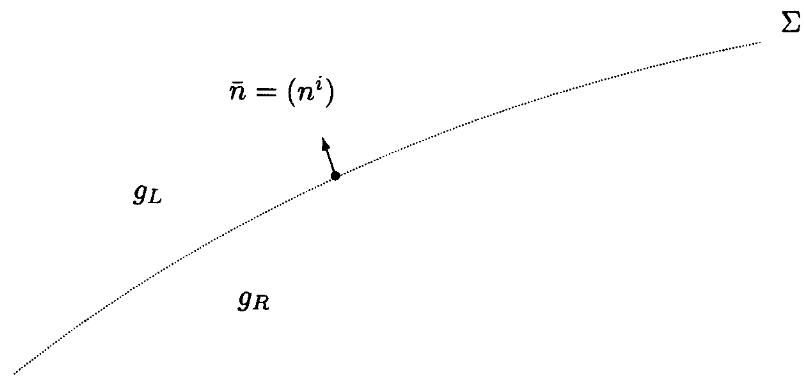

FIGURE 1

THEOREM 6. *If the extrinsic curvature K is continuous across Σ, then* (8) *holds.*

This theorem is generalized in Theorem 3, above, where we see that in the presence of spherical symmetry the jump conditions (7) reduce to the single condition

$$[T^{ij}]\, n_i n_j = 0. \tag{9}$$

In fact, in the presence of spherical symmetry, (9) holds if and only if the extrinsic curvature is continuous across Σ.

4. In this section, we shall apply the general results discussed in §3, to obtain explicit shock-wave solutions of the Einstein equations, which model shock-waves exploding into the general relativistic version of a static, singular, isothermal sphere. We shall also describe how this solution can be used in stellar dynamics to model the birth of a star.

Before proceeding with the construction of our exact shock-wave solution, it is instructive to see what (9) implies in the Oppenheimer-Snyder model discussed above. Thus, in this case, we have $\bar{p} \equiv 0 \equiv \bar{\rho}$, so (9) gives

$$p\rho\big[(1-\theta)p + \rho\big] = 0. \tag{10}$$

Thus if $\rho > 0$, then $p \equiv 0$, because $\theta \le 1$; see [10]. Thus, in the (OS) model, there is no solution for non-zero pressure!

Now the idea in [8,9,10] is to fix the (OT) outer solution: $\bar{\rho}$, $\bar{p}$, and $M(\bar{r})$, where $\bar{p} = \bar{p}(\bar{\rho})$. Then we seek equations for the unknown quantities $R(t)$ and $\rho(t)$ in the inner (FRW) solution. These equations are (cf. [11]):

$$\dot{R}^2 = \frac{8\pi\mathcal{G}}{3}\rho R^2 - k, \tag{11}$$

$$p = -\frac{d}{dt}\left(\rho R^3\right)/3R^2\dot{R}. \tag{12}$$

Note that if we are given an equation of state $p = p(\rho)$, then the system (11), (12) becomes a system of two equations in the two unknowns R and ρ, which has a unique solution, given any initial condition. In this case, generally speaking, we cannot be sure that the conservation equation (9) will hold. We thus take a different approach, allow p to be another variable, and we take the three (autonomous) equations (10), (11), and (12) to determine the three unknowns, R, ρ, and p.

The procedure outlined here leaves open a few important questions; for example

1. Is $p \geq 0$?
2. Do $p(t)$ and $\rho(t)$ give a reasonable equation of state $p = p(\rho)$; for example, is the sound speed less than 1, the (normalized) speed of light?
3. Are p and ρ larger behind the shock-wave?
4. Do the Lax characteristic conditions hold (cf. [3,6])?

In general, the answer to these questions depends on the initial conditions, as well as on the given outer (OT) equation of state $\bar{p} = \bar{p}(\bar{\rho})$. However, when $\bar{p} = \bar{\sigma}\bar{\rho}$, and $k = 0$, we obtain an *explicit* solution, by assuming that the inner (FRW) equation of state is of the form $p = \sigma\rho$; (cf [11]).

We now construct an explicit (OS) type shock-wave with $p \equiv 0$, which satisfied the conservation constraint (10). To do this, choose $k = 0$ in the (FRW) metric, and choose the equation of state in the outer (OT) metric to be of the form

$$\bar{p} = \bar{\sigma}\bar{\rho}, \quad 0 < \bar{\sigma} < 1.$$

We have the following theorem, [9].

THEOREM 7. *Under these assumptions, the* (FRW) *equation of state* $p = \sigma\rho$, *where* σ *is a constant is determined by the algebraic equation*

$$\sigma = \frac{\bar{\sigma}(\bar{\sigma}+7)}{3(1-\bar{\sigma})}.$$

The solution is a shock-wave which satisfies the conservation constraint (9) , *and is given explicitly by*

$$M(\bar{r}) = 4\pi\gamma\bar{r}, \quad B(\bar{r}) = \bar{r}^{4\bar{\sigma}/(1+\bar{\sigma})}, \quad \rho(\bar{r}) = \frac{\gamma}{\bar{r}^2},$$

where

$$\gamma = \frac{1}{4\pi\mathcal{G}}\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right),$$

and

$$R(t) = R_0\left(\frac{\rho(t)}{\rho_0}\right)^{-1/3(1+\sigma)}, \quad \rho(t) = \frac{3\gamma}{\left[\sqrt{18\pi\mathcal{G}\gamma}\,(1+\sigma)(t-t_0)+\bar{r}_0\right]^2}.$$

The shock position is given by

$$\bar{r}(t) = \sqrt{18\pi\mathcal{G}\gamma}\,(1+\sigma)(t-t_0)+\bar{r}_0,$$

or

$$r(t) = \bar{r}_0 R_0^{-1}\left(\bar{r}(t)/\bar{r}_0\right)^{(1+3\sigma)/(3+3\sigma)}.$$

The above solution is an outgoing shock-wave and satisfied $\rho = 3\bar{\rho}$ on the shock surface. Note too that there is a singularity in backward time

$$t_* = t_0 - \frac{\bar{r}_0}{\sqrt{18\pi\mathcal{G}\gamma}\,(1+\sigma)},$$

and as $t \to t_*$, $\bar{r} \to 0$; ρ, $\bar{\rho}$, p and $\bar{p}$ all tend to infinity; and R and r tends to zero. If we take this as a cosmological model, then $t = t_*$ represents the initial big-bang singularity in which a shock-wave emerges from $\bar{r} = 0$. The solution describes an expanding shock-wave, (a blast wave), as t increases; cf. Figure 2.

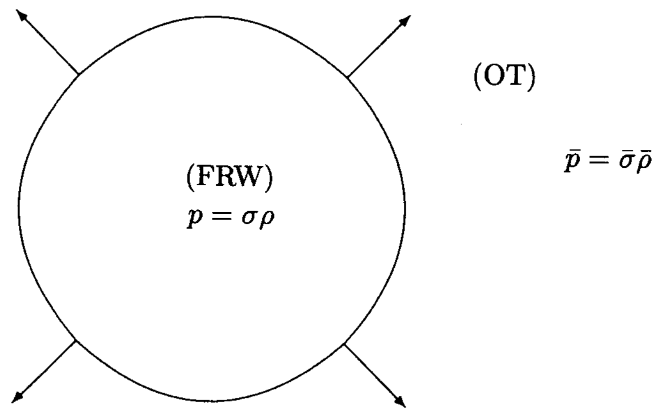

FIGURE 2

Note that the outer (OT) solution, with $p = \bar{\sigma}\bar{\rho}$, implies that $\bar{\rho}(\bar{r}) = \gamma/\bar{r}^2$, so $\bar{\rho} \to \infty$ as $\bar{r} \to 0$; this is referred to as a static, singular isothermal sphere. (For our shock-wave solution, the density is finite after some initial time $t > 0$, since $\bar{r} = 0$ is not in the domain of the solution after this initial time.)

The outer static solution is called a *static isothermal sphere* because the metric entries are time independent, and the constant sound speed models a gas at constant temperature. It is *singular* because it has an inverse-square density profile, and thus the density and pressure tend to ∞ at the center of the sphere. The Newtonian version of a static singular isothermal sphere is well known and is important in theories of how stars form from gaseous clouds [4]. The idea in the Newtonian case goes as follows: a star begins as a diffuse cloud of gas, which slowly contracts under its own gravitational force by radiating energy out through the gas cloud as gravitational potential energy is converted into kinetic energy. This contraction continues until the gas cloud reaches the point where the mean free path for transmission of light is small enough that light is scattered, instead of transmitted, through the cloud. The scattering of light within the gas cloud has the effect of equalizing the temperature within the cloud. At this point the gas begins to drift toward the most compact configuration of the density that balances the pressure when the equation of state is isothermal; namely, it drifts toward the configuration of a static, singular, isothermal sphere. Since this solution of the Newtonian case is also an inverse square in density and pressure, the density tends to infinity at the center of the sphere, and this ignites thermonuclear reactions. The result is a shock-wave explosion emanating from the center of the sphere, and this signifies the birth of a star. The explicit solution, which we present here, is an exact, *general relativistic version* of such a shock-wave explosion.

Finally, we have the following theorem; [9].

THEOREM 8. *There exist numbers $\sigma_1 < \sigma_2$, $0 < \sigma_1 = .458 < .744 = \sigma_2 < 1$ such that*

(i) *The Lax characteristic condition holds iff $0 < \sigma < \sigma_1$, and*
(ii) *The shock speed is less than the speed of light iff $0 < \sigma < \sigma_2$.*

We conclude that if $\sigma_1 < \sigma < \sigma_2$, a new type of shock-wave appears, in which the shock speed exceeds the characteristic (sound) speeds on each side of the shock. Note that $\sqrt{\sigma_1} \approx .677$, and $\sqrt{\sigma_2} = .863$. Thus, a fluid with sound speed near $\sqrt{\sigma_2} = .863$ can drive shock waves having speeds arbitrarily close to the speed of light.

5. Our dynamical shock-wave solution of the Einstein equation opens up intriguing possibilities in cosmology, as well as in the classical theory of shock-waves. For cosmology, our solution opens up the possibility, (with an explicit model), that there is a shock-wave at the edge of the universe. Since shock-waves are time irreversible solutions of the equations, because of the increase of entropy (in a generalized sense, see [3,6]), we infer from the mathematical theory of shock-waves that many solutions must decay time asymptotically to the same shock-wave. Thus, in contrast to the pure (FRW) solution, or to the (OS) solution, in our model we should not expect a unique time reversal of the solution all the way back to the initial big-bang singularity.

In the recent paper [10], we extend our results and show that our theory generates a large class of physically meaningful multi-dimensional shock-waves that model blast waves in a General Relativistic setting. In addition, we obtain formulas for the physical quantities that evolve according to the equations. These formulas are important for the numerical simulation of these solutions.

We believe that the new techniques which we have introduced have opened up a great many exciting possibilities for obtaining *multi-dimensional* shock-wave solutions for the relativistic Euler equations, as well as for the classical Euler equations by taking the limit $c \to \infty$. The implementation of this program is just beginning, because we have only recently obtained *explicit, closed-form* solutions of the equations. In addition, the rather "clean" form of the differential equations, which we have just recently derived, will enable us to easily implement them on a computer, and we can track the global evolution of the solutions, as t increases. This clearly opens up the possibility of finding new shock-wave phenomena. For example, we will be able to study such questions as: do singularities form in the equations as $t \to \infty$? (If so, are they related to "gravitational collapse"?) Do the fluid variables (pressure, density, sound speeds) remain well-behaved?

Finally, because no derivatives fall on the fluid variables in Einstein's equations (1), it suggests that the Einstein equations represent a sort of "hyperbolic regularization" of the compressible Euler equations, in a manner somewhat analogous to the role played by the Hamiliton-Jacobi equations for scalar conservation laws. In [8], we showed that the *smoothness* of the metric translates over to *conservation* of the sources, and we continue to investigate the possibility that there could actually be numerical advantages to computing with Einstein-like regularizations of the Euler equations — analogous to the viscous regularizations that have played a fundamental role to date.

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