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MULTI-DIMENSIONAL SHOCK-WAVES FOR RELATIVISTIC FLUIDS

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Abstract

We discuss the issues in our recent paper [14] in which we study multi-dimensional shock waves via a study of the relativistic Euler equations which appear as a subsystem of the Einstein gravitational field equations of general relativity. In particular, we discuss the general theory of shock waves in this setting, and we apply this theory to explicitly construct spherically symmetric shock-wave solutions of the Einstein equations that generalizes the famous Oppenheimer-Snyder model for gravitational collapse in stars, thereby solving the problem first posed by Oppenheimer and Snyder in 1939 of generalizing their model to the case when the pressure is non-zero. This solution opens up intriguing possibilities in cosmology, but in terms of the classical theory of shock waves, there are remarkable simplifying features of the Einstein equations over the classical Euler equations that enable us to explicitly construct shock-wave solutions within the framework of general relativity. Our solution models an explosion which has no known classical analog. From the point of view of the theory of general relativity, it is interesting that the O-S solution is a contact discontinuity, and hence is time reversible, while in contrast, our shock-wave solution in which $p \neq 0$ is an irreversible solution of the Einstein gravitational field equations in which the irreversibility, loss of information, and increase of entropy in the fluids puts irreversibility into the dynamics of the gravitational field.

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1 Introduction

The modern theory of shock waves and conservation laws grew out of a study of the classical Euler equations for 3-dimensional compressible flow in gas dynamics. The Euler equations can be written in the form

$$\begin{aligned} &\rho_t + div\{\rho u\} = 0, \\ &(\rho u)_t + div\{\rho u^i u^j + \delta^i_j p\} = 0, \end{aligned} \tag{1.1}$$

where ρ denotes the density, $\mathbf{u} = (u^1, \cdots, u^3)$ the velocity, and p the pressure of the fluid, and div denotes the divergence taken with respect to the spatial coordinates $\mathbf{x} = (x^1, x^2, x^3)$. We write the equations (1.1) in the simple conservation form

$$DivT = 0, \tag{1.2}$$

where Div denotes the divergence taken (on each row of T) in the combined variables (t, x^1, x^2, x^3), and T denotes the 4×4 matrix of entries in $(\rho, \mathbf{u}, p)$ that makes (1.1) agree with (1.2); the components of T in a given coordinate system are denoted by T_{ij}, $i, j = 0, ..., 3$. It is well known that shock waves form in solutions of (1.1), and these model the steep fronts that propagate in the underlying problem. Shock waves represent the fundamental physics in solutions of (1.1), but it is well known that the presence of shocks introduces many difficulties for both computing and analyzing solutions. This is because at a shock-wave, the fluid variables in T become discontinuous, and thus any scheme for approximating the derivatives in $DivT = 0$ must involve differencing discontinuous functions, and these produce delta function singularities in the derivatives. This is what leads to the well known Gibbs type oscillations that typically appear in numerical calculations of shock waves by finite difference schemes. Although there is a well developed (though highly incomplete!) theory of shock waves when (1.1) is restricted to one space and one time dimension, it is fair to say that a general mathematical theory for shock waves in more than one space dimension is essentially non-existent. The geometry of shock-wave propagation in several space dimensions can be exceedingly complicated, and most of the work in this field has involved the numerical simulation of special solutions-there being few general principles emerging from such studies. Our idea here is to approach the theory of shock waves for (1.1) through the study of shock-wave propagation in the Einstein field equations of general relativity. At first it appears that the Einstein equations are much more complicated than the Euler equations since they include the effects of gravity, but in fact the Einstein equations have some remarkable simplifying features that have for the most part been overlooked in the subject of shock waves. One simplifying feature is that the Einstein equations are based on the highly developed theory of Riemannian geometry, a theory that allows one to work in arbitrary coordinate systems $(t, \mathbf{x})$. This can be used to advantage because geometrical complications can appear simple in special coordinate systems that change from point to point. Indeed, in our generalized Oppenheimer-Snyder construction, we obtain explicit formulas for the shock surface and shock speed, and we can interpret these physically in terms of a global conservation of mass principle. However, due to the covariance (coordinate freedom) of the Einstein equations, we succeed in this without ever having to construct a single coordinate system in which the solutions on each side of the shock are simultaneously expressed. Thus the Einstein equations are a natural setting in which the powerful techniques of Riemannian geometry can be brought to bear upon the problem of constructing multi-dimensional shock waves. But there is a more subtle and more remarkable simplification that occurs in the Einstein equations over the Euler equations, and this simplification is one of the miracles that led Einstein to discover the field equations in the first place. The idea here is that to get the Einstein equations from the Euler equations, one introduces the gravitational metric potentials g_{ij} that give the properties of the underlying spacetime. The Einstein equations are then equations in the fluid variables $(\rho, \mathbf{u}, p)$, together with the gravitational potentials g_{ij}, and these equations take the form

$$G = \kappa T, \tag{1.3}$$

where G is the Einstein curvature tensor, (the simplest 4×4 matrix of entries constructed from second order derivatives of g_{ij} that is automatically divergence free, and has the same transformation properties as T under arbitrary $(t, \mathbf{x})$ coordinate transformations), T is the relativistic version of the T in (1.2),

$$T_{ij} = (p + \rho c^2) u_i u_j + p g_{ij}, \tag{1.4}$$

and $\kappa = 8\pi/c^2$, where c denotes the speed of light. The great subtlety of (1.3) is that the relativistic Euler equations $DivT = 0$ follow *as a consequence of* (1.3), and needn't be imposed on solutions! This is because G is constructed from the Riemann curvature tensor so as to be *identically* divergence free as a consequence of the well known Bianchi identities of geometry. Said differently, since (1.3) does not involve derivatives of T, using (1.3) one can solve shock-wave problems without ever taking a derivative of the fluid variables in T, the variables (ρ, u, p) that become discontinuous and non-differentiable at the shock[1]. Now because G involves second derivatives of the metric potentials g_{ij}, it is natural to ask whether the difficulties in solving $DivT = 0$ when T becomes discontinuous at shock waves is replaced by corresponding difficulties in the Einstein equations due to the formation of discontinuities in the first derivatives of the metric variables g_{ij}. The results we present on the general theory of shock waves in Section 3 of [14] shows that this is *not* the case. (These results are based on the work of Israel, [3].) Indeed, we show that the Einstein tensor G, and hence the stress tensor T itself, are free of delta function sources at a shock *in every coordinate system* if and only if there exists some coordinate system in a neighborhood of the shock in which the metric potentials are continuously differentiable functions of these coordinates, with Lipschitz continuous second derivatives, (i.e., $C^{1,1}$ functions)[2]. Thus, in numerically simulating the second derivatives that appear in the Einstein tensor G when we solve $G = \kappa T$, we need only difference Lipschitz continuous functions at a shock-wave, while in numerically simulating $DivT = 0$, we must difference discontinuous functions, which produces delta function singularities in the derivatives of the terms that appear in each component of T, together with the disturbing Gibbs type oscillations

[1]The gravitational potentials g_{ij} play a role similar to the vector potential A in the theory of electro-magnetism. In the latter, choosing $F = dA$ has the effect of making the Maxwell equations $dF = 0$ hold automatically, in the same way that choosing $G = \kappa T$ has the effect of making the Euler equations $DivT = 0$ hold automatically. More precisely, $divG = 0$ is a geometric identity, independent of the Einstein equations, and holds as a consequence of the Bianchi identities, and thus $DivT = 0$ holds as a consequence of the identity $divG = 0$ once the Einstein equations $G = \kappa T$ are solved.

[2]The point here is that in an arbitrary coordinate system in which the metric components g_{ij} are Lipschitz continuous functions of the spacetime coordinates (t, x) across a shock-wave, the terms that are differenced in G can have delta function singularities in the second derivatives, but these cancel out in each component of G, if and only if there exists a $C^{1,1}$ transformation of the spacetime coordinates such that, in the new coordinates, the metric potentials are $C^{1,1}$ functions across the shock. This leads to the conclusion that delta function singularities can never appear in the second derivatives of *any* of the terms that are differenced in G when these derivatives are calculated in these special coordinates. This is a covariant version of a result first given by Israel, [3]

that typically destroy the accuracy of numerical computations.

The advantage of having the Euler equations $DivT = 0$ as identities in the Einstein equations is also essential in our construction of the Oppenheimer-Snyder shock-wave solutions in sections 4 and 5 of [14]. In this setting, the presence of the metric potentials g_{ij} allows us to solve the problem by first matching *the metric* across the shock, and this gives us directly an explicit formula for the shock position without requiring that we solve the implicit jump conditions (the weak form of $DivT = 0$) for the fluids across the shock. Moreover, after the matching is done, we have shown in [14] that the jump conditions reduce in complexity from two to one nontrivial constraint. This reduction of the jump conditions after the metrics are matched, also represents a constraint on the weak solutions of $DivT = 0$, which follows because $DivT = 0$ is an identity on solutions of $G = \kappa T$.

To summarize, in [14] we use these simplifying features of (1.3) over (1.2) to construct an explicit spherically symmetric shock-wave solution of (1.3) that solves the longstanding open problem of generalizing the Oppenheimer-Snyder model of gravitational collapse to the case of non-zero pressure,[9]. We know of no classsical analog of our solutions, but said in plain terms, we have constructed a solution which models a spherically symmetric explosion described by an expanding shock-wave that is driven by the expansion behind the shock into a fixed ambient fluid. In this construction we heavily exploit the coordinate freedom of the Einstein equations and use the fact that it is easier to match the metric potentials g_{ij} across a shock than it is to satisfy the Rankine-Hugoniot jump conditions directly.

Our solution of the Oppenheimer-Snyder problem also opens up intriguing possibilities in cosmology which we discuss below, but in terms of the classical theory of shock waves, we plan to investigate whether the classical limit $c \to \infty$, (c denotes the speed of light), will yield new descriptions of classical shock waves as well.

2 Statement of Results

We now state the results in [14], which sets out a general theory of shock waves, (based on work of Israel, [3]), as well as a generalization of the Oppenheimer-Snyder (O-S) solution. We refer the reader to our paper for details. Our work here is best introduced in the first paragraph of our paper:

In their classic 1939 paper, Oppenheimer and Snyder [9] introduced the first mathematical model for gravitational collapse of stars based on spherically symmetric solutions of the Einstein gravitational field equations. In this pioneering paper, Oppenheimer and Snyder gave the first rigorous results describing gravitational collapse of stellar objects, and the remarkable conclusion of this work was that "black holes" could form from gravitational collapse in massive stars. In his comprehensive article on the history of the subject of gravitation, ([4], page 226, paragraph 4), Israel references the Oppenheimer-Snyder paper as having "strong claims to be considered the most daring and uncannily prophetic paper ever published in the field". Indeed, the paper appeared a quarter of a century before the process of gravitational collapse was widely accepted as the explanation for a variety of astronomical events. The Oppenheimer-Snyder paper also provided the first example in which a solution of the Einstein equations having interesting dynamics, was constructed by using the covariance of the equations to match two simpler solutions across an interface. However, it is well known that the Oppenheimer-Snyder model requires the simplifying assumption that the pressure be identically zero. In this paper we obtain a generalization of the Oppenheimer-Snyder model describing gravitational collapse which extends their model to the case when the pressure is non-zero. Our idea is to treat the case $p \neq 0$ by replacing the boundary surface of the star in the Oppenheimer-Snyder model by a shock-wave interface across which mass and momentum are transported. In the limit $p = 0$ we obtain the Oppenheimer-Snyder solution, and in this limit we observe that the interface reduces to what is referred to as a "contact discontinuity" in the mathematical theory of shock waves, a degenerate discontinuous solution in which neither mass nor momentum crosses the interface, [1, 5, 12].

Our mathematical procedure is to simultaneously find a (t, r) coordinate transformation and a shock surface $r = r(t)$ such that the Robertson-Walker (R-W) metric and the Interior Schwarzschild (I-S) metric, (two well known exact solutions of the Einstein equations $G = kT$ that are spherically symmetric in radial coordinate r, [16]), match in a Lipschitz continuous fashion across the shock surface. The (R-W) metric describes a uniformly expanding solution of the Einstein equations that is used in cosmology as a model for the universe as a whole. The R-W metric is determined by one unknown function $R(t)$, the *cosmological scale factor*, and in the universe model, the function $R(t)$ alone determines the redshift factor for electromagnetic radiation emmision from distant stars. The I-S metric is a time independent spherically symmetric solution of $G = \kappa T$, which has been used to model the interior of a stable star. Both metrics are determined by a system of ordinary differential equations that close when a particular equation of state $p = p(\rho)$ is imposed, (c.f [14], page 33). In our dynamically matched solution, we imagine the R-W metric as an exploding inner core (of a star or the universe as a whole), and the boundary of this inner core is a shock surface that is driven into the static I-S solution, which we imagine as the outer layers of a star, or the outer regions of the universe. Since the solution is time reversible for any equation of state, we can reverse the solution and equivalently obtain a shock-wave that collapses in on itself. We anticipate that the expanding shock is stable, while the collapsing shock solution is actually the analog of an unstable rarefaction shock, (in the sense of Lax [5]), except in the limit case $p = 0$ treated by Oppenheimer and Snyder. In the case $p = 0$, the shock interface reduces to a contact discontinuity, and in this degenerate case both the expanding and contracting solutions should be stable, (at least to radial perturbation). In this case, Oppenheimer and Snyder showed that the contracting shock-wave collpses to a "black hole", and they made the now well known paradoxical observation that the collapse to the black hole takes an infinite time as measured by the observer at infinity, but takes only a finite proper time as measured by the observer sitting on the shock-wave itself.

In [14] we construct a dynamical shock-wave solution of the Einstein equations that consists of the R-W metric on the inside connected to the I-S solution on the outside, with a shock-wave interface connecting the two, and we construct such a solution assuming arbitrary equations of state $p = p(\rho)$ and $\bar{p} = \bar{p}(\bar{\rho})$ in the R-W and I-S solutions, respectively. Our matching procedure is complicated by the fact that the coordinate systems for the R-W and I-S metrics must be treated differently on the inside and outside of the shock separately, (see [14]). An explicit formula for the shock position in terms of the jump in density emerges from the analysis, and this allows us to identify a *global* conservation of energy principle, despite the fact that, in general, conservation is only locally valid in Einsteins theory due to the effects of the curvature in the underlying metric g. For our general result here, the metrics match in a Lipschitz continuous fashion across the shock interface. There then arises the interesting issue of whether the weak, local form of conservation holds across this shock-wave. In Section 3 of our paper

[14], we present a self-contained treatment of the general theory of shock waves for the Einstein equations that re-organizes and extends previous results of Israel. (See also [8] and references therein where this topic is referred to by the heading *Junction Conditions*). We consider the general problem of allowing the metric to be only Lipschitz continuous across a hypersurface Σ in spacetime. In this case, the metric loses continuity in the first derivative. It turns out that the issue of conservation across an arbitrary Lipschitz continuous shock-wave is intimately related to when there exist delta function sources in G on the surface, and also intimately related to when there exists a $C^{1,1}$ coordinate transformation defined in a neighborhood of the shock that removes the jump in the derivatives of the metric potentials g_{ij} across the shock. We present a complete resolution of these issues in Theorem 3, Section 3 of our paper, which can be stated as follows:

Theorem 1 *Let Σ denote a smooth, 3-dimensional shock surface in spacetime with spacelike normal vector* $\mathbf{n}$. *Assume that the components g_{ij} of the gravitational metric g are smooth on either side of Σ, (continuous up to the boundary on either side separately), and Lipschitz continuous across Σ in some fixed coordinate system. Then the following statements are all equivalent:*

(i) $[K]=0$ at each point of Σ.

(ii) The curvature tensors R^i_{jkl} and G_{ij}, viewed as second order operators on the metric components g_{ij}, produce no delta function sources on Σ.

(iii) There exist locally Lorentzian coordinate frames at each point $P \in \Sigma$.

(iv) For each point $P \in \Sigma$ there exists a $C^{1,1}$ coordinate transformation defined in a neighborhood of P, such that, in the new coordinates, (which can be taken to be the Gaussian normal coordinates for the surface), the metric components are $C^{1,1}$ functions of these coordinates.

Moreover, if any one of these equivalencies hold, then the Rankine-Hugoniot jump conditions

$$[G]^\sigma_i n_\sigma = 0, \tag{2.5}$$

(which express the weak form of conservation of energy and momentum across Σ when $G=\kappa T$), hold at each point on Σ, a covariant statement.

Here $[K]$ denotes the jump in the second fundamental form (extrinsic curvature) K across Σ, (this being determined by the metric separately on each side of Σ because g_{ij} is only Lipschitz continuous across Σ), and by $C^{1,1}$ we mean that the first derivatives are Lipschitz continuous. Theorem 1 should be credited mostly to Israel, who obtained these results in Gaussian normal coordinates. Our contribution was to identify the covariance class of $C^{1,1}$ transformations, and to thereby obtain precise coordinate independent statements for (ii)-(iv). As a consequence of this, we obtain the result that the Ricci scalar curvature R *never* has delta function sources at a Lipschitz continuous matching of the metrics, as well as the following theorem that partially validates the statement that shock wave singularities in the source free Einstein equations $R_{ij} = 0$ or $G_{ij} = 0$ can only appear as coordinate anomalies, and can be transformed away by coordinate transformation:

Theorem 2 *If a smooth shock surface Σ forms in weak solutions of $R_{\alpha\beta} = 0$ or $G_{\alpha\beta} = 0$ posed in some given coordinate system y, such that the y-components $g_{\alpha\beta}$ of the metric tensor g are Lipschitz continuous across Σ, and C^k functions of y on either side of Σ (continuous up to the boundary on either side separately), then there exists a regular $C^{1,1}$ coordinate transformation taking $y \to x$, such that, the components g_{ij} of g in x-coordinates are actually C^k functions of x in a neighborhood of each point on the surface Σ.*

Note that when there are delta function sources in G on a surface Σ, the surface should be interpreted as a surface layer (because $G = \kappa T$), and not a shock wave, [3, 8]. In [14] we show that for spherically symmetric solutions, the weak form of conservation of energy and momentum (2.5) implies the absence of surface layers, so long as the areas of the spheres of symmetry match smoothly at Σ. We use this result in our construction of the shock waves that extend the Oppenheimer-Snyder model to the case of non-zero pressure.

We now discuss how we explicitly construct spherically symmetric shock-wave solutions of the Einstein equations (1.3) for a perfect fluid (1.4) by matching the Robertson-Walker (R-W) metric,

$$(R-W) \qquad ds^2 = -dt^2 + R^2(t)\left\{\frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2\right\}. \tag{2.6}$$

and Interior Schwarzschild (I-S) metric,

$$(I-S) \qquad d\bar{s}^2 = -B(\bar{r})d\bar{t}^2 + \left(1 - \frac{2M(\bar{r})}{\bar{r}}\right)d\bar{r}^2 + \bar{r}^2 d\Omega^2, \tag{2.7}$$

at shock wave interfaces across which the metrics are Lipschitz continuous, and the conditions (i)-(iv) of Theorem 1 above hold. Here, the quantity $d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2$ denotes the standard metric on the 2-sphere. The R-W metric is homogeneous at each fixed t, and expands or contracts in time according to the Cosmological Scale Factor $R(t)$. The fluid is assumed to be co-moving with the metric, c.f. [16], and the Einstein equations (1.3) reduce to a system of two ODE's in two unknowns when an equation of state $p = p(\rho)$ is specified. The R-W metric has been accepted as a model for the universe as a whole. In this interpretation, $R(t)$ determines the red-shift factors for far away objects, and the existence of a singularity in the R-W metric in backwards time has been interpreted as the original *big bang*. The I-S metric is a time independent solution of (1.3) that is used to model the interior of a star. In this case the stress tensor is again taken to be that of a perfect fluid that is co-moving with the metric, and the functions $B(\bar{r})$ and $M(\bar{r})$, (the total mass inside radius $\bar{r}$), are also determined by an autonomous system of two equations in two unknowns when an equation of state $\bar{p} = \bar{p}(\bar{\rho})$ is chosen, and the Einstein equations (1.3) are imposed.

The idea in the matching is to define a coordinate transformation mapping $(t, r) \rightarrow (\bar{t}, \bar{r})$ such that, under this identification of coordinates, the metrics R-W and I-S agree Lipschitz continuously on a 3-dimensional shock surface. We set $\bar{r} = Rr$ in order that the sphere's of symmetry match smoothly at the surface, and with this it remains to define $\bar{t} = \bar{t}(t, r)$. The following general theorem states that a Lipschitz continuous matching of metrics can be achieved for arbitrary R-W and I-S metrics,[14]

Theorem 3 *Let $B(\bar{r}), M(\bar{r}), \bar{\rho}(\bar{r}), \bar{p}(\bar{r})$ denote any I-S solution of the Einstein equations (1.3), and let $R(t), \rho(t), p(t)$ denote any R-W solution of (1.3). Then, (under non-degeneracy assumptions, [14]), there exists a coordinate mapping $(t, r) \rightarrow (\bar{t}, \bar{r})$ of the form $\bar{r} = Rr$ and $\bar{t} = \bar{t}(t, r)$ such that, under this identification, the metrics agree and are Lipschitz continuous at the shock surface*

$$M(\bar{r}) = \frac{4\pi}{3}\rho(t)\bar{r}^3. \tag{2.8}$$

The equation (2.8) defines the shock surface $\bar{r} = \bar{r}(t)$ implicitly so long as $\bar{\rho}'(\bar{r}) < 0$.

Note that for this theorem, arbitrary equations of state $p = p(\rho)$ and $\bar{p} = \bar{p}(\bar{\rho})$ can be assigned. The identity (2.8) can be interpreted as a global conservation of mass principle for such matchings: in words, (2.8) says that the total mass that the I-S solution would see inside the shock interface $\bar{r} = \bar{r}(t)$ if it were continued on into the origin $\bar{r} = 0$, equals the total mass inside a sphere of radius $\bar{r} = \bar{r}(t)$ and constant density $\rho(t)$.

A further constraint on the metrics must be imposed to insure that the Rankine-Hugoniot jump conditions (2.5) for conservation hold across a shock. By the general theory, it suffices to match the second fundamental forms $[K] = 0$ across the surface, and this alone will insure that the surface is not a surface layer, and that conditions of Theorem 1 will then all apply at the shock surface. In [14] we show that in the presence of spherical symmetry, $[K] = 0$ is equivalent to the single condition

$$[T^i_j]n_i n^j = 0. \tag{2.9}$$

Using this, we derive a system of two autonomous ordinary differential equations in the shock position $r(t)$ and the cosmological scale factor $R(t)$ that determine the R-W metrics that will match any given I-S metric at the shock surface (2.8) such that energy and momentum are conserved across the shock. In order to impose the extra constraint (2.9), we must allow the R-W pressure $p(t)$ to be determined by the equations, rather than by an equation of state. This is summarized in the following theorem,[14]:

Theorem 4 *Let $B(\bar{r}), M(\bar{r}), \bar{\rho}(\bar{r}), \bar{p}(\bar{r})$ denote any fixed I-S solution of the Einstein equations. Then, (under non-degeneracy assumptions), the R-W metric $R(t), \rho(t), p(t)$ will satisfy conservation across the shock surface (2.8), (under the identification $\bar{r} = Rr, \bar{t} = \bar{t}(t,r)$), if and only if $(r(t), R(t))$ solve the system of ODE's*

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - k, \tag{2.10}$$

$$\alpha \dot{r}^2 + \beta \dot{r} + \gamma = 0, \tag{2.11}$$

where all of the functions $\rho, \alpha, \beta, \gamma$ appearing in (2.10) and (2.11) can be expressed in terms of the unknowns $(r(t), R(t))$ through the shock surface equation (2.8) and the identity $\bar{r} = Rr$. The R-W pressure is then given by

$$p = -\frac{\frac{d}{dt}(\rho R^3)}{3R^2\dot{R}}. \tag{2.12}$$

Thus system (2.10), (2.11) gives the conservation constraint in terms of a first order *autonomous* ODE in these unknowns, and the solutions $(r(t), R(t))$ of this ODE determine the R-W metrics that will match a given I-S metric across a shock surface, such that the additional condition of conservation is maintained across the shock. The solutions that solve this ODE reduce to the Oppenheimer-Snyder solutions when M is constant and the I-S metric reduces to the empty space Schwarzschild metric. In the O-S case, the solution of (2.10), (2.11) reproduces $p \equiv 0$. Note that for a given I-S solution, we can in principle construct a shock wave at radius $\bar{r} = Rr$ with arbitrary R-W pressure p assigned at that radius by specifying the appropriate initial conditions for r and R.

3 Astrophysics and Cosmology

We now discuss possible applications of our Oppenheimer-Snyder type shock waves to astrophysics and cosmology. First of all, our solutions can be taken as a dynamical model for an evolving star which is either collapsing or exploding within the core, but which shows no visible signs of this collapse on the surface of the stationary I-S solution which sits on the outside. The possibility of such a dynamical solution raises interesting questions concerning limiting radii of stable stars as first discussed by Oppenheimer-Volkov and Chandrasekhar ([10]). In particular, when the I-S solution is taken as a model for the interior of a star, there is a theorem that states that for essentially any possible equation of state within the star, the I-S metric must have an infinite pressure at the center $r = 0$ whenever the entire mass of the star is compressed within 9/8 of it's Schwarzschild radius (c.f. [15], page 231). This implies that there are no stable configuations of stars that are sufficiently compressed, as it would require an infinite pressure at the center to hold up such a star once it falls within 9/8 of it's Schwarzschild radius. This has interesting cosmological consequences. In particular, a calculation shows that the maximum redshift factor[4] for light emitted from the surface of a stable star can be no more than 2, (see [15]). This becomes all the more interesting because of the discovery of quasars, (relatively bright stellar

[4]The redshift factor is the ratio given by the frequency of light received by an observer in a stationary solution, divided by the frequencey of light emitted by the source.

objects which are point sources of electro-magnetic radiation which are redshifted all the way into the radio wave frequencies), which have a red shift factor much larger than 2. Now redshifting of the light emitted by a stellar object can arise in one of two ways: either the object is far away, so that the expansion of the universe moves the object away at a high velocity, or else the light is climbing out of a large gravitational potential. Since the stability theorem appears to rule out the latter, it has been conjectured that quasars are far away, and this leads to the incredible conclusion that the quasars must be fantastically large objects in order to explain their relatively bright emissions. But our model applies to highly compressed I-S solutions; specifically, to solutions whose total mass $M(r)$ lies well within the stability limit of 9/8 of the associated Schwarzschild radius. Since our procedure enables us to replace the inner core of any I-S solution with a dynamical expanding smooth solution, we can apply this to I-S solutions having a singular infinite pressure at $\bar{r} = 0$, and thereby obtain a solution with no singularity at $\bar{r} = 0$. Thus the conclusion we can make based upon our model is that a shock-wave in the core can generate the enormous pressures necessary to hold up the outer layers of the star even when the outer layers are well within $9/8$ of the Schwarzschild radius. (For comparison, the Schwarzschild radius of the sun is about 3 kilometers.) Thus our *dynamical* shock-wave solution allows for the possibility of a stellar object that appears to us as a point source of light in the heavens that is emitting highly redshifted frequencies of radiation, and, according to our model, the surface of such a star would show no signs of the dynamics within the interior. Moreover, since (if the calculation of Oppenheimer-Snyder carries over to the case of non-zero pressure), the collapse of the shock-wave may well take an infinite time as measured by an observer far away from the star, our dynamical solution allows for the possibility that such an object would be stable for long periods of time. Thus our dynamical model supports the possibility that certain objects emitting highly red-shifted frequencies of electromagnetic radiation could really be dynamical, quasi-stable objects that are relatively nearby, thus resolving the paradox of their brightness.

Another intriguing possibility which our Oppenheimer-Snyder type shocks lead us to, is the possibility of a new model for the universe at large. The widely held view is that the universe as a whole can be described by an expanding solution of the Einstein equations which can be approximately modeled by the Roberson-Walker metric. Under these assumptions, the cosmological scale factor $R(t)$ in the R-W metric is the only physical parameter to be determined, and this is determined by the average mass-energy density contained within the universe at large. The Hubble constant gives a measure of the expansion in this model, and all of the information relating stellar distances to red-shift factors is based on the assumption of this model. One of the great open questions in physics involves the determination of the average energy density of the universe, and this is complicated by an inability to determine the amount of *dark matter*, or un-illuminated matter in the universe. Our shock-wave solution that connects the R-W metric on the inside to the I-S metric on the outside, opens the possibility that there is a shock-wave at the edge of the universe. It is interesting to pursue the possibility that the 2.7 degree Kelvin background radiation observed throughout the universe represents the black-body radiation from the boundary of such a shock. Assuming this, we can guess the energy density on the I-S side of the shock, and thereby, from our model, we should be able to determine the average energy density in the universe from certain theoretically observable parameters in the model, such as the distance to the shock-wave and the speed of the shock.[5] Even if this model is not exactly correct, these relations which are determined by the model may have an interest that transcends the model itself.

These possible physical applications make the problem of determining the large time dynamics of our Oppenheimer-Snyder shocks all that much more interesting.

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