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SYSTEMS OF CONSERVATION LAWS WITH COINCIDING SHOCK AND RAREFACTION CURVES*

BLAKE TEMPLE

1. INTRODUCTION

Systems of conservation laws which have coinciding shock and rarefaction curves arise in the study of oil reservoir simulation, nonlinear wave motion in elastic strings, as well as in multicomponent chromatography [1, 4, 5, 6, 9, 11, 12]. These systems have many interesting features. The Riemann problem for these equations can be explicitly solved in the large, and wave interactions have a simplified structure, even in the presence of a nonconvex flux function. For this reason, these systems represent some of the few examples for which the Cauchy problem has been solved for arbitrary data of bounded variation. Also, hyperbolic degeneracies appear in each of these systems. In the present paper we are concerned with locating the class of equations that exhibit the phenomenon of coinciding shock and rarefaction curves. For $n \times n$ systems, we give necessary and sufficient conditions for a shock curve to coincide with a rarefaction curve. We use these general results to write down explicitly the class of 2×2 conservation laws which have shock and rarefaction curves that coincide.

A system of conservation laws in one space dimension is a set of partial differential equations of the form

$$U_t + F(U)_x = 0. \tag{1A}$$

Here $-\infty < x < \infty$, $t \geq 0$, and U and F are vector valued functions, $U = (u_1,\ldots,u_n) \equiv U(x,t)$, $F(U) = (f_1(U),\ldots,f_n(U))$. The Cauchy problem is the natural problem to pose for system (1), and it is commonly known that discontinuities can form in the solutions of (1). For this reason we look for weak solutions $U(x,t)$; i.e., solutions that satisfy the following integral equation [7] for any smooth function $\psi(x,t)$ with compact support:

$$\iint_{\substack{-\infty<x<+\infty \\ t\geq 0}} U\psi_t + F(U)\psi_x + \int_{-\infty}^{\infty} U(x,0)\psi(x,0)dx = 0. \tag{1B}$$

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Two important systems of conservation laws arise in applications, and have been studied in [1, 4, 5, 6, 9, 11, 12].

$$\begin{aligned} u_t + \{u\phi(u,v)\}_x &= 0 \\ v_t + \{v\phi(u,v)\}_x &= 0. \end{aligned} \tag{2}$$

$$\begin{aligned} u_t + \left\{\frac{u}{1+u+v}\right\}_x &= 0 \\ v_t + \left\{\frac{\kappa v}{1+u+v}\right\}_x &= 0. \end{aligned} \tag{3}$$

System (2) arises in problems of oil reservoir simulation, as well as in elasticity theory [4, 6, 9, 12]. For example, in the reservoir simulation problem [4, 6], u is the saturation of water in the reservoir and v is the concentration of a polymer in the water, so that $0 \le u \le 1$, $0 \le v \le 1$. The system is determined by specifying the function $\phi(u,v)$, but the structure of the solutions is determined by qualitative properties of ϕ which can be verified experimentally.

System (3) arises in the study of two component chromatography [1, 5, 11]. Here u and v are transformations of the concentrations of the two solutes, and x and t are transformations of the actual space and time variables (see Aris & Amundson [1], pp. 268). The domains of the variables u and v can be taken to be $u \ge 0$, $v \ge 0$, and $\kappa \in (0,1)$ is determined by adsorption properties of the stationary phase.

Systems (2) and (3) are remarkable because for both systems, the shock curves and rarefaction curves coincide. This leads us to study the phenomenon of coinciding shock and rarefaction curves in general. To say this precisely, assume first that system (1) is hyperbolic; i.e., that the eigenvalues (wave speeds) of dF (the matrix defined by $F(U)_x = dF \cdot U_x$) are real, but not necessarily distinct. Let λ denote an eigenvalue, and R a corresponding eigenvector of dF. We call (λ,R) a "characteristic family" or "characteristic field" for system (1) if $\lambda(U)$, $R(U)$ are defined and C^3 in some neighborhood N of U-space. Let $S \subset N$ be the integral curve of R through some point $U_0 \in N$. S is called the λ-rarefaction curve of U_0 in N. Rarefaction curves are the one-dimensional sets that smooth solutions to system (1) can take values on. For example, if the range of a smooth solution $U(x,t)$ of system (1) lies on a one-dimensional curve in U-space, then that curve must be a rarefaction curve. An analagous one-dimensional curve in U-space applies to the study of discontinuous solutions of (1). The Hugoniot locus of a point U_0 is defined to be the set of points U such that

$$\sigma[U] = [F(U)] \tag{4}$$

for some scalar $\sigma = \sigma(U,U_0)$, $[U] = U - U_0$, $[F(U)] = F(U) - F(U_0)$. A state U_1 is in the Hugoniot locus of U_0 if and only if the discontinuous function

$$U(x,t) = \begin{cases} U_0 & \text{for } x < \sigma t \\ U_1 & \text{for } x > \sigma t \end{cases} \tag{5}$$

satisfies the weak form (1B) of system (1) [2, 7]. Under very general conditions, there corresponds to each family (λ,R) a one parameter subset of the Hugoniot locus of U_0 that has C^2 contact with the integral curve of R at U_0 [7]. This is called the λ-shock curve. (Often this term is reserved for that portion of the curve that determines the physically acceptable solutions in (5).)

DEFINITION. We say that the λ-shock curve coincides with the λ-rarefaction curve on S if the Hugoniot locus of each point on S, contains S.

In [13] we prove Theorem 1 (details are omitted here) which states that the λ-shock curve and λ-rarefaction curve coincide on S if and only if either S is linear in U-space or λ is constant on S; and this occurs if and only if the equations reduce to a scalar conservation law on S. We call (λ,R) a "contact" family if λ is constant on each integral curve of R, and we call (λ,R) a "line" family if each integral curve of R is a straight line in U-space. In the next section we derive the class of 2×2 equations that have either a contact or a line family. (In this case we let $U \equiv (u,v)$ and for convenience we assume that a contact family satisfies $\nabla\lambda = (\frac{\partial\lambda}{\partial u}, \frac{\partial\lambda}{\partial v}) \neq 0$ in N, and that a line family satisfies $\nabla q \neq 0$ in N, where $q(u,v)$ is the slope $\frac{dv}{du}$ of the integral curve of R through (u,v). Weaker assumptions can be made). All of the characteristic families in systems (2) and (3) are then seen to be either line or contact families. In this way the phenomenon of coinciding shock and rarefaction curves is observed from the explicit form of the equations.

2. COINCIDING SHOCK AND RAREFACTION CURVES FOR 2×2 SYSTEMS

Consider an arbitrary system of 2×2 conservation laws.

$$\begin{aligned} u_t + f(u,v)_x &= 0 \\ v_t + g(u,v)_x &= 0, \end{aligned} \tag{2.1}$$

where we take $U \equiv (u,v)$, $F = (f,g)$. We now locate the class of such 2×2 equations that have either a contact or a line field in a region N of U-space. These are generically the only fields that have coinciding shock and rarefaction curves, as indicated by Theorem 1.

We let $q = q(u,v)$ denote a Riemann invariant for a contact or line field in N. A Riemann invariant for a family (λ,R) is a function which is constant on the integral cuves of R. We assume that $\nabla q \neq 0$, and because it suffices to prove our results locally, we always assume that $q = \text{const.}$ determines a unique integral curve of R in N, which has a finite slope $\frac{dv}{du}$.

First assume that $q(u,v)$ is the wave speed of a contact field for a system of 2×2 equations, $\nabla q \neq 0$. The function q is thus a Riemann invariant for the contact field given by $\lambda = q$, $R = (-q_v, q_u) \equiv \nabla q^{\perp}$. Since (λ,R) is a contact field in N, we have by Theorem 1 that the λ-shock curves coincide with the λ-rarefaction curves in N, so the curve $q = \text{const.}$ must be contained within the Hugoniot locus of every point on that curve. Thus by (4), if $[q] \equiv q(U) - q(U_0) = 0$, then also

$$\begin{aligned} \sigma[u] &= [f], \\ \sigma[v] &= [g], \end{aligned} \tag{2.2}$$

and in the contact case, $\sigma \equiv \sigma(U,U_0) = q(U_0)$ (c.f. [7]). Conversely, if (2.2) holds in N when $[q] = 0$ for some smooth function $q(u,v)$, $\nabla q \neq 0$, then $(\sigma(U_0,U_0), \nabla q^{\perp}(U_0))$ must be a characteristic field for dF. To see this, note that (2.2) implies that

$$\begin{aligned} \sigma(U_0,U_0) &= \lim_{U \to U_0} \frac{[f]}{[u]} = f_u + f_v \frac{dv}{du} \\ &= \lim_{U \to U_0} \frac{[g]}{[v]} = g_v + g_u \frac{du}{dv}, \end{aligned}$$

where the vector $(1, \frac{dv}{du})$ is parallel to $\nabla q(U_0)$. Therefore, we can verify that

$$dF \cdot (1, \tfrac{dv}{du})^{tr} = \sigma(U_0,U_0)(1, \tfrac{dv}{du}). \tag{2.3}$$

Thus the statement that (2.2) holds with $\sigma = q$ when $[q] = 0$ in N, is equivalent to the statement that $(\lambda,R) = (q, \nabla q^{\perp})$ is a contact field in N. But (2.2) holds when $[q] = 0$ if and only if

$$\begin{aligned} [f - uq] &= 0, \\ [g - vq] &= 0, \end{aligned} \tag{2.4}$$

when $[q] = 0$; and (2.4) holds if and only if $f = uq + F(q)$ and $g = vq + G(q)$ for some smooth functions F and G. We have the following theorem:

THEOREM 2. A system of 2×2 conservation laws (2.1) has a contact field in a domain N of uv-space if and only if f and g satisfy

$$\begin{aligned} f(u,v) &= uq + F(q), \\ g(u,v) &= vg + G(q), \end{aligned} \tag{2.5}$$

in N, for some smooth functions q, F and G, $\nabla q \neq 0$. In this case $q(u,v) = \lambda$ where λ is the wave speed of the contact family.

Next assume that (λ, R) is a line family for a system of 2×2 equations defined in a region N of U-space. Let $q(u,v)$ be the slope $\frac{dv}{du}$ of the integral curve of R through the point $(u,v) \in N$, $\nabla q \neq 0$. The function q is a Riemann invariant of R. Moreover, q is a smooth nonconstant solution to Burger's equation

$$q_u + q\, q_v = 0 \tag{2.6}$$

since ∇q is orthogonal to the vector $(1,q)$ at every point in N. Since (λ, R) is a line field, Theorem 1 again implies that a curve defined by $q = \text{const.}$ contains the Hugoniot locus of each point on that curve. Thus, when $[q] = 0$,

$$\sigma[u] = [f]$$
$$\sigma[v] = [g].$$

Dividing we obtain that, when $[q] = 0$,

$$q = \frac{[v]}{[u]} = \frac{[g]}{[f]}, \tag{2.7}$$

so that

$$[fq - g] = 0. \tag{2.8}$$

By (2.3) this is equivalent to the statement that (λ, R) is a line field in N. But (2.8) holds if and only if

$$g = fq + H(q)$$

for some smooth function H. We have proven the following theorem:

THEOREM 3. A system of 2×2 conservation laws has a line family in N if and only if f and g satisfy

$$g = fq + H(q) \tag{2.9}$$

in N, for some smooth function H, where $q(u,v)$ is a smooth solution to Burger's equation (2.6) with $\nabla q \neq 0$. In this case q is a Riemann invariant for the line family.

Theorem 2 applies to system (2) with $F = G \equiv 0$, and thus $(\phi, \nabla\phi^{\perp})$ must be a contact family for system (2). Since system (2) also satisfies

$$g(u,v) = v\phi(u,v) = \frac{v}{u} f(u,v),$$

and $\frac{v}{u}$ is a smooth solution to Burger's equation in $u > 0$, $v > 0$, we have by Theorem 3 that $q(u,v) = \frac{v}{u}$ is the Riemann invariant of a linear family for system (2). Moreover, we can use Theorems 2 and 3 to locate the class of 2×2 equations that have both a line and a contact field; i.e., we say that system (1) has both a contact and a line field in N if system (1) has two Riemann invariants q and p that satisfy Theorem 3 and 4 respectively, such that u and v are smooth functions of (p,q) off a closed set of measure 0 in N. By Theorems 2 and 3,

$$vq + G(q) = uqp + F(q)P + H(p). \tag{2.10}$$

Formally differentiating (2.10) with respect to u holding p fixed yields

$$v + G'(q) = up + F'(q)p \tag{2.11}$$

since $\frac{\partial}{\partial u} v(u,p) = p$ because p is a smooth solution of (2.6). Differentiating (2.10) with respect to q holding p fixed gives

$$\frac{\partial}{\partial q} V(p,q) + v + G'(q) = up + \frac{\partial}{\partial q} u(p,q)qp + F'(q)p. \tag{2.12}$$

Therefore, substituting (2.11) into (2.12) we obtain

$$\frac{\partial}{\partial q} v(p,q) = p \frac{\partial}{\partial q} u(p,q). \tag{2.13}$$

Now differentiate (2.11) with respect to q holding p fixed and obtain

$$\frac{\partial}{\partial q} v(p,q) + G''(q) = P \frac{\partial}{\partial q} u(p,q) + F''(q)p, \tag{2.14}$$

which by (2.13) is

$$G''(q) = F''(q)p. \tag{2.15}$$

Finally, differentiating (2.15) with respect to p holding q fixed, we conclude

$$F''(q) = G''(q) = 0, \tag{2.16}$$

or

$$F(q) = aq + c, \quad G(q) = bq + d \tag{2.17}$$

for some constants a, b, c, d.

The assumptions made in (2.11) to (2.17) are that v is a differentiable function of $u(p,q)$, $\frac{\partial}{\partial u} q(u,p) \neq 0$, and $q \neq 0$. But by (2.6) these must hold off a closed set of measure 0 in N, so (2.17) must hold everywhere in N. Moreover since the addition of a constant to the flux functions f and g in (2.1) does not affect the solutions, we can take $c = d = 0$. Substituting (2.17) into (2.11) then gives

$$p = p(u,v) = \frac{v + b}{u + q},$$

and the constraint in (2.10) yields

$$H(p) = -bp. \tag{2.18}$$

We have thus proven the following corollary:

COROLLARY 1. System (2.1) has both a line and a contact field if and only if

$$f = (u + a)q,$$
$$g = (v + b)q, \tag{2.19}$$

for some smooth function $q = q(u,v)$, and some constants a and b. In this case q is the wave speed of the contact family, and

$$p = \frac{v + b}{u + a}. \tag{2.20}$$

Now consider an arbitrary 2×2 system that has two distinct line families. By Theorem 3, there exist two distinct solutions p and q of Burger's equation such that (2.9) holds; i.e., such that

$$g = fp + H_1(p), \tag{2.21}$$

$$g = fq + H_2(q), \tag{2.22}$$

for some smooth functions H_1 and H_2. Equating (2.10) and (2.11) gives

$$f = \frac{H_1(p) - H_2(q)}{q - p}.$$

This proves the following corollary:

COROLLARY 2. System (2.1) has two line families if and only if

$$f = \frac{H_1(p) - H_2(q)}{q - p},$$
$$g = \frac{gH_1(p) - pH_2(q)}{q - p}, \tag{2.23}$$

where p and q are smooth solutions of (2.6).

For system 3 one can verify that

$$\frac{u}{1+u+v} = \frac{H(p) - H(q)}{q - p}, \quad \frac{Kv}{1+u+v} = \frac{gH(p) - pH(q)}{q - p}, \tag{2.24}$$

where

$$H(z) = \frac{z - \kappa z}{z + 1}, \tag{2.25}$$

and p,q are the two solutions of Burger's equation which satisfy

$$uz^2 + \{\kappa(u+1) - (v+1)\}z - \kappa v = 0 \tag{2.26}$$

in z, and are smooth in $u > 0$, $v > 0$. This verifies that system (3) has a pair of line families with integral curves given by $p = \text{const.}$ and $q = \text{const.}$

Finally, note that a system of 2×2 equations generically has two characteristic families. The functions f and g given in (2.5) and (2.9) involve explicitly a Riemann invariant of one family. The following result, which is easily verified, determines both characteristic families given a Riemann invariant of one family. This general result is simple and explains many of the calculations in [4, 12].

THEOREM 4. Let $q = q(u,v)$ be a Riemann invariant for a system of 2×2 conservation laws (2.1), and let $\overline{f}(u,q) = f(u,v)$ and $\overline{g}(v,q) = g(u,v)$ be smooth. Then (λ_i, R_i), $i = 1,2$ are characteristic fields for the system, where

$$\begin{aligned} \lambda_1 &= \overline{f}_u, R_1 = \nabla q^{\perp} \\ \lambda_2 &= \overline{f}_u + \overline{f}_q q_u + \overline{g}_q q_v, R_2 = (\overline{f}_q, \overline{g}_q). \end{aligned} \tag{2.27}$$

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DEPARTMENT OF MATHEMATICAL PHYSICS
THE ROCKEFELLER UNIVERSITY
NEW YORK, NEW YORK 10021