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Solutions of the Oppenheimer–Volkoff Equations Inside $9/8^{ths}$ of the Schwarzschild Radius

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Abstract: We refine the Buchdahl $9/8^{ths}$ stability theorem for stars by describing quantitatively the behavior of solutions to the Oppenheimer–Volkoff equations when the star surface lies inside $9/8^{ths}$ of the Schwarzschild radius. For such solutions we prove that the density and pressure always have smooth profiles that decrease to zero as the radius $r \to 0$, and this implies that the gravitational field becomes *repulsive* near $r = 0$ whenever the star surface lies within $9/8^{ths}$ of its Schwarzschild radius.

1. Introduction

In General Relativity, the interior of a star is modeled by solutions of the Oppenheimer–Volkoff (OV) equations which describe the pressure gradient inside a static fluid sphere. In this paper we describe the global behavior of the density, pressure, and gravitational field when the surface of the star lies within $9/8^{ths}$ of its Schwarzschild radius. The well-known Buchdahl stability theorem, [1], states, loosely speaking, that when the surface of a star lies within $9/8^{ths}$ of its Schwarzschild radius, then the star is unstable to gravitational collapse, and this result is essentially independent of the equation of state. This places a maximum red-shift factor of 2 on the possible emission spectrum from the surface of a spherically symmetric, static stellar object. The precise statement of Buchdahl's theorem is as follows, ([2], p. 332). Let $\rho(r)$ and $p(r)$ denote the density and pressure, respectively, and let $M(r)$ denote the mass function at radius $r < R$, where R denotes the surface of the star. (We call ρ the density so that ρc^2 is the energy-density

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of the fluid, and c denotes the speed of light.) Assume that these functions satisfy the Oppenheimer–Volkoff equations, ((2.1), (2.2) below), and that the following conditions hold:

(A) The radius $R > 0$ of the star is fixed, and the density $\rho(r)$ and pressure $p(r)$ are arbitrary bounded positive functions defined on $0 \leq r < +\infty$, such that $\rho(r) = 0 = p(r)$ for $r \geq R$. The metric is assumed to be attached smoothly to the empty space Schwarzschild metric at $r = R$.

(B) The mass function $M(r)$ is given by

$$M(r) = \int_0^r 4\pi\rho(s)s^2 ds,$$

so that the total mass of the star is given by

$$M_0 = \int_0^R 4\pi\rho(s)s^2 ds.$$

(C) The metric coefficient A, defined by

$$A(r) \equiv 1 - \frac{2\mathcal{G}M(r)}{c^2 r},$$

where $\mathcal{G}$ denotes Newton's gravitational constant, satisfies

$$A(r) > 0.$$

(D) The density $\rho(r)$ does not increase outward:

$$\rho'(r) \leq 0.$$

Then, assuming (A)–(D), the conclusion of the Buchdahl theorem is that, if $\rho(r)$, $p(r)$ and $M(r)$ satisfy the OV equations, the surface $r = R$ must satisfy

$$R > \frac{9}{8}R_s(M_0),$$

where $R_s(M_0) = \frac{2\mathcal{G}}{c^2}M_0$ denotes the Schwarzschild radius of a star of total mass M_0. Here $\mathcal{G}$ denotes Newton's gravitational constant. The stability limit for stars is obtained from this theorem by concluding that if the boundary surface of a star satisfies $R \leq \frac{9}{8}R_s(M_0)$, then one of the above assumptions must fail. However, no information is given about exactly how (A)-(D) fail in this case. For example, can $A \to 0$ for some $r > 0$? (This would correspond to the formation of a black-hole.) Can $p \to \infty$ for some $r \geq 0$? Can $M(0) = 0$ fail, or does the solution fail to exist on the entire interval $[0, R]$ for some other reason? In addition, what is the behavior of the solutions as $A(R) \to 0$; i.e., as the star surface tends to its Schwarzschild radius? In this paper we describe the global behavior of solutions of the OV equations starting from initial data satisfying $R_s(M_0) < R \leq \frac{9}{8}R_s(M_0)$, and as a corollary we obtain a refinement of Buchdahl's theorem.

We have been led to study such solutions in detail because of our earlier work, [3, 4], in which we constructed shock-wave solutions of the Einstein equations by attaching a Friedmann-Robertson-Walker metric to the inside of an arbitrary static metric determined by the Oppenheimer–Volkoff equations, such that the interface between them is an outward moving shock-wave. In the forthcoming paper [7] we study shock-wave solutions of the Einstein equations arbitrarily close to the Schwarzschild radius by placing an outgoing shock-wave inside the static solutions that we analyze here. In such a construction the shock-wave stabilizes the solution by supplying the pressure required to "hold the star up"even when $R_s(M_0) < R \leq \frac{9}{8}R_s(M_0)$.

In order to make the exposition as simple as possible, we assume throughout that a baryotropic equation of state of the form $p = p(\rho)$ is given, where the function $p(\rho)$ satisfies the conditions that $\frac{p}{\rho}$ and $p'(\rho)$ are bounded above and below by positive constants. Note that in this case $\sqrt{p'}$ is the sound speed, which for physical reasons should be bounded by c. Our approach is to start with initial conditions at $r = r_0 > 0$, and in terms of this data we estimate the solution for $0 < r < r_0$. This contrasts with the standard approach which is to assume conditions at $r = 0$.

We prove that any solution of the OV equations starting from initial data at $r = r_0$, and satisfying $r_0 \leq \frac{9}{8}R_s(M(r_0))$, will necessarily exist all the way into $r = 0$, and $A(r) > 0$ for all $r \geq 0$. Moreover, we show that the pressure p and density ρ *never* tend to ∞, and actually are bounded and tend to zero smoothly as $r \to 0$. (This contrasts with the case when $r_0 > \frac{9}{8}R_s(M(r_0))$, in which case we can have $p \to \infty$, cf. [4].) We prove that what always happens is that the mass function M hits zero at some $r_1 > 0$, then goes negative for $r < r_1$, and $M'(r)$ remains positive for all $r \geq 0$. Moreover, $M(r) \to M(0)$ as $r \to 0$, where $-\infty < M(0) < 0$. Indeed, we show that the density ρ and pressure p increase as r decreases until they reach a critical value $r = r_2$, $0 < r_2 < r_1$, (so that $M(r_2) < 0$), and then ρ and p decrease to zero as $r \to 0$. Moreover, we also prove that $\lim_{r\to 0} \rho'(r) = \lim_{r\to 0} p'(r) = 0$, which implies that ρ and p have smooth profiles at $r = 0$. Thus we conclude that in the presence of positive density and pressure, a *repulsive* gravitational effect appears, (i.e., $p' > 0$ near $r = 0$), due to a negative mass function inside $r = r_1$.

In light of the above, our results show that hypotheses (C) and (D) are actually consequences of the other assumptions in Buchdahl's theorem because (B) implies that $M(r) \geq 0$ for all $r \geq 0$. Moreover, when $M_0 \equiv M(r_0) \leq \frac{9}{8}R_s(M(r_0))$, we show that the region of the solution where $M(r) \geq 0$ accumulates in a thin layer that tends to $r = r_0$ as r_0 tends to its Schwarzschild radius $R_s(M(r_0))$, and we obtain sharp estimates for the width of this layer. Note finally that the hypotheses of the Buchdahl theorem do not explicitly assume the existence of an equation of state. Although in our treatment here we assume the equation of state is of the form $p = p(\rho)$, we could be more general by assuming only that $\mu(r) = \frac{p}{\rho}$ and $\sigma(r) = \frac{p'}{\rho'}$ are any given positive functions that are bounded above and below by positive constants; c.f. [6].

The main results of this paper are summarized in the following theorem which gives a refinement of Buchdahl's result. In what follows we utilize the variable z defined by

$$z \equiv \frac{\rho}{\bar{\rho}}, \tag{1.1}$$

where $\bar{\rho}(r)$ is the average density inside radius r, defined by

$$\bar{\rho} \equiv \frac{3}{4\pi}\frac{M(r)}{r^3}. \tag{1.2}$$

Theorem 1. *Let* $(r_1, r_0]$, $0 \leq r_1 \leq r_0$, *be the maximal interval of existence of a positive smooth solution,* $\rho(r) > 0$, $p(r) > 0$, *and* $M(r) > 0$, *of the OV system, (given in (2.1), (2.2) below), starting from positive initial data at* $r = r_0$ *which satisfies*

$$0 < A(r_0) \equiv 1 - \frac{2\mathcal{G}M(r_0)}{c^2 r_0} < 1.$$

Then $M'(r) > 0$ and $A(r) > 0$ throughout $(r_1, r_0]$, $M(r_1) = 0$, and the following hold:

(i) If $r_1 = 0$, then $A(r_0) > \frac{1}{9}$, or equivalently $r_0 > \frac{9}{8}R_s(M(r_0))$.

(ii) If $r_1 > 0$, then the functions $\rho(r)$, $p(r)$ and $M(r)$ can be continued to the interval $[0, r_1]$ as bounded smooth solutions of the OV system, such that ρ, p, A and M' remain positive, but $M(r)$ is negative on $[0, r_1)$. Moreover, there exists a unique point $r_2 \in (0, r_1)$ such that the density ρ and pressure p increase on the interval $[0, r_2)$ and decrease on the interval $(r_2, r_0]$, and the following equalities hold:

$$\lim_{r\to 0} \rho(r) = \lim_{r\to 0} p(r) = \lim_{r\to 0} \rho'(r) = \lim_{r\to 0} p'(r) = 0, \tag{1.3}$$

and

$$\lim_{r\to 0} M(r) = M(0), \tag{1.4}$$

where $M(0)$ is a finite negative number.

(iii) Assume that the initial values satisfy the further conditions that

$$0 < z_0 < 1, \tag{1.5}$$

$$0 < A_0 \leq \frac{1}{9}. \tag{1.6}$$

Then $r_1 > 0$, and there exists a unique point r_*, $r_1 < r_* < r_0$, such that $z(r_*) = 1$, $z(r) < 1$ for $r > r_*$, $z(r) > 1$ for $r < r_*$, and the following inequalities hold:

$$1 > \frac{r_*}{r_0} > \sqrt{\frac{1 - 9A(r_0)}{1 - A(r_0)}}, \tag{1.7}$$

and

$$\rho(r) < \rho(r_*) \leq \frac{3}{8\pi\mathcal{G}r_0^2}\left(\frac{1 - A_0}{1 - 9A_0}\right), \tag{1.8}$$

for all r in the interval $r_* \leq r < r_0$.

(iv) For fixed $r_0 > 0$ and $z_0 > 0$, $r_1 \to r_0$ as $A_0 \to 0$.

Note that whenever $M(r)$ tends to a finite negative number at $r = 0$, the metric must have a singularity at $r = 0$ because $A(r) = 1 - \frac{2\mathcal{G}M(r)}{r}$. We will show below that such singularities in solutions of the OV equations are non-removable, and we will use the results in [3] to show that this singularity corresponds to a delta fuction source of negative mass at $r = 0$.

As a consequence of this theorem, it follows that for any solution of the OV system, the pressure can tend to ∞ only at the origin $r = 0$; i.e., by (ii), p is uniformly bounded if $r_1 > 0$, so p can tend to ∞ only at $r = 0$.

Note that part (i) refines the Buchdahl result because it implies that if the mass $M(r)$ ever gets within $9/8^{ths}$ of the Schwarzschild radius $R_s(M(r))$, then $r_1 > 0$, so M must go negative before $r = 0$, thereby violating the definition of M given in (B). Also, since $\rho'(r) > 0$ for r near zero, we see that (D) is also violated. Note too that in our theorem, the critical $9/8'$ths limit applies at any radius interior to the star, while in Buchdahl's argument the $9/8'$ths limit applies only at $r = R$, the surface of the star. Moreover, the fact that A stays positive is a *theorem* in our treatment, not an assumption, and we demonstrate the failure of (D) when $r_0 \leq \frac{9}{8}R_s(M(r_0))$, in which case (ii) and (iii) give the global behavior of solutions that start inside $9/8^{ths}$ of the Schwarzschild radius. Theorem 1 also rules out the possibility that $p \to \infty$ as $r \to 0$ in the critical case when r_0 is exactly $\frac{9}{8}R_s(M(r_0))$, because when $r_0 = \frac{9}{8}R_s(M(r_0))$, Theorem 1 implies that $r_1 > 0$. (See [2], p. 334, where $p \to \infty$ as $r \to 0$ and $r_0 = \frac{9}{8}R_s(M(r_0))$, but in this case $\rho \equiv const$, and so this example violates our assumption that p/ρ remains bounded.) Note also that since $r_1 \to r_0$ as $A_0 \to 0$, and $M(r_1) = 0$, it follows that the entire portion of the solution in which the mass M is positive, accumulates in a thin layer that tends to $r = r_0$ as A_0 tends to zero. In [7] we use our detailed description of this layer to analyze dynamical solutions in which a shock-wave inside the layer supplies the pressure required to hold the layer up when A_0 is arbitrarily close to zero.

Statement (1.3) implies that the density $\rho(r)$ and pressure $p(r)$ are everywhere positive and have smooth profiles that tend to zero as $r \to 0$, and this implies that the gravitational field becomes *repulsive* near $r = 0$ (when $M(r)$ is negative). Note that $M(r) < 0$ for $r > 0$ is not ruled out in general relativity, (so long as the density and pressure are positive), because $M(r)$ is not an invariant quantity. This issue is discussed in the final section of this paper.

2. Statement of Results

Theorem 1 is a consequence of the results stated in this section; in the next section we will supply the proofs of the theorems in the order that they are presented here.

The Oppenheimer–Volkoff (OV) system is, (cf. [2]),

$$-r^2\frac{dp}{dr} = \mathcal{G}M\rho\left(1+\frac{p}{\rho c^2}\right)\left(1+\frac{4\pi r^3 p}{Mc^2}\right)A^{-1}, \tag{2.1}$$

$$\frac{dM}{dr} = 4\pi\rho r^2, \tag{2.2}$$

where

$$A \equiv A(r) = 1 - 2\frac{\mathcal{G}}{c^2}\frac{M(r)}{r}. \tag{2.3}$$

Equations (2.1), (2.2) form a system of two ODE's in the unknown functions $p = p(r)$, $\rho = \rho(r)$, and $M = M(r)$, where p denotes the pressure, ρc^2 denotes the mass-energy density, c denotes the speed of light, $M(r)$ denotes the total mass inside radius r, and $\mathcal{G}$ denotes Newton's gravitational constant. The last three factors in (2.1) are the general-relativistic corrections to the Newtonian theory, [2].

Solutions of (2.1) and (2.2) determine a Lorentzian metric tensor g of the form

$$ds^2 = -B(r)d(ct)^2 + A(r)^{-1}dr^2 + r^2\left(d\theta^2 + sin^2(\theta)d\phi^2\right), \tag{2.4}$$

that solves the Einstein equations

$$G = \frac{8\pi\mathcal{G}}{c^4}\mathcal{T}, \tag{2.5}$$

when G is the Einstein tensor, and $\mathcal{T}$ is the stress-energy tensor for a perfect fluid,

$$\mathcal{T}_{ij} = (p + \rho c^2)u_i u_j + p g_{ij}. \tag{2.6}$$

Here i and j are indices that run from 0 to 3, $A(r)$ is defined by (2.3), and the function B satisfies the equation

$$\frac{B'}{B} = -2\frac{p'}{p + \rho c^2}. \tag{2.7}$$

The metric (2.4) is spherically symmetric, time independent, and the fluid 4-velocity is given by $u_t = \sqrt{B}$ and $u_r = u_\theta = u_\phi = 0$, so that the fluid is fixed in the (t, r, θ, ϕ)-coordinate system, [2].

We assume that, (cf. [6]),

$$\mu = \frac{p}{\rho}, \tag{2.8}$$

and

$$\sigma = \frac{dp/dr}{d\rho/dr}, \tag{2.9}$$

satisfy the apriori bounds

$$0 \leq \mu < \mu_+ < \infty, \tag{2.10}$$

and

$$0 < \sigma_- < \sigma < \sigma_+ < \infty. \tag{2.11}$$

Note that if an equation of state of the form $p = p(\rho)$ is given, then the bounds (2.10) and (2.11) are implied by the usual physical requirements on the function $p(\rho)$, (cf. [6]).

Our results rely on a regularity theorem, (Theorem 2 below), for solutions of (2.1), (2.2) that satisfy (2.10) and (2.11). The results are stated in terms of the variables z and A, where z is defined above in (1.1). That is, in [6] we showed that on the maximal interval $(r_1, r_0]$ over which $M(r) > 0$, the OV system (2.1), (2.2) is equivalent to the system

$$\frac{dz}{dr} = -C\frac{z}{A}\left(\frac{1-A}{r}\right), \tag{2.12}$$

$$\frac{dA}{dr} = (1 - 3z)\left(\frac{1-A}{r}\right), \tag{2.13}$$

where

$$C \equiv \frac{(1 + \frac{\mu}{c^2})(1 + \frac{3\mu z}{c^2})}{2\frac{\sigma}{c^2}} - 3(1 - z)\frac{A}{(1 - A)}. \tag{2.14}$$

In terms of z and A, Eq. (2.7) becomes

$$\frac{B'}{B} = \frac{1}{r}\left(1 + 3\frac{\mu z}{c^2}\right)\left(\frac{1-A}{A}\right). \tag{2.15}$$

The regularity theorem that we need is the following theorem proved in [6].

Theorem 2. *Let $(z(r), A(r))$ denote the smooth, (i.e., C^1), solution of (2.12), (2.13), defined on a maximal interval $(r_1, r_0], 0 \leq r_1 < r_0 < \infty$, satisfying the initial conditions $z(r_0) = z_0,\ A(r_0) = A_0$, where*

$$0 < z_0 < \infty, \quad 0 < A_0 < 1. \tag{2.16}$$

Assume that (2.10) and (2.11) hold. Then $(z(r), A(r))$ satisfies the following inequalities for all $r \in (r_1, r_0]$:

$$0 < z(r) < \infty, \tag{2.17}$$

$$0 < A(r) < 1, \tag{2.18}$$

$$B(r) > 0, \tag{2.19}$$

$$0 < M(r) < M(r_0), \quad M'(r) > 0, \tag{2.20}$$

and

$$\lim_{r \to r_1+} M(r) = 0. \tag{2.21}$$

Moreover, if $r_1 > 0$, then

$$\lim_{r \to r_1+} z(r) = +\infty, \tag{2.22}$$

$$\lim_{r \to r_1+} A(r) = 1, \tag{2.23}$$

$$\lim_{r \to r_1+} B(r) = B(r_1) > 0. \tag{2.24}$$

If $r_1 = 0$, then

$$0 \leq z(r) \leq 1, \tag{2.25}$$

for all $r \in (0, r_0]$, and if $\rho(r)$ has a finite limit at $r_1 = 0$, then (2.23) and (2.24) also hold.

The original variables ρ and p of the OV system (2.1), (2.2) satisfy the inequalities

$$0 < \rho(r_0) < \rho(r) < \rho(r_1) < \infty, \quad \rho'(r) < 0, \tag{2.26}$$

and

$$0 < p(r_0) < p(r) < p(r_1) < \infty, \quad p'(r) < 0, \tag{2.27}$$

for all $r, r_1 < r < r_0$.

We remark that (2.21) and (2.22) show that z can only tend to infinity at a value $r_1 > 0$ where $M(r_1) = 0$. Furthermore, it follows that when $r_1 > 0$, the values of $\rho(r)$ and $p(r)$ are bounded on the closed interval $r_1 \le r \le r_0$. Thus, solutions of the OV system (2.1),(2.2), actually exist on a larger interval containing $[r_1, r_0]$, but $M \ge 0$ is violated.

Our first result is given in the following theorem which describes the continuation of an OV solution to values $0 \le r \le r_1$ in the case when $r_1 > 0$. We then show that r_1 is always positive when $r_0 \le \frac{9}{8} R_s(M(r_0))$; that is, $r_1 > 0$ if r_0 is within $9/8^{ths}$ of the Schwarzschild radius.

Theorem 3. *Let $(z(r), A(r))$ denote the smooth, (i.e., C^1), solution of (2.12), (2.13), defined on a maximal interval $(r_1, r_0], 0 \le r_1 < r_0 < \infty$, satisfying the initial conditions (2.16), and assume that (2.10) and (2.11) hold, so that the hypotheses of Theorem 2 hold. Assume that $r_1 > 0$. Then the functions $\rho(r), p(r)$ and $M(r)$ can be extended as a smooth solution of the OV system (2.1), (2.2), to values r satisfying $0 \le r < r_0$. Moreover, for $r < r_1$,*

$$-\infty < M(0) < M(r) < 0, \tag{2.28}$$

where

$$\lim_{r \to 0} M(r) = M(0), \tag{2.29}$$

$A(r) > 0$, $M'(r) > 0$, and $p(r)$ and $\rho(r)$ are positive and bounded for all $r \in [0, r_0]$. Furthermore, there exists a unique value r_2, $0 < r_2 < r_1$, such that the functions $p(r)$ and $\rho(r)$ assume their maximum values at $r = r_2$, and

$$\lim_{r \to 0} p(r) = \lim_{r \to 0} \rho(r) = \lim_{r \to 0} p'(r) = \lim_{r \to 0} \rho'(r) = 0. \tag{2.30}$$

Finally, the component B in the metric (2.4) satisfies

$$B(r) = O(r^{-1}) \ \ as \ \ r \to 0, \tag{2.31}$$

and the tensor invariant $\mathcal{R} \equiv R_{ijkl}R^{ijkl}$ of the Riemann curvature tensor determined by the metric (2.4) satisfies

$$\mathcal{R} \ge \frac{const.}{r^6} \ \ as \ \ r \to 0, \tag{2.32}$$

so that there is a non-removable singularity in the metric (2.4) at $r = 0$ when $r_1 > 0$.

The next theorem will be used to show that r_1 tends to r_0 as the initial condition $A(r_0) = A_0$ tends to zero. That is, as the initial condition is taken closer and closer to the Schwarzschild radius, the point r_1 at which $M(r_1) = 0$ tends to r_0. Since by (2.21), $M = 0$ at $r = r_1$, and $M(r_0)$ tends to $\frac{c^2 r_0}{2\mathcal{G}}$ as A_0 tends to zero, we conclude that all of the mass accumulates in a surface layer near $r = r_0$ as A_0 tends to zero. Our analysis is based on estimating, explicitly in terms of A_0, the position $r = r_*$ of the unique point where $\frac{M(r)}{r^3}$ assumes its maximum. A calculation (below) shows that at $r = r_*$, we also have $\rho(r_*) = \bar{\rho}(r_*)$, so $z(r_*) = 1$, and moreover, $\rho > \bar{\rho}$ for $r_* < r < r_0$, and $\rho < \bar{\rho}$ for $r_1 < r < r_*$.[3]

[3] The point r_* also plays an important role in the shock-wave matching problem set out in, [3, 4, 5]. Indeed, we showed in [5] that outgoing shocks, modeling explosions, can be constructed from any outer OV solution so long as $\rho > \bar{\rho}$. We will use these results in a future paper to study shock-waves near the Schwarzschild radius.

Theorem 4. *Let $(z(r), A(r))$ be a smooth solution of (2.12),(2.13) starting from initial values (z_0, A_0) and defined on a maximal interval $(r_1, r_0]$. Assume that the initial values satisfy*

$$0 < z_0 < 1, \tag{2.33}$$

$$0 < A_0 \leq \frac{1}{9}. \tag{2.34}$$

Then $r_1 > 0$, and there is a unique point r_*, $r_1 < r_* < r_0$, such that $z(r_*) = 1$, $z(r) < 1$ for $r > r_*$, $z(r) > 1$ for $r < r_*$, and the following inequalities hold:

$$1 > \frac{r_*}{r_0} > \sqrt{\frac{1 - 9A(r_0)}{1 - A(r_0)}}, \tag{2.35}$$

and

$$\rho(r) < \rho(r_*) \leq \frac{3}{8\pi\mathcal{G}r_0^2}\frac{1 - A_0}{1 - 9A_0}, \tag{2.36}$$

for all r, $r_* \leq r < r_0$.

The estimate (2.35) gives a rate at which $\frac{r_*}{r_0} \to 1$ as $A_0 \to 0$, and we will use this to demonstrate that $\frac{r_1}{r_0} \to 1$, as $A_0 \to 0$.

Note that the hypothesis $0 < A_0 \leq \frac{1}{9}$ implies that r_0 is outside the Schwarzschild radius $R_s(M_0)$, but inside $9/8^{ths}$ of $R_s(M_0)$.

Theorem 1 of the introduction follows directly from Theorems 2-4, together with the following corollary which generalizes the Buchdahl theorem:

Corollary 1. *If $r_1 = 0$, then $A_0 > \frac{1}{9}$, or equivalently*

$$r_0 > \frac{9}{8}R_s(M(r_0)).$$

To see this, note that if $r_1 = 0$, then $M(0) = 0$, and so $M(r) = \int_0^r 4\pi\rho(s)s^2 ds$. Now suppose that $A_0 \leq \frac{1}{9}$. Then by (2.35), $r_* > 0$. But if $r_1 = 0$, then $\rho' < 0$ implies $\rho \leq \overline{\rho}$ so $z \leq 1$ when $r_1 = 0$. (Theorem 3). Thus $r_1 = 0$ is impossible when $r_* > 0$ because the latter implies $z > 1$ for $r < r_*$, a contradiction.

The next corollary shows that $r_1 \to r_0$ as $A_0 \to 0$, thereby demonstrating that all of the mass accumulates in a layer that tends to r_0 as r_0 tends to the Schwarzschild radius.

Corollary 2. *If r_0 and z_0 are fixed, then*

$$\lim_{A_0 \to 0} \frac{r_1}{r_0} = 1. \tag{2.37}$$

The final theorem estimates the size of the surface layer $r_* < r < r_0$, (where $z < 1$), from above in terms of the initial data (z_0, A_0). Our estimate for the width of the layer depends on the value $B(r_*)$, but this value depends on the initial condition for $B(R)$ at the surface of the star $r = R$. Thus in this case we shall assume that the solution is defined for $r_1 < r \leq R$, and that $lim_{r \to R} z(r) = 0$, and $B(R) = A(R)$. (Note here that the OV solution will not go continuously to a vacuum at $r = R$, ($z(R) = 0$, $\rho(R) = 0$), unless $\sigma \to 0$ as $r \to R$. This follows directly from (2.12) because, if σ is bounded away from zero, then the system (2.12), (2.13) is regular, and has a unique solution through $r = R$, namely, the Schwarzschild solution. Allowing $\sigma \to 0$ as $r \to R$, is not a problem in the arguments to follow.)

Theorem 5. *Let $(z(r), A(r)$ be a smooth solution of (2.12),(2.13) starting from initial values (z_0, A_0) and defined on a maximal interval $(r_1, R]$, $0 < r_1 < r_0 < R$, where we assume the initial values satisfy (2.33), (2.34), together with*

$$lim_{r\to R} z(r) = 0, \tag{2.38}$$

and

$$z(r) = 0 \;\; and \;\; B(r) = A(r) \;\; for \;\; r \geq R. \tag{2.39}$$

Then the following inequality holds:

$$\frac{r_*}{r_0} \leq \frac{1 - A_0}{1 - B(r_*)}. \tag{2.40}$$

Moreover, if A is sufficiently small so that C in (2.12) satisfies $C > 0$ for $r \in (r_*, r_0)$, (for example $A < \frac{1}{9}$ and $\sigma < 2$ will suffice), then $B(r_*)$ satisfies

$$B(r_*) = B(R) e^{-\int_{z_0}^{1} \frac{1+3\mu z}{Cz} dz}. \tag{2.41}$$

Note that to estimate $B(r_*)$ by using (2.41), (which by (2.40) yields an estimate for $\frac{r_*}{r_0}$ from below), we need to estimate the function C in (2.14) and this essentially requires knowledge of the equation of state.

3. Proofs of Theorems

In this section we supply the proofs of Theorems 3–5 stated in Sect. 3. From here on we always assume that the speed of light c is unity.

Proof of Theorem 3: Assume $r_1 > 0$. By Theorem 2,

$$\lim_{r\to r_1} M(r) = 0,$$

and ρ and p have finite positive limits $\rho(r_1)$, $p(r_1)$, at $r = r_1$, respectively. Thus by defining $M(r_1) = 0$, we have a continuous extension of the OV solution to $r = r_1$. Moreover,

$$M'(r_1) = 4\pi\rho(r_1) r_1^2 > 0;$$

thus there is an extension of the OV solution to a neighborhood $(r_1 - \epsilon, r_1]$, and we choose ϵ sufficiently small so that, on this neighborhood, $p(r) > 0$ and $\rho(r) > 0$ but $M(r) < 0$. Now let $I \equiv (r_3, r_1]$ denote the largest interval over which the solution of the OV equations starting from initial data at $r = r_1$, exists, is smooth, and both ρ and p are positive. The OV equation (2.1) can be rewritten in the form

$$-\rho' = \frac{\mathcal{G}(1+\mu)}{r^2\sigma} \rho (M + 4\pi\mu r^3) \frac{1}{1 - \frac{2\mathcal{G}M}{r}}. \tag{3.1}$$

Let

$$D(r) \equiv M(r) + 4\pi p(r) r^3. \tag{3.2}$$

Claim 1. ρ and M are bounded on $[r_3, r_1]$.

Proof of Claim 1. Using (3.1) we have that for $r \in I$,

$$-\rho' \leq K_1 \frac{\rho}{r^2}(4\pi p r^3)\frac{r}{2\mathcal{G}|M|} \leq K_2 \rho^2 r^2 \frac{1}{|M|}, \tag{3.3}$$

for some positive constants K_1 and K_2. But $M_\epsilon \equiv M(r-\epsilon) < 0$. Thus, since $M'(r) > 0$ on $I_\epsilon \equiv (r_3, r_1 - \epsilon]$, we have

$$-\rho' \leq \frac{K_2}{|M_\epsilon|}\rho^2 r^2 \leq K\rho^2 r^2,$$

for some positive constant K. Then integrating from $r > r_3$ to $r_1 - \epsilon$ gives

$$\rho(r) \leq \frac{1}{\rho(r_1-\epsilon) + \frac{K}{3}[r^3 - (r_1-\epsilon)^3]} < \text{Const},$$

and this proves Claim 1.

Using the claim we conclude that $D(r_2) = 0$ for some $r_2 \in I$. Indeed, if $D(r) \neq 0$ for all $r \in I$, then since $\rho' < 0$ and ρ is bounded, it follows that ρ, p and M would have finite positive limits at $r = r_3$ if $r_3 \neq 0$, so we must have $r_3 = 0$ in order not to contradict the maximality of the interval I. But if $r_3 = 0$, then clearly $D(r) = M + 4\pi p r^3$ is negative for r sufficiently close to $r = 0$.

Now let r_2 be any point in I for which $D(r_2) = 0$. Then

$$\frac{d}{dr}D(r_2) = M'(r_2) + 4\pi p'(r_2) r_2^3 + 12\pi p(r_2) r_2^2 > 0,$$

since $p'(r_2) = 0$. It follows from this that there exists a unique $r_2 \in I$ at which $D(r_2) = 0$. For $r < r_2$, note that $\rho'(r) > 0$ and $p'(r) > 0$.

Claim 2. $r_3 = 0$.

Proof of Claim 2. Using (3.1) we can write

$$\rho' = \frac{\mathcal{G}(1+\mu)}{\sigma r^2}\rho(-M - 4\pi\mu\rho r^3)\frac{1}{A} < K\frac{1}{r^2}\rho(-M)\frac{r}{-M} < K_+\frac{\rho}{r},$$

for some positive constants K and K_+. Integrating from $r < r_2$ to r_2 gives

$$\rho(r) > \rho(r_2)\left(\frac{r}{r_2}\right)^{K_+},$$

so that $\rho(r) \geq 0$ for all $r \geq r_3$. We conclude that either $r_3 = 0$ or else we contradict the maximality of I. This proves Claim 2.

Claim 3. $\lim_{r\to 0} \rho(r) = 0$.

Proof of Claim 3. Note first that

$$D'(r) = M'(r) + 4\pi p'(r)r^3 + 12\pi p(r)r^2 \geq 0,$$

for all $r \in (0, r_2]$. It follows that

$$-D(r) > -D(r_2 - \epsilon) \equiv K_\epsilon,$$

$0 < r < r_2 - \epsilon$, for some small positive number ϵ. Thus from (3.1) we obtain for $0 < r < r_2 - \epsilon$,

$$\rho' \geq \frac{K}{r^2}\rho K_\epsilon \frac{1}{1 + \frac{\mathcal{G}|M|}{r}},$$

so that

$$\rho' \geq K_- \frac{\rho}{r},$$

where $K_- > 0$. Thus for such r we have

$$\rho(r) < \rho(r_2 - \epsilon)\left(\frac{r}{r_2 - \epsilon}\right)^{K_-},$$

and this shows that $\rho(r) \to 0$ as $r \to 0$, which proves Claim 3.

Next we show that

$$\lim_{r \to 0} \rho'(r) = 0. \tag{3.4}$$

To see this, note that for r near $r = 0$, we obtain from (3.1) that

$$\rho' = \frac{\mathcal{G}(1+\mu)}{\sigma r^2}\rho(|M| + O(r))\frac{r}{2\mathcal{G}|M|}(1 + O(r)),$$

which we can rewrite as

$$\rho'(r) = \frac{(1+\mu)}{2\sigma}\frac{\rho}{r}(1 + O(r)).$$

Since $\lim_{r\to 0}\rho(r) = \lim_{r\to 0} p(r) = 0$, we may write this last equation as

$$\rho'(r) = \frac{(1+\mu(0))}{2\sigma(0)}\frac{\rho}{r}(1 + O(r)) \ \ as \ \ r \to 0. \tag{3.5}$$

Now integrating from $r < \epsilon$ to $r = \epsilon$, (where ϵ is near zero), we obtain

$$\rho(r) = \rho(\epsilon)\left(\frac{r}{\epsilon}\right)^{K_0} e^{-K_0 O(\epsilon)},$$

where

$$K_0 = \frac{1 + \mu(0)}{2\sigma(0)}.$$

But, $\mu(0) = \lim_{\rho\to 0}\frac{p(\rho)}{\rho} = p'(0) = \sigma(0)$. Thus,

$$K_0 = \frac{1 + \sigma(0)}{2\sigma(0)} > 1,$$

because σ, the sound speed squared, is less than unity. We conclude that

$$\lim_{r\to 0}\frac{\rho(r)}{r} = 0,$$

and hence

$$\rho'(0) = \lim_{r\to 0}\frac{\rho(r)-\rho(0)}{r-0} = 0.$$

Finally we verify (2.31) and (2.32). For (2.31) note that we have

$$\frac{B'}{B} = -\frac{2p'}{p+\rho}, \tag{3.6}$$

and using an argument similar to the derivation of (3.5), we obtain that near $r = 0$,

$$p' = \frac{1+\mu}{2}\rho\left(\frac{1}{r} - \frac{1}{2\mathcal{G}|M(0)|} + O(r)\right). \tag{3.7}$$

Substituting this for p' in (3.6), we see that for r near zero,

$$\frac{B'}{B} = -\frac{\rho}{p+\rho}\left(\frac{1}{r} + O(1)\right). \tag{3.8}$$

Now integrating from $r < \epsilon$ to $r = \epsilon$ yields

$$B(r) = B(\epsilon)\left(\frac{\epsilon}{r}\right)(1 + O(\epsilon)). \tag{3.9}$$

This shows that $B(r) = O\left(\frac{1}{r}\right)$ near $r = 0$.

To verify (2.32), a calculation using MAPLE yields

$$\mathcal{R} = \frac{[2ABB'' - A(B')^2 + BA'B']^2}{4B^4} + \frac{2A^2(B')^2}{r^2B^2} + \frac{2(A')^2}{r^2} + \frac{4(1-A)^2}{r^4}.$$

Thus

$$\mathcal{R} \geq \frac{4(1-A)^2}{r^4} = 16\mathcal{G}^2\frac{M(r)^2}{r^6} \to \infty \ \ as \ \ r \to 0,$$

since $M(0) \neq 0$. This completes the proof of Theorem 3.

We can use the shock-wave matching techniques developed in [3] to show that the non-removable singularity that appears in the metric at $r = 0$ in the case when $r_1 > 0$ really does represent a delta function source of negative density. Indeed, a Friedmann-Robertson-Walker (FRW) metric can only be matched Lipschitz continuously to a metric of type (2.4) if the following condition holds, (cf. [3]):

$$M(r) = \frac{3}{4\pi}\bar{\rho}r^3, \tag{3.10}$$

where $\bar{\rho}$ denotes the FRW density behind the interface between an FRW metric inside radius r and a metric of type (2.4) outside radius r. Thus if $M(r) < 0$, then only FRW metrics with negative density can be matched to (2.4) at radius r. In the limit that $r \to 0$, $M(r) \to M(0) < 0$, and thus by (3.10) FRW density $\bar{\rho}$ tends to a negative delta function source of magnitude $M(0)$ centered at $r = 0$. In other words, replacing the ball of radius $r = \epsilon$ by an FRW space at fixed time has the effect of regularizing the singularity at $r = 0$ at that time. But by (3.10), the FRW solution inside radius $r = \epsilon$ determines a sequence whose density converges to a delta-function of negative mass $M(0)$ as $\epsilon \to 0$.

We now show that a solution of the OV equation starting from initial values $M(r_0) < 0$ and $p(r_0) > 0$, cannot reach $p = 0$ for some $R > r_0$ without having $M(R) \geq 0$. To see this note that if $lim_{r\to R}p(r) = 0$, we must have $p'(r_k) < 0$ on a sequence $r_k \to R$, so long as $p > 0$ for $r < R$. But if $M < 0$, then $A > 1$, and so the OV equation (2.1) implies that

$$0 \leq lim_{r_k\to R}\left(M(r_k) + 4\pi p(r_k)r_k^3\right) = lim_{r_k\to R}M(r_k),$$

and so in fact, since $M'(r) > 0$ when $p > 0$, we must have $M(R) \geq 0$. Thus negative total masses will *never* be observed at the surface of a star $r = R$, (or beyond), if $\rho(r) > 0$ at any $r < R$ outside the Schwarzschild radius (i.e., the solution is not the empty space Schwarzschild solution with negative mass).

Proof of Theorem 4. We begin by proving the following:

Lemma 1. *Let $(z(r), A(r))$ denote the solution of (2.12), (2.13) defined on the maximal interval $(r_1, r_0]$, starting from initial data $z(r_0) = z_0$, $A(r_0) = A_0$, where*

$$0 < z_0, A_0 < 1,$$

(so that the hypotheses of Theorem 2 hold). Assume that $r_1 > 0$. Then there exists a unique point r_*, $r_1 < r_* < r_0$, such that $z(r_*) = 1$.

Proof of Lemma. Since $z(r_0) < 1$, and by Theorem 2, $z(r) \to +\infty$ as $r \to r_1$, we see that there exists an r_* for which $z(r_*) = 1$. On the other hand, by (2.12), $z'(r) < 0$ if $z \geq 1$, so we see that r_* is unique. This completes the proof of the lemma. $\square$

Now differentiating the average density,

$$\bar{\rho} = \frac{3}{4\pi}\frac{M(r)}{r^3},$$

we obtain

$$\bar{\rho}' = \frac{3}{r}(\rho - \bar{\rho}) = \frac{3\bar{\rho}}{r}(z-1), \tag{3.11}$$

so we see that $\bar{\rho}$ takes a unique maximum at $r = r_*$, and thus

$$\bar{\rho}'(r) < 0 \;\; if \;\; r_* < r < r_0, \tag{3.12}$$

$$\bar{\rho}'(r) > 0 \;\; if \;\; r_1 < r < r_*. \tag{3.13}$$

We now estimate $\frac{r_*}{r_0}$ when $A_0 < \frac{1}{9}$. As a first step, we prove the following lemma, which implies (2.35) in the special case when r_0 is the boundary surface of the star, and the Schwarzschild solution is attached to the OV solution at $r = r_0$. (Note here that the OV solution will not go continuously to a vacuum at $r = R$, namely, $z(R) = 0$, $\rho(R) = 0$, unless $\sigma \to 0$ as $r \to R$. This follows directly from (2.12) because, if σ is bounded away from zero, then the system (2.12), (2.13) is regular, and has a unique solution through $r = R$, namely, the Schwarzschild solution. Allowing $\sigma \to 0$ as $r \to R$, is not a problem in the arguments to follow because, for any $\tilde{r} < R$, $\rho(\tilde{r}) \neq 0$, $\sigma > 0$, and our regularity results Theorems 2 and 3 are valid for $r \leq \tilde{r}$.)

Lemma 2. *Assume the hypotheses of Theorem 4, and in addition assume that*

$$\rho(r) = 0 = p(r),$$

and

$$B(r) = A(r),$$

for all $r \geq r_0$. Then inequality (2.35) holds.

Proof of Lemma 2. From Weinberg, [2], p. 333, we have the following identity that holds on solutions of the OV system:

$$\left(\frac{1}{r}\sqrt{A}(\sqrt{B})'\right)' = \mathcal{G}\frac{B}{A}\left(\frac{M}{r^3}\right)', \tag{3.14}$$

where prime denotes differentiation with respect to r. (Note that by Theorem 2, $A(r)$ and $B(r)$ are both positive on $(r_1, r_0]$.) Now from (3.11) and (3.12), $\left(\frac{M}{r^3}\right)' < 0$ for $r > r_*$, (and this holds when $r_* = 0$ because in this case $r_1 = 0$, and thus from (3.11), $\overline{\rho}' < 0$ for all $r > 0$), so that, from (3.14),

$$\left(\frac{1}{r}\sqrt{A}\left[\sqrt{B}\right]'\right)' < 0,$$

holds for $r_* < r < r_0$. Integrating we obtain for such r

$$0 > \int_r^{r_0}\left(\frac{1}{s}\sqrt{A}(\sqrt{B})'\right)' ds = \frac{1}{r_0}\sqrt{A(r_0)}\left[\sqrt{B(r_0)}\right]' - \frac{1}{r}\sqrt{A(r)}\ \left[\sqrt{B(r)}\right]',$$

or

$$\frac{r}{\sqrt{A(R)}}\frac{1}{r_0}\sqrt{\frac{A(r_0)}{B(r_0)}}\frac{B'(r_0)}{2} < \left[\sqrt{B(r)}\right]'. \tag{3.15}$$

But note that by assumption $B(r_0) = A(r_0)$, and moreover,

$$B'(r_0) = A'(r_0) = \frac{2\mathcal{G}M(r_0)}{r_0^2}. \tag{3.16}$$

Indeed, for the second equality we use $M'(r_0) = 4\pi\rho(r_0)r^2$ and $\rho(r_0) = 0$. For the first equality, we substitute the expression for p' given in the OV equation (2.1) into (2.7) and again use the fact that $\rho(r_0) = p(r_0) = 0$, and $A(r_0) = B(r_0)$.

Integrating (3.15) from r_* to r_0 and using the fact that $B'(r_0) = A'(r_0)$, gives

$$\sqrt{B(r_0)} - \sqrt{B(r_*)} > \frac{\mathcal{G}M(r_0)}{r_0^3}\int_{r_*}^{r_0}\frac{rdr}{\sqrt{1-\frac{2\mathcal{G}M(r)}{r}}} \geq \frac{\mathcal{G}M_0}{r_0^3}\int_{r_*}^{r}\frac{rdr}{\sqrt{1-\frac{2\mathcal{G}M_0}{r_0^3}r^2}},$$

because

$$M(r) = \frac{2\pi}{3}\overline{\rho}(r)r^3 \geq \frac{4\pi}{3}\overline{\rho}(r_0)r^3 = \frac{M_0}{r_0^3}r^3.$$

Now making the substitution $u = 1 - \frac{2\mathcal{G}M_0}{r_0^3}r^2$, in the last integral, gives

$$3\sqrt{A(r_0)} > \sqrt{1 - \frac{2\mathcal{G}M_0}{r_0^3} r_*^2}. \tag{3.17}$$

In particular, this implies that $r_* > 0$ because $r_* = 0$ would imply that $A_0 > \frac{1}{9}$, in violation of our hypothesis. But, if $r_* > 0$, then $z(r) > 1$ for $r < r_*$ by (2.12). Now using Theorem 2, we see that if $r_1 = 0$, then $z(0) \leq 1$, and this is a contradiction. Thus $r_1 > 0$. Now simplifying (3.17) yields (2.35) in the case when $r = r_0$ is attached to the empty space Schwarzschild solution. This completes the proof of Lemma 2. $\square$

To complete the proof of (2.35) it remains only to extend Lemma 2 to the case when the initial conditions at $r = r_0$ are the general conditions (2.33), (2.34); that is, this is the case when we do not assume that the solution is attached to the empty space Schwarzschild metric at $r = r_0$; i.e., we assume that $\rho(r_0) > 0$. To accomplish this, we will extend the definition of the equation of state function $p(\rho)$ to values of ρ smaller than the value $\rho(r_0)$ in such a way that the extension of the solution to $r > r_0$, (r near r_0), hits $\rho = 0$ at an arbitrarily small distance from $r = r_0$. The extension of $p(\rho)$ to values of $\rho < \rho(r_0) \equiv \rho_0$ does not affect the solution for $r \in (r_1, r_0]$ because in this range, $\rho'(r) < 0$, and hence $\rho > \rho(r_0)$. Thus (2.35) will follow in full generality by passing to the limit.

To carry out this program, let $0 < \delta < \rho_0$ be given and let $p_\delta(\rho)$ be an extension of $p(\rho)$ to values of $\rho < \rho_0$ such that the following conditions hold:

$$\begin{aligned} p_\delta(\rho) &= p(\rho), \quad for \quad \rho \geq \rho_0, \\ p_\delta(\rho) &= \delta\rho, \quad for \quad 0 \leq \rho \leq \rho_0 - \delta, \end{aligned} \tag{3.18}$$

and we let p_δ be a smooth interpolation of p between the values $\rho = \rho_0$ and $\rho = \rho_0 - \delta$. For this extension p_δ of p, we now show that the extension of the solution by the OV equation to values of $r > r_0$, satisfies $\rho'(r) < 0$, and $\rho(r) = 0$ for some $r \in (r_0, r_0 + \epsilon)$ for $\epsilon = \epsilon(\delta) \to 0$ as $\delta \to 0$. To this end, note that for r sufficiently close to $r = r_0$, it is not difficult to see that using the OV equation (2.1), we can obtain the following estimate:

$$\rho'(r) \leq -K \frac{\rho(r)}{p'_\delta(\rho(r))}, \tag{3.19}$$

where K is a constant independent of δ, (uniform over a fixed r-interval about r_0, and depending only on values of the solution near $r = r_0$). Now fix $\epsilon << 1$; we show that there exists a δ such that the solution of the OV system starting from initial data at $r = r_0$ to values $r > r_0$, (using equation of state p_δ), must satisfy $\rho(r) = 0$ for some r, $r_0 < r < r_0 + \epsilon$. To this end, assume $\rho(r) > 0$ on this interval for all $\delta << 1$. We show that this is impossible. Indeed, integrating (3.19) from r_0 to $r_0 + \epsilon$ gives

$$\int_{\rho_0}^{\rho(r_0+\epsilon)} \frac{p'_\delta}{\rho} d\rho \leq -K \int_{r_0}^{r_0+\epsilon} dr = -K\epsilon.$$

But

$$\begin{aligned} \int_{\rho_0}^{\rho(r_0+\epsilon)} \frac{p'_\delta(\rho)}{\rho} d\rho + \int_{\rho_0}^{\rho_0-\delta} \frac{p'_\delta(\rho)}{\rho} d\rho + \int_{\rho_0-\delta}^{\rho(r_0+\epsilon)} \frac{p'_\delta(\rho)}{\rho} d\rho \\ = O(\delta) + \delta\rho(r_0 + \epsilon). \end{aligned}$$

Thus we get

$$O(\delta) + \delta\rho(r_0 + \epsilon) \leq -K\epsilon. \tag{3.20}$$

Since ϵ is fixed, we see from (3.20) that $\rho(r_0 + \epsilon)$ cannot be positive for δ sufficiently small. This proves that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\rho(r_\epsilon) = 0$ for $r_0 < r_\epsilon < r_0 + \epsilon$, when $p_\delta(\rho)$ is taken as the equation of state. Thus for each $\epsilon << 1$, we can match the (extended) OV solution determined from initial data (2.33), (2.34), to the empty space Schwarzschild solution, at $r = r_\epsilon$. Thus, by applying the last lemma we conclude that

$$1 > \left(\frac{r_*}{r_0}\right) > \sqrt{\frac{1 - 9A_\epsilon}{1 - A_\epsilon}},$$

where

$$A_\epsilon = A(r_\epsilon) = 1 - \frac{2\mathcal{G}M(r_\epsilon)}{r_\epsilon}.$$

Since $M(r_\epsilon) \to M(r_0)$ as $\epsilon \to 0$ because

$$M(r_\epsilon) - M(r_0) = \int_{r_0}^{r_\epsilon} 4\pi\rho(r)r^2 dr \to 0,$$

as $\epsilon \to 0$, we conclude that indeed estimate (2.35) must hold in full generality.

To complete the proof of Theorem 4 it remains only to prove (2.36). To this end, we have

$$M(r_*) = \frac{4\pi}{3}\bar{\rho}(r_*)r_*^3 = \frac{4\pi}{3}\rho(r_*)r_*^3,$$

so that

$$A(r_*) = 1 - \frac{2\mathcal{G}M(r_*)}{r_*} = 1 - \frac{8\pi\mathcal{G}}{3}\rho(r_*)r_*^2,$$

and hence

$$1 - A(r_*) = \frac{8\pi\mathcal{G}}{3}\rho(r_*)r_*^2 > \frac{8\pi\mathcal{G}}{3}\rho(r_*)r_0^2\left(\frac{1 - 9A_0}{1 - A_0}\right),$$

where we have used (2.35). Thus

$$0 < A(r_*) < 1 - \frac{8\pi\mathcal{G}}{3}\rho(r_*)r_0^2\left(\frac{1 - 9A_0}{1 - A_0}\right),$$

and simplifying yields (2.36) because $\rho'(r) < 0$ on $r_* < r < r_0$. This completes the proof of Theorem 4.

We now give the proof of Corollary 2. For this, consider a solution of (2.12), (2.13) defined on the maximal interval (r_1, r_0), starting from initial data (z_0, A_0) that satisfies $0 < z_0, A_0 < 1$. Now fix z_0 and r_0 and let $A_0 \to 0$. Then we know from Theorem 4 that $r_* \to r_0$ as $A_0 \to 0$. We also show that $r_1 \to r_0$ as $A_0 \to 0$. To this end, assume not. Then (at least for some subsequence of A_0's), there exists an interval $(\tilde{r}_1, r_0)$ such that $r_1 \leq \tilde{r}_1$ for all $A_0 \to 0$ in this subsequence. We show that this implies that $z(r) \to \infty$

for all $r \in (\tilde{r}_1, r_0)$ as A tends to zero along this subsequence. This would give the desired contradiction because $z = \rho/\overline{\rho}$, and

$$\overline{\rho}(r) = \frac{3}{4\pi}\frac{M(r)}{r^3}$$

is bounded away from zero as $A_0 \to 0$, so $z \to \infty$ implies that $\rho(r) \to \infty$ as $A_0 \to 0$. The contradiction then is that

$$M(r_0) = \int_{r_1}^{r_0} 4\pi\rho(r)r^2 dr > \int_{\tilde{r}_1}^{r_0} 4\pi\rho(r)r^2 dr \to \infty,$$

as $A_0 \to 0$, but $M(r_0) < \infty$. (We use the fact that the integral of a sequence of positive functions tends to infinity if the sequence tends to infinity pointwise.) Thus we need only show that $z(r) \to \infty$ as $A_0 \to 0$. To see this, note first that $z > 1$ for all A_0 sufficiently small because for A_0 sufficiently small, $r_* > r$ and hence $z(r) > 1$ because $z' < 0$ for $r < r_*$. Thus (2.14) implies that

$$C \geq \bar{C}$$

for some positive constant $\bar{C}$ that is independent of A_0. Moreover, solving for $\frac{1-A}{r}$ in (2.13) and substituting into (2.12), and using the fact that $z > 1$ and that

$$\left|\frac{z}{1-3z}\right| \geq \frac{1}{3},$$

we obtain the inequality

$$z' \leq \frac{\bar{C}}{3}\frac{A'}{A},$$

which holds for all $r \in (\tilde{r}_1, r_*)$. Integrating between r and r_* yields

$$z(r) \geq 1 + \frac{\bar{C}}{3}\ln\left(\frac{A(r)}{A(r_*)}\right). \tag{3.21}$$

Notice now that

$$M(r_*) = M(r_0) - \int_{r_*}^{r_0} 4\pi\rho(r)r^2 dr.$$

But since (2.36) shows that $\rho(r)$ is uniformly bounded on the interval (r_*, r_0), we see that this latter integral tends to zero as $A_0 \to 0$ because $r_* \to 0$. Thus $M(r_*) \to M(r_0)$ as $A_0 \to 0$ which implies $A(r_*) \to 0$ as $A_0 \to 0$. But $A(r)$ is uniformly bounded away from zero because $A' = \frac{(1-3z)(1-A)}{r}$ is bounded above by a nonzero negative constant when $z > 1$. In light of this, (3.21) shows that $z(r) \to \infty$ as $A_0 \to 0$ for all $r \in (\tilde{r}, r_0)$, the condition we sought. This proves Corollary 2.

Proof of Theorem 5. We first verify (2.41). From (2.15), (2.12) and (2.14), if the function C given in (2.14) satisfies $C > 0$, then z is a monotone function of r, so we have

$$\frac{d\ln(B)}{dz} = \frac{1}{B}\frac{dB}{dz} = \frac{1}{B}\frac{dB}{dr}\frac{dr}{dz} = -\frac{1+3\mu z)}{C}\frac{1}{z}.$$

Thus integrating from z_0 to $z = 1$ gives (2.41).

We also shall need the following lemma:

Lemma 3. *The metric coefficients $B(r)$ and $A(r)$ determined by a solution of the OV equations satisfy*

$$\frac{d}{dr}\left[\ln\left(\frac{A}{B}\right)\right] = -\frac{(1+\mu)}{A}8\pi\mathcal{G}\rho r < 0. \tag{3.22}$$

Proof of Lemma. First write

$$\frac{d}{dr}\left[\ln\left(\frac{A}{B}\right)\right] = \frac{A'}{A} - \frac{B'}{B},$$

and use (2.13) together with the OV equation (2.1) to write

$$\frac{A'}{A} - \frac{B'}{B} = \frac{(1-3z)(1-A)}{rA} - \frac{(1-A)}{rA}\left(1+\frac{4\pi pr^3}{M}\right),$$

from which (3.22) follows upon noticing that

$$3z = \frac{4\pi\rho r^3}{M}.$$

This completes the proof of the lemma. □

To prove Theorem 5, we see from (3.14) together with the last lemma, (which implies that $\frac{A}{B} > 1$ since $B(R) = A(R)$), that we may write

$$\left(\frac{1}{r}\sqrt{A(r)}(\sqrt{B(r)})'\right)' \geq \mathcal{G}\left(\frac{M(r)}{r^3}\right)',$$

for all $r \in (r_*, R)$. Integrating this expression from $r \in (r_*, R)$ to R yields

$$\frac{1}{R}\sqrt{A(R)}\frac{B'}{2\sqrt{A(R)}} - \frac{\sqrt{A(r)}}{r}\sqrt{B(r)}' \geq \mathcal{G}\left(\frac{M(R)}{R^3} - \frac{M(r)}{r^3}\right).$$

Using (3.16) and simplifying gives

$$\sqrt{B(r)}' \leq \frac{\mathcal{G}M(r)}{r^2\sqrt{A(r)}},$$

so integrating from r_* to R gives

$$\int_{r_*}^{R}\left(\sqrt{B(r)}\right)' dr \leq \int_{r_*}^{R}\frac{\mathcal{G}M(r)}{r^2\sqrt{A(r)}}dr,$$

or

$$\sqrt{B(R)} - \sqrt{B(r_*)} \leq \int_{r_*}^{R}\frac{\mathcal{G}M(r)}{r^2}\frac{1}{\sqrt{1-\frac{2\mathcal{G}M(r)}{r}}}dr. \tag{3.23}$$

Now to estimate the integral on the right hand side of (3.23), use the fact that

$$M(r) \leq M(R),$$

and

$$\frac{1}{\sqrt{1-\frac{2\mathcal{G}M(r)}{r}}} \le \frac{1}{\sqrt{1-\frac{2\mathcal{G}M(R)}{r}}},$$

to obtain

$$\int_{r_*}^{R} \frac{\mathcal{G}M(r)}{r^2} \frac{1}{\sqrt{1-\frac{2\mathcal{G}M(r)}{r}}} dr \le \int_{r_*}^{R} \frac{\mathcal{G}M(R)}{r^2} \frac{1}{\sqrt{1-\frac{2\mathcal{G}M(R)}{r}}} dr. \tag{3.24}$$

Using the substitution

$$u = 1 - \frac{2\mathcal{G}M(R)}{r}, \quad du = \frac{2\mathcal{G}M(R)}{r^2} dr$$

we obtain from (3.24) the estimate

$$\sqrt{B(R)} - \sqrt{B(r_*)} \le \sqrt{A(R)} - \sqrt{1 - \frac{2\mathcal{G}M(R)}{r_*}}.$$

Finally, since $B(R) = A(R)$, a straightforward calculation gives (2.40). This completes the proof of Theorem 5. $\Box$

4. Concluding Remarks

The issue of negative mass functions raises an interesting question. Recall that, for spherically symmetric solutions, it is only the total mass $M(R)$, which is the total mass measured in the far field, that has an intrinsic physical meaning in general relativity. That is, in the Newtonian theory, $M(r) = \int_0^r 4\pi\rho(s)s^2 ds$ must be interpreted as the total mass inside radius r because the underlying space is Euclidean; but in general relativity, the mass function enters indirectly through the metric coefficient $A(r)^{-1}$, the coefficient of the dr^2 term in the gravitational metric tensor, via the formula $M(r) = \frac{rc^2}{2\mathcal{G}}(1 - A(r))$. In general relativity, only the equation $M'(r) = 4\pi\rho r^2$ follows from the Einstein equations, and the integration constant is not specified. Said differently, in general relativity, there is no intrinsic physical interpretation for the function $M(r)$ when $r < R$ because the spacetime inside radius r is not fixed apriori as in the Newtonian theory.

Since the density and pressure are everywhere positive but the mass $M(r)$ is negative for $0 < r < r_1$ in the solutions constructed here, we pose the question as to whether a region $0 \le r < \tilde{r} < r_1$ in an OV solution can be replaced by a perfect fluid solution that is singularity free inside radius $\tilde{r}$, such that the density and pressure are everywhere positive. This introduces the following dichotomy. Namely, if such a matching is possible, then the gravitational field can have a repulsive effect, in light of the fact that $p' > 0$ near $r = 0$. If such a matching cannot be made, then the following conjecture must hold: *Conjecture: No singularity free metric that solves the Einstein equations for a perfect fluid can be matched Lipschitz continuously to the negative mass portion of an OV metric in such a way that the interface between the metrics describes a fluid dynamical shock-wave, and such that the matched solution is singularity free, and has everywhere positive density and pressure.*

We showed above (before the proof of Theorem 4) that the conjecture is correct for matching to a Friedmann-Robertson-Walker metric; cf. [3].

In light of this dichotomy, we find it interesting that, as we proved above, the *invariant* quantity $\lim_{r\to\infty} M(r) = M(R)$ must satisfy $M(R) \geq 0$ at the surface of the star $r = R$, even when $M(r)$ is negative at some interior point $r < R$. Therefore we conclude that negative mass $M < 0$ would never be seen by an observer beyond the surface of the star, (consistent with the positive mass theorem, [8]).

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