# Shock waves near the Schwarzschild radius and stability limits for stars

Joel Smoller
*Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109*

Blake Temple
*Department of Mathematics, University of California, Davis, Davis, California 95616*

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We analyze the time dilation associated with the propagation of a shock wave that moves outward into a static fluid sphere in general relativity. In this way we investigate the possibility that a shock wave inside a star could supply the pressure required to hold up a highly collapsed static outer layer. In such a model, one would observe a highly redshifted, time-independent emissions spectrum during the time interval between the time when the shock wave is formed, and the time when it reaches the star surface. Our conclusion is that the time it takes a shock wave to pass through a surface layer and reach the surface of a star, as seen by an observer in the far field, is on the order of the total mass of the star times a function that tends to infinity as the outer boundary of the star tends to its Schwarzschild radius. [S0556-2821(97)04312-9]

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## I. INTRODUCTION

The well-known stability limit of Buchdahl [1,2], lends support to the belief that large redshifts in stellar objects must be due to large recessional velocities, as opposed to strong gravitational fields near the surface of a star. The Buchdahl result is derived from an analysis of the Oppenheimer-Volkoff (OV) equations that describe the pressure gradient in a static fluid sphere when the gravitational forces are modeled by Einstein's theory of general relativity. Buchdahl's theorem states that if the radius of a star of mass $M$ ever gets within 9/8ths of the Schwarzschild radius for mass $M$, then no static configuration can supply a pressure at the center of the star sufficient to support the mass $M$ at that radius. This result places a limit of 2 on the redshift factor for emissions from the surface of a static star [2]. The result applies for any equation of state of the form $p=p(\rho)$ (where $p$ denotes the pressure, $\rho c^2$ the mass-energy density, and $c$ denotes the speed of light), and is based on the assumptions that the fluid is time independent, spherically symmetric, and that the pressure $p$ at a given radius is exactly that required to balance the gravitational force of the fluid outside that radius in a general relativistic sense. In [3], we described the global behavior of solutions of the OV equations starting from initial data inside 9/8ths of the Schwarzschild radius. We showed that black holes never form in solutions of the OV equations, but rather, the entire mass of a star accumulates in a thin layer that tends to the outer surface of the star as the surface of the star tends to its Schwarzschild radius.

In this paper we use the surface layers described in [3] to analyze shock-wave solutions of the Einstein equations in which the shock wave is arbitrarily close to the Schwarzschild radius. We imagine that the shock wave represents the outer edge of an explosion inside a star, which propagates supersonically outward through a static outer layer modeled by solutions of the OV system. We estimate the time $\Delta t$ it would take a shock wave starting inside such a surface layer to reach the surface of the star, as measured by a distant observer, when the surface of the star lies inside 9/8ths of its Schwarzschild radius. In this case one expects a large time dilation due to the strong gravitational field near the star's surface, but since the layer width tends to zero as the surface is compressed into its Schwarzschild radius, it is not *a priori* clear, and requires estimates to determine whether $\Delta t$ tends to zero, or is bounded, or tends to infinity in this limit. Furthermore, shock waves always travel at supersonic speed, but as the layer width depends on the sound speed as well, it is also not *a priori* clear whether or not making the shock and sound speeds small can significantly increase $\Delta t$. In this paper we answer these questions definitively by giving sharp estimates for $\Delta t$ from above and below. Our conclusion is that the time it takes the shock wave starting inside such a layer to reach the surface of the star is on the order of the total mass of the star times a function that tends to infinity as the shock wave is started closer and closer to the Schwarzschild radius. Moreover, it is somewhat surprising that making the shock and sound speeds small does not significantly increase $\Delta t$.

By placing a shock wave inside a surface layer that otherwise would be unstable, we explore the idea that a shock wave could supply the pressure required to "hold up" a surface layer when it is inside 9/8ths of the Schwarzschild radius. Our analysis in this paper can thus be viewed as an investigation into the validity of the 9/8ths stability limit in a *dynamical* setting. The results we obtain imply that the time it takes such a shock wave to reach the surface can tend to infinity as the shock is placed closer and closer to the Schwarzschild radius. It follows that, in principle, this model allows for the possibility of a star whose surface lies inside 9/8ths of the Schwarzschild radius, such that the star surface emits a time-independent emissions spectrum for an arbitrarily long time, as measured by an observer in the far field. This suggests the possibility of stellar emission spectrums which have arbitrarily large redshifts due to gravitational fields at their surface. In actuality, however, we show that either the mass must be exceedingly large, or the layer exceedingly narrow, in order for $\Delta t$ to be on the order of 1 yr. Thus, for stars whose solar mass is on the order of one solar mass, one can interpret our results as a further justification of the 9/8ths stability limit in a dynamical setting, thus supporting the conclusion that, for such stars, observed high redshifts in stellar emissions are due to large recessional velocities as opposed to strong gravitational fields. On the other hand, for masses much larger than the mass of a typical galaxy, this interpretation is not clear. Our results in this paper are quite general in the sense that they apply whenever it is reasonable to model the outer layer of a star by a solution of the OV equations.

In our final result (theorem 3 below) we construct solutions of the Einstein equations that agree with an OV solution outside some positive radius, have positive density and mass, and are singularity-free for some interval of time. This construction applies to surface layer solutions of the OV system that lie arbitrarily close to the Schwarzschild radius. On the other hand, Buchdahl's theorem implies that static, spherically symmetric, time-independent solutions of the Einstein equations must have a singularity if the surface lies within 9/8ths of its Schwarzschild radius. Theorem 3 implies that this singularity theorem does not carry over to dynamical models.

To do the construction for theorem 3, we use the "shock matching" methods developed in [4,5] to explicitly construct Lax-admissible shock-wave solutions of the Einstein equations (c.f. [6,7]). This is done by placing a shock-wave interior to the surface layers constructed in [3] which lie arbitrarily close to the Schwarzschild radius. We show that these shock-wave solutions exist for a finite interval of time, and this proves that an OV solution given for, say, $r \geq \bar{r}$ at a fixed time, can be extended as initial conditions defined on $r<\bar{r}$ in such a way that the Einstein constraint equations are satisfied, cf. [10], and this can be done even when the OV solutions lie arbitrarily close to the Schwarzschild radius. In this construction, the shock waves tend to a standing shock wave moving at the speed of light, right at the Schwarzschild radius, as the shock is started closer and closer to the Schwarzschild radius. In this limit, the shock wave appears to be stationary as viewed by an outer observer, and also moves at zero speed relative to the fluid behind the shock. (It is somewhat paradoxical that, in this limit, the shock trajectory is lightlike because the shock is fixed at the Schwarzschild radius, but yet the shock moves with zero speed relative to the fluid behind the shock. The point is that the coordinate system for OT metric becomes singular at the Schwarzschild radius, and thus the shock-matching techniques break down in the actual limit.)

To be precise, recall that in [3] we showed that the surface layers that form in solutions of the OV system when the surface lies within 9/8ths of the Schwarzschild radius are determined by the values of three independent parameters $r_0$, $z_0$, and $A_0$, where $z_0$ and $A_0$ can be taken to be the initial conditions for the OV system at radius $r_0$. Here, $A_0 = 1-2\mathcal{G}M_0/c^2 r_0$, and $z_0=\rho_0/\bar{\rho}_0$, where $M_0$, $\rho_0$, and $\bar{\rho}_0 \equiv (3/4\pi)(M_0/r_0^3)$, denote the mass, density, and average density at radius $r=r_0$, respectively. The constants $\mathcal{G}$ and $c$ denote Newton's gravitational constant and the speed of light. The condition that the outer surface lies inside 9/8ths of the Schwarzschild radius $R_S(M_0)$, where $R_S(M_0) = 2\mathcal{G}M_0/c^2$, is equivalent to the condition that $A_0<1/9$. Thus, let $\Delta t$ denote the time, as measured by an observer in the far field, that it would take for a shock wave to propagate out through such a surface layer described by a solution of the OV equations, starting from initial data $0<z_0<1$ and $0<A_0<1/9$. In this paper we always assume that the density behind the shock is smaller than the average density; this restricts the surface layer to the region $0<z\leq 1$. [One could be more general, but the shock-matching results in [5] require $z \leq 1$ for outgoing shock waves. Moreover, since, by Eq. (2.13) below, the time dilation factor $A$ takes a minimum value at $z=1/3$, it follows that the dominant effects of time dilation occur within the region where $0<z\leq 1$.] We prove, under the assumptions that the sound speed of the fluid is uniformly bounded away from zero, and that the shock speed is less than the speed of light, that the time $\Delta t$ is bounded below by a function that is proportional to the the total mass $M_0$ of the star, times the factor $\ln(1/z_0)$. It follows, then, that $\Delta t$ tends to infinity as $z_0 \to 0$ or $M_0 \to \infty$. Moreover, trying to make $\Delta t$ large by making the shock speed small does not succeed, because, since the shock is supersonic, a small shock speed implies a small sound speed, and this, in turn, has the effect of making the layer so narrow as to cancel out the effect of the small shock speed on the value of $\Delta t$; c.f. Eq. (1.1).

Now at first glance it is somewhat surprising that by making $z_0$ small we can create a large time dilation. However, assuming that $r_0$ and $M_0$ are fixed, $z_0 \to 0$ is equivalent to $\rho_0 \to 0$. Thus, what is really happening, is that the assumption that the sound speed is bounded from below as $\rho \to 0$ has the effect in the OV system of creating a surface layer inside $r = r_0$. It turns out that within this layer, the metric coefficient $A(r)$, the function that governs the time dilation, stays small for a long enough subinterval within the layer so as to make $\Delta t$ tend to infinity, even though the width of the layer tends to zero. However, because the logarithm grows so slowly, it follows that the surface layer must be exceedingly narrow for the factor $\ln(1/z_0)$ to significantly affect $\Delta t$. In fact, the shock wave would need to be essentially right at the Schwarzschild radius, or else the mass of the star would need to be on the order of galaxies, in order for the lifetime $\Delta t$ to be on the order of a year. Indeed, our conclusion implies the following: The time $\Delta t$ it would take a shock wave (or sound wave) to propagate from radius $r=r_*$, where $z(r_*)=1$, out to radius $r_0$, where $z(r_0)<1$, is estimated above and below by

$$\frac{1}{5}\frac{\sigma_{\min}}{c^2}\frac{2\mathcal{G}}{c^2}M(r_0)\ln\frac{1}{z_0} \leq c\Delta t \leq 9\frac{\sqrt{\sigma_{\max}}}{c}\frac{2\mathcal{G}}{c^2}M(r_0)\ln\frac{1}{z_0}, \tag{1.1}$$

where $\sigma^2 = dp/d\rho$ denotes the square of the sound speed of the fluid. If we assume the equation of state for free particles in the extreme relativistic limit, namely $p=(c^2/3)\rho$, [2], and evaluate the physical constants, we obtain the following estimates for $\Delta t$ in years:

$$2.08\times 10^{-14} N \ln\frac{1}{z(r_0)} \leq \Delta t \leq 8.25\times 10^{-13} N \ln\frac{1}{z(r_0)}, \tag{1.2}$$

for a mass $M_0$ equal to $N$ solar masses. The logarithm term tends to infinity as $z_0 \to 0$, but this term cannot significantly increase the value of $\Delta t$ without making the size of the layer unrealistically narrow.

The plan of the paper is as follows. In Sec. II we describe the surface layers based on our earlier mathematical results on solutions of the OV system. In Sec. III we state the new results in this paper. In Sec. IV we prove the consistency of placing shock waves arbitrarily close to the Schwarzschild radius by establishing the existence of Lax admissible shock waves based on matching Oppenheimer-Tolman (OT) metrics (determined by solutions of the OV system) to Friedmann-Robertson-Walker (FRW) metrics, across shock interfaces, cf. [3]. That is, it follows from the Lax shock conditions that all speeds in the problem are bounded by the speed of light, and that the shock is supersonic relative to the fluid in front of the shock and subsonic relative to the fluid behind the shock. In Appendix A we summarize the prior results needed for the shock-matching problem in Sec. III, and in Appendix B we include a derivation of the volume of the FRW space that lies inside the shock wave in the shock-matching problem.

## II. SURFACE LAYERS NEAR BLACK HOLES

In this section we summarize the results in [9,8] which describe solutions of the OV equations in the region $0<r<r_0$, starting from initial data $0<A_0<1/9$, $0<z_0<1$, at $r=r_0$.

The Oppenheimer-Volkoff (OV) system is (cf. [2])

$$-r^2\frac{dp}{dr}=\mathcal{G}M\rho\left(1+\frac{p}{\rho c^2}\right)\left(1+\frac{4\pi r^3 p}{Mc^2}\right)A^{-1}, \quad (2.1)$$

$$\frac{dM}{dr}=4\pi\rho r^2, \quad (2.2)$$

where

$$A=A(r)\equiv 1-2\frac{\mathcal{G}}{c^2}\frac{M(r)}{r}. \quad (2.3)$$

Equations (2.1) and (2.2) form a system of two ODE's in the unknown functions $p=p(r)$, $\rho=\rho(r)$, and $M=M(r)$, where $p$ denotes the pressure, $\rho c^2$ denotes the mass-energy density, $c$ denotes the speed of light, $M(r)$ denotes the total mass inside radius $r$, and $\mathcal{G}$ denotes Newton's gravitational constant. The last three factors in Eq. (2.1) are the general-relativistic corrections to the Newtonian theory [2].

Solutions of (2.1) and (2.2) determine a Lorentzian metric tensor $g$ of the form

$$ds^2=-B(r)d(ct)^2+A(r)^{-1}dr^2+r^2[d\theta^2+\sin^2(\theta)d\phi^2], \quad (2.4)$$

that solves the Einstein equations

$$G=\frac{8\pi\mathcal{G}}{c^4}T, \quad (2.5)$$

when $G$ is the Einstein tensor, and $T$ is the stress-energy tensor for a perfect fluid:

$$T_{ij}=(p+\rho c^2)u_iu_j+pg_{ij}. \quad (2.6)$$

Here $A(r)$ is defined by Eq. (2.3), and the function $B$ satisfies the equation

$$\frac{B'}{B}=-2\frac{p'}{p+\rho c^2}. \quad (2.7)$$

The metric (2.4) is spherically symmetric, time independent, and the fluid four-velocity is given by $u_t=\sqrt{B}$ and $u_r=u_\theta=u_\phi=0$, so that the fluid is fixed in the $(t,r,\theta,\phi)$ coordinate system [2].

We assume that the equation of state is of the form $p=p(\rho)$, and satisfies the bounds (c.f. [8,3])

$$0<\mu<c^2 \quad (2.8)$$

and

$$0<\sigma_{\min}<\sigma<c^2, \quad (2.9)$$

where

$$\mu=\frac{p}{\rho}, \quad (2.10)$$

and $\sigma$, the sound speed squared, is defined by

$$\sigma=\frac{dp/dr}{d\rho/dr}. \quad (2.11)$$

Note that since an equation of state of the form $p=p(\rho)$ is assumed, the bounds (2.8) and (2.9) are implied by the usual physical requirements on the function $p(\rho)$ (cf. [8]).

In [8] we showed that on the maximal interval $(r_1,r_0]$ over which $M(r)>0$, the OV system (2.1) and (2.2) is equivalent to the system

$$\frac{dz}{dr}=-C\frac{z}{A}\left(\frac{1-A}{r}\right), \quad (2.12)$$

$$\frac{dA}{dr}=(1-3z)\left(\frac{1-A}{r}\right), \quad (2.13)$$

where

$$C\equiv\frac{(1+\mu/c^2)(1+3\mu z/c^2)}{2\frac{\sigma}{c^2}}-3(1-z)\frac{A}{1-A}. \quad (2.14)$$

In terms of $z$ and $A$, Eq. (2.7) becomes

$$\frac{B'}{B}=\frac{1}{r}\left(1+3\frac{\mu z}{c^2}\right)\left(\frac{1-A}{A}\right). \quad (2.15)$$

Our results here rely on an analysis of the surface layers that are described by solutions of the OV system when $A_0\leq 1/9$. These layers are quantitatively described in the following theorem which summarizes results proved in [8,3].

*Theorem 1.* Let $(z(r),A(r))$ be a smooth solution of Eqs. (2.12) and (2.13) starting from initial values $z_0>0$, $A_0>0$, and defined on a maximal interval $(r_1,r_0]$. Then

$$\lim_{r \to r_1} z(r) = +\infty,$$

$$\lim_{r \to r_1} M(r) = 0,$$

and $\rho$ and $p$ are positive, $\rho'$ and $p'$ are negative on $[r_1, r_0]$.

Assume further that the initial values satisfy

$$0 < z_0 < 1, \tag{2.16}$$

$$0 < A_0 \leq \frac{1}{9}. \tag{2.17}$$

Then $r_1 > 0$, and there is a unique point $r_*$, $r_1 < r_* < r_0$, such that $z(r_*) = 1$, $z(r) < 1$ for $r > r_*$, $z(r) > 1$ for $r < r_*$, and the following inequalities hold:

$$1 > \frac{r_*}{r_0} > \sqrt{[1 - 9A(r_0)]/[1 - A(r_0)]} \tag{2.18}$$

and

$$\rho(r) < \rho(r_*) \leq \frac{3}{8\pi \mathcal{G} r_0^2} \frac{1 - A_0}{1 - 9A_0}, \tag{2.19}$$

for all $r$, $r_* \leq r < r_0$. Finally, if $r_0$ and $z_0$ are fixed, then

$$\lim_{A_0 \to 0} \frac{r_1}{r_0} = 1. \tag{2.20}$$

The point $r_*$ plays an important role in the shock-wave-matching problem considered in [4,9,5] because outgoing shocks modeling explosions can be constructed by matching FRW metrics to OT metrics only at OT positions $r > r_*$, where $\rho > \bar{\rho}$. Note that Eq. (2.20) implies that the width of the layer tends to zero as $A_0 \to 0$, and the estimate (2.18) gives a rate at which $r_*/r_0 \to 1$ as $A_0 \to 0$. Note too that the hypothesis $0 < A_0 \leq \frac{1}{9}$ implies that $r_0$ is outside the Schwarzschild radius $R_s(M_0)$, but inside 9/8ths of $R_s(M_0)$.

The following corollary of theorem 1 gives a refinement of the Buchdahl 9/8ths theorem [3].

*Corollary 1.* If $r_1 = 0$, then $A_0 > \frac{1}{9}$, or equivalently

$$r_0 > \frac{9}{8} R_s[M(r_0)].$$

In what follows, we study the problem in which an interior shock wave supplies the pressure required to hold up a surface layer that lies between $r = r_*$ and $r = r_0$. We are interested in determining the possible "lifetimes" of such a solution by estimating the time it would take a shock wave to pass through a surface layer that lies close to the Schwarzschild radius, as seen by a distant observer. In this treatment we assume that the shock wave lies outside radius $r_*$, which is equivalent to assuming that the density behind the shock is smaller than the average density behind the shock because $0 \leq z \leq 1$ for $r \geq r_*$. This condition was required for the construction of outgoing shocks in [5]. Moreover, it is quite reasonable to restrict to the layers that lie between $r_*$ and $r_0$ because, by Eq. (2.13), $A$ takes its minimum value at $z = 1/3$, and thus the time dilation factor $A$ has its greatest effect within the region between $r_*$ and $r_0$.

## III. STATEMENT OF RESULTS

In this section we consider the case of a shock wave which propagates outward into a time-independent OV metric between the radial positions $r = r_*$ and $r = r_0$, where $0 \leq z \leq 1$. We imagine that there is a shock wave inside the star, blasting outward toward the boundary of the star. In [4,9,5] we have shown how to construct explicit examples of shock waves which match an arbitrary OT metric to a Friedmann-Robertson-Walker (FRW) metric across a shock interface such that mass energy and momentum are conserved. Before turning to this matching procedure for the OV solutions discussed above, we first estimate the time it would take for an arbitrary wave inside the star to reach the surface of the star when $A_0$ is small. Since shock waves always move at speeds supersonic relative to the fluid in front of the shock, at this stage we assume only that the wave propagates outward at speed $c\nu$ less than the speed of light $c$. That is, if the interior of the star gets into a configuration where the pressure balances the gravitational force at some instant of time, such that the stationary OV solution is on the outside (for example, due to the rebound after gravitational collapse), then we shall give an estimate for the time it would take for a wave to reach the surface as measured by an observer in the far field. This gives an estimate for the "lifetime" of the star in the sense that it estimates the time interval over which the outer layer of the star will appear time independent to a far away observer. In particular, as $A_0$ tends to zero (so that the boundary surface tends to the Schwarzschild radius of the star), we show that the time interval over which it would take a wave to travel between $r = r_*$, where $z(r_*) = 1$, out to $r_0$, where $z(r_0) = z_0 < 1$ (cf. theorem 1), tends to infinity due to the time dilation effect resulting from the strong gravitational field in the outer layer.

In order to make this precise, assume that a spherical wave propagates outward into a fluid, which we assume is described by an OT metric (2.4). We assume too that for sufficiently large values of $r$ the solution reduces to the empty space Schwarzschild metric, which is asymptotically Minkowskian in the limit $r \to \infty$; i.e., both $A$ and $B$ tend to unity in the far field. Assume that the wave propagates from radius $r = r_*$ at time $t = t_*$ to $r = r_0$ at time $t = t_0$, where $r_* < r_0$, $t_* < t_0$. (From now on we shall suppress the $\phi$ and $\theta$ coordinates which play no role.) Then, from Eq. (2.4), the coordinate speed of the wave is given by

$$\frac{dr}{dt} = c\nu\sqrt{AB},$$

where $\nu$ denotes the "dimensionless" speed of the wave, as measured in a local Minkowski frame $(c\bar{t}, \bar{r})$ in which $d\bar{t} = \sqrt{B}dt$ and $d\bar{r} = (1/\sqrt{A})dr$. Let $t_*^\infty$ and $t_0^\infty$ denote the respective OT coordinate times at which an observer in the far field located at $r_\infty \gg 1$ receives light signals emitted at $(t_*, r_*)$ and $(t_0, r_0)$, respectively. Since we have assumed that the OT metric is asymptotically Minkowskian, the time change $\Delta t \equiv t_0^\infty - t_*^\infty$ measures proper time at $r_\infty$. Now from the OT metric (2.4), the lightlike radial geodesics travel at coordinate speed

$$\frac{dr}{dt} = c\sqrt{AB}.$$

Thus integrating, the time, $\Delta t$, it takes the wave to travel from $r_*$ to $r_0$ as measured by an observer at $r_\infty$ is

$$c\Delta t = c(t_0 - t_*) + \int_{r_*}^{r_0} \frac{dr}{\sqrt{AB}},$$

where

$$c(t_0 - t_*) = \int_{r_*}^{r_0} \frac{dr}{\nu\sqrt{AB}}.$$

Thus we have

$$c\Delta t = \int_{r_*}^{r_0} \frac{1+\nu}{\nu} \frac{dr}{\sqrt{AB}}.$$

Using Eq. (2.13) to rewrite the integral in terms of the variable $z$ in favor of $r$ we obtain the following theorem.

*Theorem 2.* Let $(z(r),A(r))$ denote a solution of system (2.12) and (2.13), starting from intial data $0<A_0<1$, $0<z_0<1$, where $C$ is given by Eq. (2.14). Assume that the solution is defined on the interval $I \equiv (r_*, r_0)$ over which $0<z_0<z<z_*=1$, and such that $C>0$, [which holds for sufficiently small $A$, and guarantees that $z'(r)>0$ in view of Eq. (2.12)]. Then the time $\Delta t$ it takes a wave moving outward at dimensionless speed $\nu$, to travel from $r_*$ to $r_0$ (as measured by an observer in the far field) is given by

$$c\Delta t = \int_{z_0}^{1} \frac{1+\nu}{\nu} \frac{1}{C} \frac{r}{1-A} \sqrt{A/B}\, \frac{dz}{z}. \tag{3.1}$$

We remark that, in view of theorem 1, $r_* \to r_0$ as $A_0 \to 0$, and thus by taking $A_0$ sufficiently small we can be sure $C>0$ on the interval $r_*$ to $r_0$. Note also that for $z<1/3$, $A<A_0$, so in order to guarantee that $C>0$ we could obtain a similar result by restricting the layer to the interval over which $z<1/3$.

We can obtain estimates for $\Delta t$ that are independent of the shock speed as follows. Assuming that the shock speed is less than the speed of light is equivalent to taking $\nu<1$, in which case Eq. (3.1) gives

$$c\Delta t \geq \int_{z_0}^{1} 2\frac{1}{C} \frac{r}{1-A} \sqrt{A/B}\, \frac{dz}{z}. \tag{3.2}$$

Using the fact that shock waves are always supersonic relative to the fluid in front of the shock, and the fact that $\sqrt{\sigma} = \sqrt{p'(\rho)}$ is the sound speed when $p=p(\rho)$, we obtain

$$c\Delta t \leq \int_{z_0}^{1} \frac{1+\sqrt{\sigma/c^2}}{\sqrt{\sigma/c^2}} \frac{1}{C} \frac{r}{1-A} \sqrt{A/B}\, \frac{dz}{z}. \tag{3.3}$$

We can apply Eq. (3.1) to the solutions described in theorem 1 that satisfy the estimate (2.18) in the limit as $A_0 \to 0$. From Eq. (2.14), it follows that if $A<1/9$ and $z<1$, then

$$\frac{1}{4}\frac{\sigma}{c^2} \leq C^{-1} \leq 8\frac{\sigma}{c^2}. \tag{3.4}$$

(In fact, we know that by Eq. (2.19), $M'$ is uniformly bounded on $[r_*, r_0]$, so theorem 1 implies that $A(r_*) \to A_0$ as $A_0 \to 0$, and thus we may assume that $A_0$ is sufficiently small so that $A<1/9$ throughout this interval.) In this case, we have the following corollary.

*Corollary 2.* Let $(z(r),A(r))$ be a solution of system (2.12) and (2.13) defined on interval $I=(r_*, r_0)$, $r_*<r_0$. Then

$$\frac{1}{4}\frac{\sigma_{\min}/c^2}{\nu_{\max}} r_* \ln\frac{1}{z_0} \leq c\Delta t \leq 8\sqrt{\sigma_{\max}/c^2}\, r_* \ln\frac{1}{z_0}, \tag{3.5}$$

where $\nu_{\max}$ denotes the maximum shock speed in the interval $I$.

In obtaining the right-hand inequality in Eq. (3.5), we have used the fact that the dimensionless shock speed $\nu$ is always faster than the dimensionless sound speed $\sqrt{\sigma/c^2}$.

Note that by Eq. (2.18), $r_*$ tends to $r_0$ as $A_0$ tends to zero, and since $A = 1 - 2\mathcal{G}M/c^2 r$, it follows that $r_*$ tends to $2\mathcal{G}M(r_0)/c^2$ in this limit, where $M(r_0)$ can be interpreted as the total mass of the star. Thus, the lower bound in Eq. (3.5) implies

$$c\Delta t \geq \frac{1}{5}\frac{\sigma_{\min}}{c^2}\frac{2\mathcal{G}}{c^2} M(r_0) \ln\frac{1}{z_0}, \tag{3.6}$$

and the upper bound implies

$$c\Delta t \leq 9\sqrt{\sigma_{\max}/c^2}\, \frac{2\mathcal{G}}{c^2} M(r_0) \ln\frac{1}{z_0}, \tag{3.7}$$

for $A_0$ sufficiently small, where the errors have been incorporated into the replacement of the factor $\frac{1}{4}$ by $\frac{1}{5}$ for Eq. (3.6), and the factor 8 by 9 in Eq. (3.7).

In particular, these inequalities show that the time interval $\Delta t$ increases linearly with the total mass of the star $M(r_0)$. What is remarkable, however, is the presence of the additional factor $\ln(1/z_0)$, which tends to infinity as the initial condition $z_0$ tends to zero. This is somewhat surprising. Indeed, assume, for example, that $r=r_*$ [i.e., $z(r_*)=1$]. Then, because in the above estimate (3.6) the shock speed is assumed to be the speed of light, and by Eq. (2.18) the shock layer $\Delta r$ has a width on the order of $r_0 A_0$, one might expect this factor to cancel exactly the $A$ in the time dilation factor $A^{-1}$ that comes from the OT metric. But, in fact, the estimate (3.6) demonstrates that in this case the time it takes a wave moving at the speed of light to reach the surface at $r=r_0$, tends to infinity as the initial condition $z_0$ tends to zero. In this case, the shock layer between $z=z_0$ and $z=1$, would be on the order of $z_0$. [To see this, note that from Eq. (2.18) the layer width is smaller than order $A(r_0)$, but in fact, from Eqs. (2.12) and (2.13), we see that the function $z(r)$ stays very close to $z_0$ until $A(r)$ becomes on the order of $z(r)$ in light of the factor $z/A$ in Eq. (2.12) and the fact that $A'$ is bounded away from zero when $z \ll 1$.] It is also interesting to note that the time it takes the shock to propagate through the layer cannot be increased beyond the estimate (3.7) by making the shock speed arbitrarily small. The point here is that the shock speed $\nu$ is bounded from below by the sound speed $\sqrt{\sigma}/c$ of the fluid, but as $\sigma \to 0$, the width of the layer tends to zero. The inequality (3.7) represents the net effect of these competing influences.

In practical terms, once the physical constants are substituted into the estimates, the width of the layer between $r_*$ and $r_0$ is exceedingly small when $\ln(1/z_0)$ is large enough to produce a significant time dilation.

To get a sense of how large $c\Delta t$ can be, note that when $p=p(\rho)$, the sound speed $\sigma$ satisfies $\sigma<1$, and using the values

$$c=3.00\times 10^{10} \text{ cm/sec},$$

$$\mathcal{G}=6.67\times 10^{-8} \text{ dyne cm}^2\text{ g}^{-2},$$

we obtain

$$\frac{\mathcal{G}}{c^3}=2.47\times 10^{-39} \text{ sec}.$$

Now the mass of the Sun is

$$1.99\times 10^{33} \text{ g},$$

and as there are $3.15\times 10^7$ sec in a year, we see that, in years, Eq. (3.6) yields

$$\Delta t \geq 6.24\times 10^{-14} N\sigma_{\min}\ln\frac{1}{z_0} \text{ yr},$$

while Eq. (3.7) yields

$$\Delta t \leq 4.87\times 10^{-12} N\sigma_{\min}\ln\frac{1}{z_0} \text{ yr},$$

for a mass $M(r_0)$ in Eq. (3.6) of Eq. (3.7) equal to $N$ solar masses. Statement (1.2) of the Introduction follows in the case $\sigma_{\min}=c^2/3$. Thus, without the $\ln(1/z_0)$ term, and assuming that the sound speed is comparable to the speed of light, it would take a mass equal to $10^{12}$ solar masses to get a life-time on the order of one year.[1]

In the next theorem, we show that there exist fluid dynamical shock waves that satisfy the Lax entropy condition [6,7], which lie arbitrarily close to the Schwarzschild radius $r=2\mathcal{G}M/c^2$, $A=0$. Our construction is based on the analysis in [5], which we summarize in Appendix A. The theorem demonstrates the existence of shock waves satisfying the Lax shock conditions, such that in the limit $r\to 2\mathcal{G}M/c^2$ these shock waves tend to a shock wave positioned right at the Schwarzschild radius, as seen by a distant observer in the OT coordinate system. For these shocks, the density behind the shock tends to a fixed finite value, and the density in front of the shock tends to zero, so in this limit, the shock wave tends to a contact discontinuity of Oppenheimer-Snyder type, cf. [4]. Our analysis here entails determining initial conditions at $r=r_0$ for the shock equations derived in [5] (see Appendix A), such that the Lax shock conditions hold initially. By continuity it follows that these conditions are valid in a neighborhood of $r_0$. Note that this verifies the consistency of placing a shock wave arbitrarily close to the Schwarzschild radius. However, in this simplified model, the pressure behind such a shock is larger than the pressure in front of the shock, but the pressure behind the shock is determined by the equations, and cannot be arbitrarily assigned when FRW-type metrics are assigned behind the shock, cf. [4,5]. (In the following theorem, and its development in the next section, we assume for simplicity that $c=1$.)

*Theorem 3.* Let $(z(r),A(r))$ be a solution of system (2.12) and (2.13) defined on interval $I=(r_*,r_0)$, so that we may assume Eq. (2.18) holds. Then if $z_0=z(r_0)$ is sufficiently small, there exists a matching of a Friedmann-Robertson-Walker (FRW) metric to the OT metric determined by $(z(r),A(r))$, such that the interface between the metrics defines a fluid dynamical shock wave that satisfies the Lax characteristic conditions in a neighborhood of $r=r_0$. In this construction, the shock waves propagate outward into the fixed OT solution, and in the limit $A(r_0)\to 0$, the shock position, as measured in OT coordinates, tends to $r=2\mathcal{G}M_0$, the density behind the shock tends to

$$\hat{\rho}=\frac{3}{8\pi\mathcal{G}}\frac{1}{r_0^2}=\frac{3}{32\pi\mathcal{G}^3}\frac{1}{M_0^2},$$

and the speed of the shock tends to the speed of light in the sense that the trajectory of a particle fixed at the Schwarzschild radius is lightlike. Because the shock position in this limit is right at the Schwarzschild radius, in this limit the shock appears stationary to an observer at infinity in the OT coordinates. Moreover, the limit of the pressure behind these shocks tends to zero as $A(r_0)\to 0$, and thus the limiting solution is a standing shock wave, positioned right at the Schwarzschild radius.

In this construction, the shock waves are determined by two different sets of initial conditions imposed at $r=r_0$. Each determines a Lax admissible shock wave in a neighborhood of $r=r_0$. In the first case, the position of the shock as measured in FRW coordinates behind the shock, tends to the outer boundary of the FRW space, but in the second case, the FRW shock position tends to a finite radius.

## IV. SHOCK WAVES NEAR THE SCHWARZSCHILD RADIUS

In this section we give the proof of theorem 3. Throughout this section we shall refer to results summarized in Appendix A. In order to distinguish different coordinate systems in the shock-matching problem, from here on we use barred variables to denote variables in an OT metric, and unbarred variables for variables in a Friedmann-Robertson-Walker metric. (This is consistent with the notation used in [3–5].) To start, assume throughout that $c=1$, and assume that $(z(\bar{r}),A(\bar{r}))$ is a fixed solution of system (2.12) and (2.13) defined on the interval $I=(\bar{r}_*,\bar{r}_0)$. By theorem 1, Eq.

[1]If there actually were stellar objects described by this model, then the shorter the lifetime, the more of them there would have to be in order to have a chance of observing one.

(2.18) holds, and hence $z$ decreases monotonically from $z=1$ to $z=z_0 \ll 1$, as $\bar{r}$ goes from $\bar{r}_*$ to $\bar{r}_0$. We now construct shock waves at $\bar{r}=\bar{r}_0$, by matching the OT metric (2.4) determined by this OT solution, Lipschitz continuously to a FRW metric [a metric of the form (A4)], at $\bar{r}=\bar{r}_0$, to obtain a matched metric which agrees with the OT metric for $\bar{r} \geq \bar{r}_0$, and such that the OT metric for $\bar{r}<\bar{r}_0$ is replaced by a FRW metric [3–5]; see Appendix A for a short summary of the results in these papers.

The interface between the metrics defines a shock wave whose position $\bar{r}=\bar{r}(t)$, is given implicitly by the equation (cf. [4] and theorem 5 in Appendix A)

$$M(\bar{r})=\frac{4\pi}{3}\rho(t)\bar{r}^3,$$

where $\rho(t)$ denotes the FRW mass-energy density function, and $t$ is the time coordinate for the inner FRW metric. The construction of the FRW metric that matches a given OT metric Lipschitz continuously at the shock, such that conservation of energy and momentum hold across the shock, satisfy Eqs. (A15) and the FRW pressure is then determined by Eq. (A1). For a fixed OT solution and fixed value of $k$ in the FRW metric, system (A15) is an autonomous system of two ODE's for the two unknown functions $R(t)$ and $r(t)$, $R(t)$ being the cosmological scale factor that determines the FRW metric, and $r(t)$ giving the shock position in FRW coordinates $(t,r)$. (By rescaling the variable $r$ in the FRW metric, the magnitude of $k$ can be assigned arbitrarily, but the sign of $k$ is invariant under such scalings, and thus there are canonically three distinct cases, $k=-1,0,+1$, cf. Appendix B.) The FRW sound speed $\sigma$ in a given shock-wave solution, is determined by Eq. (A19), where we assume an equation of state of the form $p=p(\rho)$ so that $\sigma=p'/\rho'$.

The shock equations (A15) allow for two initial conditions, and since we assume the shock is initially positioned at $\bar{r}=\bar{r}_0$, and the matching of the spheres of symmetry require that $\bar{r}=Rr$, there is only one degree of freedom left in the initial conditions. This final degree of freedom is fixed by a choice of $\theta=A(\bar{r})/(1-kr^2)$, cf. Eq. (A10) in Appendix A. The Lipschitz continuous matching of the metrics together with the conservation of mass and energy, alone do not guarantee that the shock speed and FRW sound speed are less than the speed of light, or that the Lax shock conditions, (A22) and (A23), hold. The shock speed is subluminous if and only if

$$\theta>\theta_-(z,\bar{\mu})\equiv\frac{2\gamma-1}{\gamma^2}=1-\left(\frac{1-z}{1+\bar{\mu}z}\right)^2, \tag{4.1}$$

the OT-Lax condition (A23) holds if and only if

$$\theta<\theta_+(z,\bar{\mu})\equiv 1-\left(\frac{1-z}{1+\bar{\mu}z}\right)^2\bar{\sigma}, \tag{4.2}$$

and when $\theta_-<\theta<\theta_+$, it follows that $p>\bar{p}\geq 0$, and $0<\mu<1$, where

$$\mu\equiv\frac{p}{\rho}=\frac{\gamma\theta z-1}{\gamma\theta-1}, \tag{4.3}$$

cf. theorem 8 in Appendix A.

We shall show that Eqs. (4.1) and (4.2) together with the FRW-Lax condition (A22) can all be met at $\bar{r}=\bar{r}_0$ in the above fixed OT solution, for suitable initial values of $\theta$, so long as $z_0=z(\bar{r}_0)$, is sufficiently small. To this end, we use the following expressions which are valid to leading order in $z$ as $z\to 0$:

$$\theta_-\sim 2(1+\bar{\mu})z, \tag{4.4}$$

$$\theta_+\sim 1-\frac{\bar{\sigma}}{(1+\bar{\mu})^2}. \tag{4.5}$$

Consider, then, the problem of assigning initial conditions for the shock equations at $\bar{r}=\bar{r}_0$ in the OT solution $(z(\bar{r}),A(\bar{r}))$. Recall that the OT solution itself is determined from initial conditions $z(\bar{r}_0)=z_0$ and $A(\bar{r}_0)=A_0$. Consider first a sequence of such OT solutions for which $z_0\to 0$, and $A_0\to 0$; i.e., $M(\bar{r}_0)\to\hat{M}=\bar{r}_0/2\mathcal{G}$. These imply $\rho(\bar{r}_0)\to\hat{\rho}\equiv(3/4\pi)(\hat{M}/\bar{r}_0^3)$, and $\bar{\mu}(\bar{r}_0)\to\hat{\mu}\geq 0$, and $\bar{\sigma}\to\hat{\sigma}\geq 0$. We now determine appropriate initial conditions for the shock equations (A15) as functions of $z_0$, so that the OT solution determined by each set of initial conditions determines a Lax admissible shock wave at $\bar{r}=\bar{r}_0$ (and thus in a neighborhood of $\bar{r}_0$), for $z_0$ sufficiently close to zero. In order to do this, we assign initial values $A_0$ and $\theta_0$ at $\bar{r}_0$, as functions of $z_0$ [call them $A(z_0)$ and $\theta(z_0)$ and assume $A(z_0)\to 0$ as $z_0\to 0$], in such a way that the shocks will be Lax admissible at $\bar{r}=\bar{r}_0$ (and hence in a neighborhood of $\bar{r}_0$), for each value of $z_0$ near zero. In this way, each value of $z_0$ will determine a unique shock wave in a neighborhood of $\bar{r}_0$.

In order to ensure that the sound speed $\sigma$ given in Eq. (A19) is finite in the limit $z\to 0$, it suffices to choose initial condition $\theta(z_0)$ so that

$$\frac{z_0}{\theta(z_0)}\to 0 \tag{4.6}$$

as $z_0\to 0$. That is, as $z\to 0$, the following asymptotic expressions hold:

$$\gamma\theta\sim\frac{1}{1+\bar{\mu}}\frac{\theta}{z},$$

$$\alpha\sim\frac{3}{z},$$

$$\beta\sim\frac{1}{z}.$$

Moreover, it follows from Eq. (A19) that if we assume $z_0/\theta(z_0)\to 0$ as $z_0\to 0$, then the following asymptotic expression for the FRW sound speed also holds:

$$\sigma=\frac{p'}{\rho'}\sim\frac{1+\bar{\mu}}{6}\left(3+\frac{1-\theta}{\bar{\sigma}}\right)\frac{z}{A\theta}, \tag{4.7}$$

as $z\to 0$. Furthermore, the dimensionless shock speed $s$ relative to the FRW fluid is given in Eq. (A16), and the FRW pressure $p$ is given in Eq. (A1). It follows from Eq. (A1) that

$$p \sim (1+\bar{\mu}-\theta)\,\frac{\hat{\rho}z}{\theta} \tag{4.8}$$

and

$$\mu=\frac{p}{\rho}\sim z, \tag{4.9}$$

as $z\rightarrow 0$; and it follows from Eq. (A16) that

$$s\sim(1+\bar{\mu})z\,\frac{\sqrt{1-\theta}}{\theta}, \tag{4.10}$$

and as $z\rightarrow 0$. Moreover,

$$\theta_{-}\sim 2(1+\bar{\mu})z \tag{4.11}$$

and

$$\theta_{+}\sim 1-\bar{\sigma}, \tag{4.12}$$

as $z\rightarrow 0$. Thus, if $\theta(z_0)$ stays smaller than $1-\bar{\sigma}$ and $z/\theta\rightarrow 0$, we must have by theorem 8 of Appendix A that $\theta_{-}<\theta<\theta_{+}$ in the limit $z_0\rightarrow 0$, and thus the OT-Lax conditions must hold and the shock speed must be less than the speed of light at $\bar{r}=\bar{r}_0$ if $z_0$ is sufficiently small.

To verify the FRW-Lax conditions, a calculation shows that Eq. (A22) is equivalent to

$$A<\frac{1}{6(1+\bar{\mu})(1-\theta)}\left(3+\frac{1-\theta}{\bar{\sigma}}\right)\frac{\theta}{z}, \tag{4.13}$$

which obviously holds for $z$ sufficiently small, assuming $z/\theta\rightarrow 0$. Finally, we see from Eq. (4.7) that we can choose $A(z_0)\rightarrow 0$ in such a way that $\sigma\rightarrow\bar{\sigma}$ for any $0\leqslant\bar{\sigma}<1$ by requiring

$$A\sim\frac{1+\bar{\mu}}{6}\left(3+\frac{1-\theta}{\bar{\sigma}}\right)\frac{1}{\bar{\sigma}}\frac{z}{\theta}. \tag{4.14}$$

As a first example, take the case $k>0$ and let $\theta(z_0)=\theta_0$ be constant, and choose $A=Kz_0$ for some constants $0<\theta_0<\theta_{+}$, and $K>0$. In this case, in the limit $z_0\rightarrow 0$, the shock position $r_0$ on the FRW side of the shock tends to $r_0=1/\sqrt{k}$, the edge of the FRW universe. Also by Eq. (A17), the speed of the OT fluid relative to the FRW fluid at the shock tends to $u=-\sqrt{1-\theta_0}>-1$.

As a second example, take $k\neq 0$, and choose $\theta=L\sqrt{z_0}$ and $A=K\sqrt{z_0}$, in which case the shock position $r_0$ on the FRW side of the shock tends to a nonzero finite value.

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## APPENDIX A: THE SHOCK-MATCHING PROBLEM

In this section we summarize the results in [4,9], in which we construct shock-wave solutions of the Einstein equations by matching a FRW metric to a given OT metric where we assume that the FRW metric is on the inside of a given OT metric. We assume in this section that the speed of light $c=1$, and we let barred variables denote OT variables, unbarred denote FRW variables, and we restrict to the case of an outgoing shock wave in which the FRW metric is placed on the inside of the OT metric. We assume in this section that the OT solution (2.4) is given for a preassigned equation of state of the form $\bar{p}=\bar{p}(\bar{\rho})$, and the FRW solution, $\rho(t)$, $p(t)$, $R(t)$, is determined by a system of two ordinary differential equations. The solution then *determines* the FRW equation of state $p=p(\rho)$ implicitly through the formulas

$$p=\frac{1-\gamma\theta z}{\gamma\theta-1}\,\rho, \tag{A1}$$

where

$$\gamma=\frac{\rho+\bar{p}}{\bar{\rho}+\bar{p}}, \tag{A2}$$

and

$$\rho=\frac{3}{4\pi}\frac{M(\bar{r})}{\bar{r}^3} \tag{A3}$$

is the FRW density, and $\bar{\rho},\bar{p}$ denote the OT density and pressure, respectively. The solutions $\rho(t)$, $p(t)$, $R(t)$ determine a solution of the Einstein equations of FRW-type, namely,

$$ds^2=-dt^2+R(t)^2\left\{\frac{dr^2}{1-kr^2}+r^2d\Omega^2\right\}. \tag{A4}$$

In this section we restrict to the case of an outgoing shock wave for which the FRW metric is inside the OT metric, $z=\bar{\rho}/\rho<1$.

The first theorem summarizes the results in Sec. IV of [4], pp. 280–285.

*Theorem 4.* Assume that we are given any FRW and OT metrics such that the shock surface $\bar{r}=\bar{r}(t)$ is defined implicitly by Eq. (A3) in a neighborhood of a point $(t_0,\bar{r}_0)$, $t_a<t_0<t_b$, $\bar{r}_a<\bar{r}<\bar{r}_b$, and $r_a<r\equiv\bar{r}/R<r_b$. Assume that

$$\frac{d\bar{p}}{d\bar{\rho}}>0, \tag{A5}$$

$$A(\bar{r}_0)\neq 0, \tag{A6}$$

and, hence,

$$1-kr_0^2>0\quad[r_0=\bar{r}_0/R(t_0)]. \tag{A7}$$

Then for any value of $\bar{r}_0$, there exists a mapping $(t,r)\rightarrow(\bar{t},\bar{r})$ (defined in Sec. 4 of [4]), which we denote by

$$\Psi(t,r)\equiv(\bar{t}(t,r),\bar{r}(t,r)),\quad \bar{r}(t,r)=R(t)r, \tag{A8}$$

such that $\Psi$ is 1-1 and regular in a neighborhood of the point $(t_0,r_0)$ and takes the open interval $(t_0,r_0)$ into the interval $(\bar{t}_0,\bar{r}_0)$.[2] Moreover, under this coordinate identification, the given FRW and OT metrics match Lipschitz continuously across the surface (A3). The condition

$$\bar{r}(t,r)=R(t)r, \tag{A9}$$

implies that the areas of the sphere's of symmetry agree in the barred and unbarred frames, and the shock surface in $(t,r)$ coordinates is given by $r(t)=\bar{r}(t)/R(t)$.

Remarks: Condition (A5) says that the OT sound speed is positive, condition (A6) says that $\bar{r}$ is not at the "Schwarzschild radius," and condition (A7) says that the value of $r_0$ is not outside the FRW universe, i.e., is inside the region of validity of the FRW coordinate system. It follows that the condition

$$\theta\equiv\frac{A(\bar{r})}{1-kr^2}\leq 1 \tag{A10}$$

must hold (see, e.g., [5]).

In this case, the results in [5] state that the only physically interesting possibility is when $\rho>\bar{\rho}$, and the constraint equation determines the FRW pressure through the formulas (A1)–(A3).

We now give the main results obtained in [5] for this case. We always assume

$$0<\bar{\mu}\equiv\bar{p}/\bar{\rho}<1, \tag{A11}$$

$$0<\bar{\sigma}\equiv\frac{d\bar{p}}{d\bar{\rho}}<1, \tag{A12}$$

and we use the notation

$$\mu=\frac{p}{\rho} \tag{A13}$$

and

$$\sigma=\frac{dp}{d\rho}\equiv\frac{\dot{p}}{\dot{\rho}}. \tag{A14}$$

The first theorem gives the ODE's that describe the time evolution of the FRW metrics that match a given OT metric across the surface (A3), such that conservation of energy and momentum hold across the shock.

*Theorem 5.* Let a fixed OT solution satisfying (A5) be given. Then any FRW metric that matches this OT metric Lipschitz continuously (on the inside) of the shock surface (A3), such that $\rho>\bar{\rho}$ and Eq. (A10) hold, and such that the Rankine Hugoniot jump conditions

$$[T^{ij}]n_i=0$$

also hold across the shock, must solve the ODE's

$$r\dot{R}=\sqrt{1-kr^2}\sqrt{1-\theta},$$

$$R\dot{r}=\frac{1}{\gamma\theta-1}\sqrt{1-kr^2}\sqrt{1-\theta}. \tag{A15}$$

Conversely, any smooth solution of Eqs. (A15) satisfying Eqs. (A6) and (A7) and (A10), will determine a solution of FRW type if we take

$$\rho=\frac{3}{4\pi}\frac{M}{\bar{r}^3},$$

and $p$ to be given by Eq. (A1); this solution will match the OT metric Lipschitz continuously across the shock surface (A3) [when we make the identification (A8)], and the Rankine Hugoniot jump conditions will hold across the shock.

The two unknowns in Eqs. (A15) are the shock position $r(t)$ and the FRW cosmological scale factor $R(t)$. $\theta$ is a function of $\bar{r}=Rr$ and $r$ obtainable from the known functions $\bar{\rho}(\bar{r}),\bar{p}(\bar{r})$, and $A(\bar{r})$ of the OT solution, and $\rho$ is determined as a function of $\bar{r}$ through the shock surface equation (A3), $\rho=(3/4\pi)(M/\bar{r}^3)$. Thus we are free to choose two initial conditions, $r$ and $R$, for the ODE's (A15). Moreover, the OT solution is determined by the choice of initial conditions $M$ and $\bar{\rho}$ at given $\bar{r}$ for arbitrary equation of state $\bar{p}=\bar{p}(\bar{\rho})$. Thus, we can determine local shock-wave solutions by arbitrarily assigning the OT equation of state, as well as $\bar{r}$, $\bar{\rho}$, $M$ and one of $r$ or $R$, thus allowing four initial conditions in all. The variables $p$, $\mu$, and $\sigma$ are all then determined by Eqs. (A15).

In this paper we take the OT solution to be given for $\bar{r}_*<\bar{r}<\bar{r}_0$ as in theorem 1, and we take our initial conditions for Eqs. (A15) at $\bar{r}=\bar{r}_0$. In this case, since $\bar{r}=rR$, we are free to assign one additional initial condition, which we can take to be the assignment of $\theta=\theta_+$ at $\bar{r}=\bar{r}_0$.

The next theorem discusses the shock speed.

*Theorem 6.* The speed of the shock determined by Eqs. (A15), as measured in a local Minkowski coordinate frame fixed with the FRW fluid element, is given by

$$s^2=\frac{1-\theta}{(1-\gamma\theta)^2}. \tag{A16}$$

The other important "speeds" in the problem, the sound, characteristic, and fluid speeds, are described in the following theorem.

*Theorem 7.* Consider any shock-wave solution determined by Eqs. (A15). Then, in the locally Minkowski coordinate frame fixed with the FRW metric, the OT fluid speed $\bar{u}$ is given by

$$\bar{u}=-\sqrt{1-\theta}, \tag{A17}$$

the OT outgoing characteristic speed $\bar{\lambda}_2^{\rm OT}$ is given by

$$\bar{\lambda}_2^{\rm OT}=\frac{\bar{u}+\sqrt{\bar{\sigma}}}{1+\bar{u}\sqrt{\bar{\sigma}}}. \tag{A18}$$

and the FRW sound speed $\sigma$ is given by

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[2]Note that the mapping $(t,\bar{r})\rightarrow(\bar{t},\bar{r})$ is 1-1 whenever the mapping $(t,r)\rightarrow(\bar{t},\bar{r})$ is 1-1, because the mapping $(t,r)\rightarrow(t,\bar{r})=(t,R(t)r)$ has Jacobian $R(t)>0$.

$$(1-\theta\gamma)^2\sigma=\frac{1}{6}\frac{\theta(1-A)}{A(1+\bar{\mu})}\left\{\alpha(z,\bar{\mu})+\beta(z,\bar{\mu})\frac{1-\theta}{\bar{\sigma}}\right\}+\frac{2}{3}\frac{\theta}{A}+\theta-\frac{5}{3}, \tag{A19}$$

where

$$\alpha(z,\bar{\mu})\equiv\frac{3-7z+5\bar{\mu}z-9\bar{\mu}z^2}{z} \tag{A20}$$

and

$$\beta(z,\bar{\mu})\equiv\frac{(1+3\bar{\mu}z)(1+\bar{\mu}z)^2}{z(1-z)}. \tag{A21}$$

The Lax shock conditions for an outgoing two-shock then read

$$\sqrt{\sigma}>s \quad \text{(FRW-Lax)} \tag{A22}$$

and

$$s>\bar{\lambda}_2^{\text{OT}} \quad \text{(OT-Lax)}. \tag{A23}$$

The following two theorems give conditions under which the Lax shock conditions hold on solutions of Eqs. (A15).

*Theorem 8.* The pressure $p\geqslant 0$ if and only if $p\geqslant\bar{p}$ if and only if

$$\theta>\theta_1(z,\bar{\mu})\equiv\frac{1}{\gamma}=\frac{\bar{\rho}+\bar{p}}{\rho+\bar{p}}=\frac{1+\bar{\mu}}{1+\bar{\mu}z}z, \tag{A24}$$

$0<\mu<1$ if and only if

$$\theta>\theta_2(z,\bar{\mu})\equiv\frac{2z(1+\bar{\mu})}{(1+z)(1+\bar{\mu}z)}, \tag{A25}$$

and the shock speed $s$ is less than the speed of light if and only if

$$\theta>\theta_-(z,\bar{\mu})\equiv\frac{2\gamma-1}{\gamma^2}=1-\left(\frac{1-z}{1+\bar{\mu}z}\right)^2. \tag{A26}$$

The OT-Lax condition holds if and only if

$$\theta<\theta_+(z,\bar{\mu})\equiv 1-\left(\frac{1-z}{1+\bar{\mu}z}\right)^2\bar{\sigma}. \tag{A27}$$

The final theorem gives an existence theorem for solutions of Eqs. (A15).

*Theorem 9.* Assume that a positive OT solution $\bar{\rho}(\bar{r})>0$, $\bar{p}(\bar{r})>0$, and $M(\bar{r})>0$ of the OV equations (2.1) and (2.2) are defined and smooth for all $\bar{r}$ in the interval

$$\bar{r}_a<\bar{r}<\bar{r}_b\leqslant\infty.$$

Assume also that the following additional conditions hold throughout the interval $(\bar{r}_a,\bar{r}_b)$:

$$0<\bar{\mu}=\frac{\bar{p}(\bar{r})}{\bar{\rho}(\bar{r})}<1, \tag{A28}$$

$$0<\bar{\sigma}=\frac{\bar{p}'(\bar{r})}{\bar{\rho}'(\bar{r})}<1, \tag{A29}$$

and

$$\rho=\frac{3}{4\pi}\frac{M(\bar{r})}{\bar{r}^3}>\bar{\rho}. \tag{A30}$$

Then the solution $(r(t),R(t))$ of the shock equations (A15) starting from initial data $(r_0,R_0)$ satisfying

$$\bar{r}_a<\bar{r}_0=r_0R_0<\bar{r}_b, \tag{A31}$$

will exist, and satisfy,

$$0<s<1, \tag{A32}$$

$$1<\mu=\frac{p}{\rho}<1, \tag{A33}$$

and the OT-Lax condition (A23) throughout the maximal subinterval of $(\bar{r}_a,\bar{r}_b)$ containing $\bar{r}_0$ on which

$$\theta_-<\theta<\theta_+, \tag{A34}$$

cf. Eqs. (A26) and (A27).

## APPENDIX B: REMARKS ON THE FRW-OT MATCHING PROBLEM

In this section we clarify the role of the scaling laws $r\rightarrow\alpha r$ in the FRW metric (A4), and use this to determine the "size" of the FRW universe that matches a given OT metric at a radius $\bar{r}$. To this end, consider a change of scale $\tilde{r}=\alpha r$; in the new radial coordinate $\tilde{r}$, Eq. (A4) reads

$$ds^2=-dt^2+\left(\frac{R(t)}{\alpha}\right)^2\left\{\frac{d\tilde{r}^2}{1-\frac{k}{\alpha^2}\tilde{r}^2}+\tilde{r}^2d\Omega^2\right\}, \tag{B1}$$

so that

$$\tilde{R}=\frac{R}{\alpha}, \tag{B2}$$

$$\tilde{k}=\frac{k}{\alpha^2}. \tag{B3}$$

That is, if $P$ is a point in the FRW spacetime manifold labeled by $(t,r,\theta,\phi)$, then the label for $P$ in the tilde coordinates is $(t,\tilde{r},\theta,\phi)$, where Eqs. (B2) and (B3) hold. Thus, $kr^2=\tilde{k}\tilde{r}^2$ and $Rr=\tilde{R}\tilde{r}$ are invariant under the scaling $r\rightarrow\alpha r$.

Now fix $k$, and assume Eq. (A4) matches an OT metric (2.4) at $\bar{r}=\bar{r}_0$. Since our shock matching allows for two initial conditions $R_0$ and $r_0$, and one of these is fixed by the identity $\bar{r}_0=r_0R_0$, we conclude that for each $k$ there is a one-parameter family of distinct FRW metrics (B1) that match Eq. (2.4) at $\bar{r}=\bar{r}_0$, these being parametrized by $r_0$. Thus, since

$$\theta=\frac{A(\bar{r})}{1-kr^2},$$

we can alternatively view the one-parameter family of distict FRW metrics that match a given OT metric at $\bar{r}=\bar{r}_0$, as being parametrized by $\theta$, $0<\theta\leqslant 1$. Since $kr^2$ is an invariant, it follows that $\theta$ is also invariant under the scaling $r\rightarrow\alpha r$.

Thus, $\theta$ is a dimensionless parameter that uniquely picks out a spacetime constructed by matching a FRW metric to a given OT metric at $\bar{r}=\bar{r}_0$. Thus, choosing a value for $k$ is equivalent to setting a scale for the radial coordinate $r$ in this unique FRW spacetime.

Now for fixed $\theta$ and any $k$, the area of the two-sphere determined by the angular coordinates at $\bar{r}=\bar{r}_0$, can be calculated to be $\tau\bar{r}_0^2$ (where $\tau$ is the area of the Euclidean two-sphere, and this can be calculated from either the inner FRW metric or the outer OT metric). That is, the FRW universe that matches a given OT metric at $\bar{r}_0$ "fits inside a ball of area $\tau\bar{r}_0^2$."

Alternatively, one can calculate the invariant radial distance, $\Delta s$, from the center $r=0$ of the FRW metric to the shock wave positioned at $\bar{r}=\bar{r}_0$ at fixed $t=t_0$; namely,

$$\Delta s=R(t_0)\int_0^{r_0}\frac{dr}{\sqrt{1-kr^2}}. \tag{B4}$$

Note that this expression is invariant under the rescaling $r\to\alpha r$. To evaluate this integral, set $r=\sin(u)$ when $k>0$, and $r=\sqrt{k}\sinh(|u|)$ when $k<0$. In the first case

$$\Delta s=R(t_0)\frac{u_0}{\sqrt{k}}=\frac{\bar{r}_0u_0}{\sqrt{kr_0^2}}=\frac{\arccos(\sqrt{1-kr_0^2})}{\sqrt{kr_0^2}}\bar{r}_0,\quad k>0. \tag{B5}$$

In the second case, $k<0$, we obtain

$$\Delta s=R(t_0)\frac{u_0}{\sqrt{|k|}}=\frac{\bar{r}_0u_0}{\sqrt{|k|r_0^2}}=\frac{\operatorname{arccosh}(\sqrt{1+|k|r_0^2})}{\sqrt{|k|r_0^2}}\bar{r}_0,\quad k<0, \tag{B6}$$

and in the case $k=0$,

$$\Delta s=\bar{r}_0. \tag{B7}$$

These expressions give the invariant distance from the center of the FRW space to the shock wave in terms of invariant quantities. We now let $w^2=|k|r^2$, and show that the function

$$\phi(w)\equiv\frac{\arccos(\sqrt{1-w^2})}{\sqrt{w^2}} \tag{B8}$$

is monotone increasing from 1 to $\pi/2$ on the interval $0<w<1$, and the function

$$\psi\equiv\frac{\operatorname{arccosh}(\sqrt{1+|k|r_0^2})}{\sqrt{|k|r_0^2}} \tag{B9}$$

is monotone decreasing from 1 to 0 on the interval $0<w<\infty$. In the case $k>0$, Eq. (B8) implies that

$$1\leq\frac{\Delta s}{\bar{r}_0}\leq\frac{\pi}{2},\quad k>0, \tag{B10}$$

and in the case $k<0$, Eq. (B9) implies

$$0\leq\frac{\Delta s}{\bar{r}_0}\leq 1,\quad k<0. \tag{B11}$$

Statements (B6)–(B11) put precise limits on the size of the FRW manifold that can be matched to an OT manifold at radius $\bar{r}=\bar{r}_0$.

To show that $\phi(w)$ increases from 1 to $\pi/2$ on $0<w<1$, we first note that

$$w^2\phi'=\frac{w}{\sqrt{1-w^2}}-\arccos\sqrt{1-w^2}. \tag{B12}$$

Letting

$$v=\frac{w}{\sqrt{1-w^2}},$$

implies $0<v<\infty$, and

$$w=\frac{v}{\sqrt{1+v^2}}.$$

Then Eq. (B12) implies that if $\phi'=0$, we must have

$$v=\arccos(\sqrt{1/(1+v^2)}), \tag{B13}$$

but as $\arccos(\sqrt{1/(1+v^2)})$ takes values in $(0,\pi/2)$ because $r_0<1/\sqrt{k}$, we see that Eq. (B13) cannot hold for $v\geq\pi/2$. Now if $0<v<\pi/2$, then Eq. (B13) implies $\tan(v)=v$, on this range, an impossibility. Thus $\phi'(w)>0$ for $0<w<1$. But it is easily verified that $\phi(1)=\pi/2$ and $\phi(0)=1$, so $\phi$ increases from 1 to $\pi/2$ for $0\leq w\leq 1$. An analogous argument shows that in the case $k<0$, the function $\psi$ decreases from 1 to 0 for $0\leq w<\infty$.


[1] W. A. Buchdahl, Phys. Rev. **116**, 1027 (1956).

[2] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).

[3] J. Smoller and B. Temple, Commun. Math. Phys. (to be published).

[4] J. Smoller and B. Temple, Arch. Ration. Mech. Anal. **128**, 249 (1994).

[5] J. Smoller and B. Temple, Arch. Ration. Mech. Anal. (to be published).

[6] P. D. Lax, Commun. Pure Appl. Math. **10**, 537 (1957).

[7] J. Smoller, *Shock-Waves and Reaction-Diffusion Equations* (Springer-Verlag, Berlin, 1983).

[8] J. Smoller and B. Temple, Arch. Ration. Mech. Anal. (to be published).

[9] J. Smoller and B. Temple, Phys. Rev. D **51**, 2733 (1995).

[10] R. Schoen and S. T. Yau, Commun. Math. Phys. **90**, 575 (1983).