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SHOCK-WAVE SOLUTIONS IN CLOSED FORM AND THE OPPENHEIMER–SNYDER LIMIT IN GENERAL REALITY*

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Abstract. In earlier work the authors derived a set of ODEs that describe a class of spherically symmetric, fluid dynamical shock-wave solutions of the Einstein equations. These solutions model explosions in a general relativistic setting. The theory is based on matching Friedmann–Robertson–Walker (FRW) metrics (models for the expanding universe) to Oppenheimer–Tolman (OT) metrics, (models for the interior of a star) Lipschitz-continuously across a surface that represents a time-irreversible, outgoing shock-wave. In the limit when the outer OT solution reduces to the empty space Schwarzschild metric and the inner FRW metric is restricted to the case of bounded expansion ($k > 0$), our equations reproduce the well-known solution of Oppenheimer and Snyder in which the pressure $p \equiv 0$. In this article we derive closed form expressions for solutions of our ODEs in all cases ($k > 0, k < 0, k = 0$) when the outer OT solution is Schwarzschild, as well as in the case of an arbitrary OT solution when the inner FRW metric is restricted to the case of critical expansion ($k = 0$). This produces a large class of shock-wave solutions given by explicit formulas. Among other things, these formulas can be useful in testing numerical shock-wave codes in general relativity.

Key words. general relativity, differential geometry, hyperbolic conservation laws/shock-waves

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1. Introduction.

In this paper we present a new class of exact shock-wave solutions of the Einstein equations of general relativity, and we derive them from a general theory in which the well-known Oppenheimer and Snyder (O-S) solution [3] comes out as a special case. The O-S solution appears in the limit of zero pressure, and in this paper we also characterize the zero pressure limit by constructing two additional solutions of O-S type, these two new solutions being qualitatively different from the one first given by Oppenheimer and Snyder [10]. We are interested in presenting these solutions to the applied mathematics community in the hope that these fluid dynamical shock-wave solutions given in closed form might find application in the numerical analysis of fluids. Since we understand that the subject of general relativity is not so familiar to researchers in applied mathematics, we begin by giving a brief non-technical introduction to the subject of general relativity, and to our previous work on shock-matching methods.

In Einstein's theory of general relativity, the gravitational field is described by a Lorentzian metric tensor defined on the four-dimensional manifold of spacetime. In this theory, free-falling objects follow geodesics of the metric, and the geodesics reduce to straight lines in *inertial* coordinate systems in which the metric tensor is diagonal, with $(-1, 1, 1, 1)$ along the diagonal. In Einstein's theory, Newton's

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gravitational "force" (say, at the surface of the Earth) is explained as the acceleration one experiences because coordinate systems fixed relative to the Earth's surface are not inertial. Indeed, when curvature is present, free-falling objects will appear to accelerate in any spacetime coordinate system, because in this case the fundamental theorem of Riemannian geometry tells us that there does not exist a coordinate system in which the components of the metric have the canonical diagonal form throughout an open neighborhood of a point at which the curvature is nonzero. It is in this sense that gravitational forces are identified with spacetime curvature in Einstein's theory of gravity. In 1915, Albert Einstein began the subject of general relativity by introducing the system of partial differential equations that describes the simultaneous evolution of the gravitational metric field together with the source of the gravitational field—the source of the gravitational field at each point in spacetime being the mass-energy and momentum densities, and their fluxes, at that point. These fundamental equations go by the name of the Einstein gravitational field equations, or just the Einstein equations.

Now the Einstein equations are not an esoteric set of equations far removed from the study of classical fluids. To the contrary, when the source of the gravitational field is assumed to be produced by a perfect fluid, we can turn this viewpoint around and understand the Einstein equations as really being just the compressible Euler equations coupled to "source terms," where the source terms are the components of the metric tensor, a source of accelerations for the Euler flow. But in this picture, the coupling occurs in a very interesting and unique way. Indeed, the Einstein equations are the simplest set of equations in which the coupling of the gravitational field to the fluid turns the Euler equations into *geometrical identities* that follow from first-order differential identities in the curvature of the underlying gravitational metric. In geometry, these differential identities go by the name of the Bianchi identities. Thus, the fluid variables appear *undifferentiated* in the Einstein equations. In this paper we present a class of exact, spherically symmetric, shock-wave solutions of the Einstein gravitational field equations for a perfect fluid, and these reduce to classical solutions of the compressible Euler equations in the limit of weak fields and low velocities.

The Einstein equations for a perfect fluid are given in component form by

$$G_{ij} = \kappa T_{i_j}, \tag{1.1}$$

$$T_{ij} = (\rho c^2 + p)u_i u_j + p g_{ij}. \tag{1.2}$$

Here G_{ij}, $i, j = 0, ..., 3$, denote the components of the Einstein curvature tensor for the gravitational metric g (G is a $(0,2)$-tensor constructed from the general Riemann curvature tensor of Riemannian geometry [10]), $\kappa = \frac{8\pi\mathcal{G}}{c^4}$, where $\mathcal{G}$ is Newton's constant, and c is the speed of light. The perfect fluid is described by the variables ρc^2, the mass-energy density p, the pressure, and u, the 4-velocity of the fluid; cf. [10]. The Bianchi identities of geometry imply that

$$\text{div} G = 0, \tag{1.3}$$

and so on solutions of (1.1) we have

$$\text{div} T = 0, \tag{1.4}$$

where we take the covariant divergence for the metric g. (The covariant divergence is just the differential operator that reduces to the classical divergence at a point in a coordinate system that canonically diagonalizes the metric to second order at that point.) In the limit of low velocities and weak gravitational fields, (1.4) reduces to the classical compressible Euler equations of gas dynamics. The connection between our exact solutions and classical fluids is demonstrated by the fact that in the limit of weak gravitational fields and small densities, our shock waves reduce to solutions of the classical Euler equations with classical gravity. It remains for us to study the correspondence with the classical limit in detail.

The fact that the Euler equations reduce to identities that follow from geometrical constraints has led us to the belief that there could be simplifying features that Einstein-like equations have over the usual Euler equations. That is, the fluid variables appear *undifferentiated* in the Einstein equations, and we show in [6] that conservation across shocks becomes automatic in coordinate systems in which the metric is sufficiently smooth (we show that $C^{1,1}$ implies conservation). Thus conservation translates into smoothness as you go from Euler to Einstein. This is quite remarkable, and we continue to believe that there could be numerical advantages to be exploited in this point of view. We hope that this paper helps bring these issues to the attention of applied mathematicians.

Another mathematical aspect of our shock-wave solutions that makes them interesting for applied mathematics is that the solution technique utilizes the full coordinate independent geometrical framework of general relativity. Indeed, the greatness of the Einstein equations lies in the fact that they are quite simple when expressed in terms of geometrical quantities defined by coordinate independent tensorial quantities, (1.1). But in any fixed coordinate system, the Einstein equations can look horrendously complicated ((1.1) gives 10 equations in 14 unknowns, and 4 more equations are needed to fix the coordinate system). Our exact shock-wave solutions take advantage of the ability to use a different coordinate system on each side of the shock. Since these shocks reduce to classical shocks in the limit of weak fields and low velocities, these solutions present a new technique for describing classical shocks—a technique that brings the *geometry* of general relativity to bear upon the problem of describing shock dynamics.

The shock-wave solutions described in this paper have a straightforward physical interpretation. Indeed, the solutions are constructed by matching together two different metrics that solve the Einstein equations across a shock interface, and these two metrics are used routinely in astrophysical models. Behind the shock is the Friedmann–Robertson–Walker (FRW) metric (the solution of the Einstein equations used to model the expanding universe), and ahead of the shock is the Oppenheimer–Tolman (OT) metric (a static, spherically symmetric model for the interior of a star). In [6], we describe in theory how to match any two such solutions across an interface, and in [7] we obtain a solution in closed form by fixing the outer solution as the general relativistic version of a classical *static, singular, isothermal sphere.* The static isothermal sphere represents the pre-ignition state of a gas in classical scenarios of star formation (see [7]), and this puts our exact solution into a physical context in which explosions are to be expected. Our model can describe blast waves on many scales. For the supernova problem, we take the outer solution to be the outer regions of a star, in which case the inner solution represents the exploding inner core, and the shock-wave moves outward through the star. For cosmology, our solutions give an alternative cosmological model in which the universe begins with an explosion instead of the popular *big bang* beginning that is described by the FRW metric alone.

2. Background. In [6] the authors set out a theory of matching gravitational metrics Lipschitz-continuously across interfaces that represent shock-waves in the theory of general relativity. This was applied to the problem of matching FRW metrics to OT metrics across a shock interface. The paper [6] concluded with the derivation of an autonomous system of two ODEs that describe the evolution of the shock position as well as the cosmological scale factor and pressure p for the FRW metrics that match a given OT solution across a shock interface, such that conservation of energy and momentum holds across the shock. In the FRW–OT matching problem, the smooth matching of the spheres of symmetry implies a reduction of the conservation constraint to a single constraint on the FRW pressure, and it follows that the FRW metric and shock position can be determined for any arbitrarily preassigned OT solution. In [7] we found an exact solution of the shock equations that describes an explosion into the general relativistic version of a static, singular, isothermal sphere. In this model the geometry of the FRW and OT metrics constrains the inner FRW equation of state in such a way that the pressure, sound speed, and temperature are all higher where the density is higher, which suggests that this is a highly physical situation. In [8] we derived a simplified system of ODEs and set out a general theory for solutions of the ODEs, including the identification of natural dimensionless parameters in terms of which the shock speed and Lax characteristic conditions [2, 4] can be expressed. For simplicity we restrict ourselves throughout to the simplest case of a perfect fluid satisfying an equation of state of the form $p = p(\rho)$, where p denotes the pressure of the fluid and ρ the mass-energy density of the fluid.

The FRW metric is isotropic, homogeneous, and maximally symmetric at each fixed time because the spatial metric has constant scalar curvature along the constant time surfaces [10]. Thus there are three cases, the cases of positive, negative, and zero scalar curvature in the constant time surfaces. The positive curvature case ($k < 0$) is the case of bounded expansion, negative curvature ($k > 0$), the case of uniform expansion, and zero curvature ($k = 0$), the case of critical uniform expansion.

In this paper we use the shock ODEs established in [8] to derive closed form expressions for solutions in the case when the OT solution reduces to the empty space Schwarzschild metric (Oppenheimer and Snyder did the case $k > 0$, but here we do all three cases $k < 0$, $k = 0$, and $k > 0$), as well as in the case of an arbitrary OT solution under the assumption of critical expansion ($k = 0$) in the FRW metric. This completes and unifies the theory first begun by Oppenheimer and Snyder in [3], and also introduces a large class of multidimensional, time-irreversible, fluid dynamical shock-wave solutions of the Einstein equations that are given in terms of explicit formulas. As an application, we use these formulas to rederive the exact solution introduced in [7], which models an explosion into a static, isothermal sphere. We also derive a general class of examples having an inverse square density profile (as does the static, singular, isothermal sphere), and we show again how the example in [6] comes out as a special case.

Our equations provide a unified approach in that they display the O-S solutions as limits of shock-wave solutions. We furthermore anticipate that the formulas derived here will be useful for testing numerical shock-wave codes in general relativity. We refer to [5, 6, 7, 8, 10] for background.

3. Preliminaries. In coordinates, the FRW metric takes the special form

$$ds^2 = -dt^2 + R^2(t)\left\{\frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2\right\}, \tag{3.1}$$

where $R(t)$ denotes the cosmological scale factor and $\mathrm{sign}(k)$ is minus the sign of the scalar 3-curvature of the $t = \text{const.}$ surfaces [10]. Here, $d\Omega$ denotes the area element on the unit 2-sphere. By rescaling the radial coordinate r one can rescale k to one of the normalized values $1, 0, -1$, the cases of bounded expansion, critical expansion, and uniform expansion; cf. [10]. Assuming a perfect fluid co-moving with the coordinates and assuming a barotropic equation of state $p = p(\rho)$, the Einstein equations are equivalent to the following system of two ODEs in the density ρ and scale factor $R(t)$ (cf. [8, 10]):

$$p = -\rho - \frac{R\dot{\rho}}{3\dot{R}}, \tag{3.2}$$

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2. \tag{3.3}$$

Here G denotes Newton's gravitational constant, and we let "dot "denote differentiation with respect to FRW time t.

The OT metric takes the special form

$$d\bar{s}^2 = -B(\bar{r})d\bar{t}^2 + A(\bar{r})^{-1}d\bar{r}^2 + \bar{r}^2 d\Omega^2, \tag{3.4}$$

where

$$A(\bar{r}) = 1 - \frac{2GM(\bar{r})}{\bar{r}},$$

and $M(\bar{r})$ denotes the total mass inside the radius $\bar{r}$. In this metric, the density $\bar{\rho}$ is assumed to be a function of $\bar{r}$ alone, so that (3.4) represents a static, spherical distribution of fluid with variable radial density and pressure. Imposing the Einstein equations on a metric of type (3.4) implies the following two ODEs in the two unknowns $M, \bar{\rho}$ once an equation of state $\bar{p} = \bar{p}(\bar{\rho})$ is given by (cf. [8, 10]):

$$\frac{dM}{d\bar{r}} = 4\pi\bar{r}^2\bar{\rho}, \tag{3.5}$$

$$-\bar{r}^2\frac{d\bar{p}}{d\bar{r}} = GM\bar{\rho}\left\{1 + \frac{\bar{p}}{\bar{\rho}}\right\}\left\{1 + \frac{4\pi\bar{r}^3\bar{p}}{M}\right\}\left\{1 - \frac{2GM}{\bar{r}}\right\}^{-1}. \tag{3.6}$$

The metric component $B \equiv B(\bar{r})$ is then determined on solutions of (3.5) and (3.6) from the equation

$$\frac{B'(\bar{r})}{B} = -2\frac{\bar{p}'(\bar{r})}{\bar{p} + \bar{\rho}}. \tag{3.7}$$

We refer to system (3.5), (3.6) as the Oppenheimer–Volkoff (OV) system.

Throughout this paper we use the barred vs. unbarred coordinates for the OT and FRW metrics, respectively, to indicate two different coordinate systems when the metrics are matched across a shock interface. In [6] we showed that for arbitrary FRW and OT metrics, there exists a coordinate mapping

$$(t, r) \to (\bar{t}, \bar{r}),$$

such that, under this coordinate identification, the FRW metric (3.1) matches the OT metric (3.4) Lipschitz-continuously across an interface $\bar{r} = \bar{r}(t)$ that is defined implicitly by the equation (see [6])

$$M(\bar{r}) = \frac{4\pi}{3}\rho(t)\bar{r}^3. \tag{3.8}$$

The coordinate mapping for $\bar{r}$ is given by

$$\bar{r} = R(t)r,$$

which implies that the areas of the spheres of symmetry change smoothly across the interface. The $\bar{t} = \bar{t}(t,r)$ mapping is proven to exist, but no explicit use of this is required for our analysis.

One further constraint on the FRW metric is required in order to rule out the possibility of the surface being a surface layer containing delta function sources of energy and momentum. In [6] (see also [1]), we derived a system of two ODEs that determine the shock position and the FRW metric that will match a given OT metric across the shock surface (3.8) such that conservation of energy and momentum hold across the shock, and such that the shock surface is free of delta function sources of energy and momentum. In [5] we showed that for such surfaces there exist a $C^{1,1}$ (differentiable with Lipschitz-continuous first derivatives) coordinate transformation, defined in a neighborhood of the shock surface, such that in the new coordinates the metric is $C^{1,1}$, and thus has one more order of smoothness than in the original coordinates. In [8] we used an analysis that rules out an unphysical pressure jump across the shock to derive the following simplified set of equations for the shock position $r(t)$ and scale factor $R(t)$ that determine the FRW metrics that match a given OT metric Lipschitz-continuously across the shock surface (3.8), such that conservation holds across the shock:

$$r\dot{R} = \pm\sqrt{1-kr^2}\sqrt{1-\theta}, \tag{3.9}$$

$$R\dot{r} = \pm\sqrt{1-kr^2}\frac{\sqrt{1-\theta}}{\gamma\theta-1}. \tag{3.10}$$

Here, θ and γ are dimensionless parameters defined by

$$\theta = \frac{A}{1-kr^2} \tag{3.11}$$

and

$$\gamma = \frac{\rho+\bar{p}}{\bar{\rho}+\bar{p}}. \tag{3.12}$$

The speed of the shock relative to the inner FRW fluid is easily shown to be

$$s \equiv \frac{d\tilde{r}}{dt} = \frac{1-kr^2}{R}\dot{r},$$

and so the equation (3.10) immediately gives the shock speed as (cf. [8])

$$s^2 = \frac{1-\theta}{(\gamma\theta-1)^2}. \tag{3.13}$$

The upper plus and lower minus signs in (3.9), (3.10) correspond to the outgoing shock and its time-reversal, an incoming shock, respectively. In general, solutions of (3.9) and (3.10) will only be Lax admissible and moving at less than the speed of light when the parameter θ lies within certain bounds that were identified in [8]. It is shown in [8] that, in general, only the outgoing shock will be a Lax admissible shock, and its time-reversal will then be an unstable Lax rarefaction shock [5, 4, 2]. The outgoing shock will generally satisfy $\rho > \bar{\rho}$ and $\gamma\theta > 1$; cf. equations (3.9), (3.10), and [8].

Solutions of (3.9) and (3.10) determine a FRW metric having pressure given by (3.2) and density given by (3.8), and since all quantities in the given OT metric are functions of $\bar{r} = rR$, the ODEs (3.9), (3.10) are autonomous in the unknowns $(r(t), R(t))$. After ruling out an unphysical pressure (see [8]), it follows that the FRW pressure is given by the simple formula

$$p = \frac{1-\gamma\theta z}{\gamma\theta - 1}\rho, \tag{3.14}$$

where

$$z \equiv \frac{\bar{\rho}}{\rho}.$$

In [8, Theorem 5], it is shown that the FRW pressure p satisfies $p > \bar{p}$ (where $\bar{p}$ is the OT pressure in front of the shock) if and only if $\gamma\theta > 1$, and this inequality holds for a large class of solutions.

4. The zero pressure limit. In this section we analyze the limit $\bar{\rho} \to 0$, $\bar{p} \to 0$, the limit in which the outer OT solution tends to the empty space Schwarzschild metric

$$ds^2 = -A(\bar{r})d\bar{t}^2 + A(\bar{r})^{-1}d\bar{r}^2 + \bar{r}^2 d\Omega^2, \tag{4.1}$$

where $A = 1 - \frac{2GM}{\bar{r}}$ and $M \equiv$ const. In this limit the shock surface reduces to a time-reversible contact discontinuity that can be interpreted as modeling the boundary surface of a spherical star of constant spatial density. (We still refer to such a surface as a "shock"surface.) Indeed, in this limit,

$$\gamma = \frac{\rho + \bar{p}}{\bar{\rho} + \bar{p}} \to \infty,$$

and thus equation (3.10) implies $\dot{r} = 0$, and hence

$$r = a \equiv \text{const.}^{1} \tag{4.2}$$

Equation (3.14) then implies that the FRW pressure is given by $p \equiv 0$, the well-known simplifying assumption made in Oppenheimer and Snyder's original paper; and from the shock surface equation we immediately obtain

$$\rho(t) = \frac{3M}{4\pi a^3}\frac{1}{R(t)^3}. \tag{4.3}$$

[1] Although we have only derived (4.2) at a formal level, equation (4.2) follows directly from the "constraint equation"(Eq. (2.29) of [8]), if we were to redo the derivation of (3.9), (3.10) as in [8] for the case $p = \bar{\rho} = \bar{p} = 0$.

In particular, this implies

$$\rho(t)R(t)^3 = \frac{3M}{4\pi a^3} = \rho_0 R_0^3, \tag{4.4}$$

because $\frac{3M}{4\pi a^3} \equiv \text{const}$. It remains then only to find a formula for $R(t)$.

To this end, note that for constant θ and ϕ, it is easy to check that $r = \text{const.}$ defines a radial geodesic of the inner FRW metric. This implies that $r = a$ also defines a radial geodesic of the outer OT metric. Indeed, geodesics satisfy the geodesic equation

$$\ddot{x}^i = \Gamma^i_{jk} x^j x^k,$$

where Γ^i_{jk} denote the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} g_{i\sigma}\{-g_{jk,\sigma} + g_{\sigma j,k} + g_{k\sigma,j}\}.$$

Here, x^i, $i = 0, ..., 3$, denotes a coordinate system, g_{ij} denote the components of the metric in this coordinate system, and we assume summation from 0 to 3 on repeated up-down indices. Now, by the general results in [6], the Gaussian normal coordinates at the shock surface are related to the original FRW or OT coordinates by a $C^{1,1}$ coordinate transformation, and in the Gaussian normal coordinates the metric is $C^{1,1}$ because the ODEs (3.9), (3.10) preclude the presence of delta function sources on the shock surface. Thus, in the Gaussian normal coordinates, the Christoffel symbols, which involve at most first derivatives of the metric components g_{ij}, are continuous functions of the coordinate variables. From this we conclude that in the Gaussian normal coordinates, the shock surface must satisfy the geodesic equation on the OT side of the shock if it is satisfied on the FRW side. By covariance and the fact that geodesics are critical points of the length functional, we conclude that the shock surface must also be a geodesic of the outer Schwarzschild metric as well, when we take the limit $\bar{\rho}, \bar{p} \to 0$.

Using the fact that $r = a$ on the shock, equation (3.9) can be written as

$$\left(a\dot{R}\right)^2 = (1 - ka^2)\left(1 - \frac{A}{1 - ka^2}\right) = \frac{2GM}{aR} - ka^2, \tag{4.5}$$

which simplifies to

$$\dot{R}^2 = \frac{2GM}{a^3}\frac{1}{R} - k. \tag{4.6}$$

We first analyze the case $k \neq 0$.

When $k \neq 0$ and we make the change of variables

$$r = \alpha\tilde{r},$$

the FRW metric becomes

$$ds^2 = -dt^2 + R(t)^2\alpha^2\left\{\frac{d\tilde{r}^2}{1 - k\tilde{r}^2} + \tilde{r}^2 d\Omega\right\}.$$

Thus, in the new $t\tilde{r}$-coordinates,

$$\tilde{R} = \alpha R, \quad \tilde{k} = \alpha^2 k.$$

We conclude that changing the value of k is equivalent to rescaling the FRW coordinate system. Indeed, we can choose k so that $R(t) = 1$ at an arbitrary time t, and since $\bar{r} = Rr$, fixing the value of k so that $R(t) = 1$ is equivalent to choosing a FRW radial coordinate so that $r = \bar{r}$ at the special time t. Therefore, when $k \neq 0$, choose

$$|k| = \frac{2GM}{a^3}, \tag{4.7}$$

so that (4.6) reduces to

$$\dot{R}^2 = |k| \left(\frac{1}{R} - 1 \right). \tag{4.8}$$

We now analyze (4.8) in the cases $k > 0$ and $k < 0$ separately.

Consider first the case $k > 0$, the case first considered by Oppenheimer and Snyder [3]. When $k > 0$, equation (4.8) reads

$$\dot{R}^2 = k \left(\frac{1}{R} - 1 \right). \tag{4.9}$$

It is easy to see from (4.9) that the outgoing solution in (4.9) increases to a maximum radius $R = 1$, at which point $\dot{R} = 0$, and the solution cannot be continued to values of R larger than 1. Thus, when $k > 0$, setting $k = \frac{2GM}{a^3}$ in (4.7) has the effect of making $\bar{r} = r$ at the maximum radius. Now initialize $t = 0$ at $R = 1$ and consider the incoming solution starting at $R(0) = 1$, $\dot{R}(0) = 0$. To the observer in the outer Schwarzschild coordinates, this corresponds to a sphere which starts at rest with surface coordinate $\bar{r} = r = a$, and then free-falls (geodesic motion) inward as the surface coordinate $\bar{r} = aR(t)$ decreases, as $R(t)$ decreases according to (4.9). The surface then hits $\bar{r} = 0$ when $R(t) = 0$, which corresponds to the *big bang* singularity in the FRW metric inside the shock. Following Weinberg [10], the solution of equation (4.9) is a cycloid which has the closed form solution given parametrically by

$$R = \frac{1}{2}(1 + \cos\psi), \tag{4.10}$$

$$t = \frac{\psi + \sin\psi}{2\sqrt{k}}. \tag{4.11}$$

Here, the solution expands from $\bar{r} = 0$, $R = 0$, out to the maximum radius $\bar{r} = r = a$, $R = 1$, as ψ goes from $-\pi$ to 0; and then the solution reverses itself and collapses back in to $\bar{r} = 0$ as ψ goes from 0 to π. Thus, for the collapse, R hits zero when $\psi = \pi$, which implies $R(t_*) = 0$ at

$$t_* = \frac{\pi}{2\sqrt{k}}. \tag{4.12}$$

Since t is proper time along the shock (which in this case is really a contact discontinuity), t_* gives the proper time it takes an observer at the surface to hit the singularity $\bar{r} = 0$.

Consider next the time it takes the shock surface to hit the Schwarzschild radius $\bar{r}_S = 2GM$. At this time,

$$\bar{r}_S = R_S a = 2GM,$$

$$R_S = \frac{2GM}{a} = \frac{1}{2}(1 + \cos\psi_S), \tag{4.13}$$

which implies

$$\cos\psi_S = \frac{2GM}{a} - 1.$$

The assumption

$$2GM < a = \bar{r}(0)$$

is required to insure that the initial surface position is outside the Schwarzschild radius of the outer Schwarzschild metric. For the incoming solution, R decreases from its initial value $R = 1$, and so

$$0 < \frac{2GM}{a} < 1,$$

$$0 < \frac{4GM}{a} - 1 < 1,$$

which insures that the solution hits the Schwarzschild radius at the unique value of $\psi_S \in (0, \frac{\pi}{2})$ where $\cos\psi = \frac{4GM}{a} - 1$, i.e.,

$$\psi_S = \cos^{-1}\left(\frac{4GM}{a} - 1\right),$$

$$t_S = \frac{\psi_S + \sin\psi}{2\sqrt{k}}.$$

Note now that the lightlike radial geodesics of the Schwarzschild metric satisfy

$$\frac{d\bar{t}}{d\bar{r}} = \frac{1}{A}. \tag{4.14}$$

Thus if light is emitted from the shock surface at $(\bar{t}_2, \bar{r}_2)$ and then again at $(\bar{t}_1, \bar{r}_1)$, where $\bar{t}_2 < \bar{t}_1$ and $2GM < \bar{r}_1 < \bar{r}_2$, then the coordinate time lag $\Delta\bar{t}$ observed between the pulses received at $\bar{r}_3$, $\bar{r}_3 > \bar{r}_2 > \bar{r}_1$, is given by

$$\begin{aligned}\Delta\bar{t} &= \bar{t}_1 + \int_{\bar{r}_1}^{\bar{r}_3} \frac{d\bar{r}}{A} - \bar{t}_2 - \int_{\bar{r}_2}^{\bar{r}_3} \frac{d\bar{r}}{A} \\ &= \bar{t}_1 - \bar{t}_2 + \int_{\bar{r}_1}^{\bar{r}_2} \frac{d\bar{r}}{A} > \int_{\bar{r}_1}^{\bar{r}_2} \frac{d\bar{r}}{A}.\end{aligned} \tag{4.15}$$

Therefore, we conclude that the time lag $\Delta\bar{t}$ (which is at least as small as the time lag observed at $\bar{r}_3$ between a lightlike signal emitted from $(\bar{t}_1, \bar{r}_1)$ and *any* signal emitted from $(\bar{t}_2, \bar{r}_2)$) tends to ∞ as the shock surface falls into the Schwarzschild radius $\bar{r} = 2GM$. Since $\bar{t}$ measures proper time for observers in the far field $\bar{r} \to \infty$, we have the conclusion first reached by Oppenheimer and Snyder that observers at $\bar{r} = \infty$ will measure an infinite time lapse $\Delta\bar{t}$ before the shock surface hits the Schwarzschild radius, while observers on the shock surface will hit $\bar{r} = 2GM$ as well as $\bar{r} = 0$ in their own finite proper time Δt.

Since the shock surface follows a radial geodesic of the Schwarzschild metric, it follows that the surface trajectory $\bar{r} = \bar{r}(t)$ is a timelike geodesic of the metric

$$ds^2 = -Ad\bar{t}^2 + A^{-1}d\bar{r}^2, \tag{4.16}$$

and the variable t of the FRW metric gives the proper time or arclength parametrization of this geodesic. Thus

$$-1 = -A\dot{\bar{t}}^2 + A^{-1}\dot{\bar{r}}^2$$

(where the minus sign accounts for the fact that the shock surface defines a radial *timelike* geodesic), so that dividing by $\dot{\bar{r}}^2$ gives

$$\frac{d\bar{t}}{d\bar{r}} = \frac{1}{A}\frac{1}{\sqrt{1-\theta}}$$

where we have used the following formula for $\dot{\bar{r}}$ obtained by adding (3.9) to (3.10) and taking the limit $\gamma \to \infty$:

$$\dot{\bar{r}}^2 = (1 - kr^2)(1 - \theta).$$

Moreover, using the variational formulation for geodesics,

$$0 = \delta \int ds = \delta \int -A\dot{\bar{t}}^2 + A^{-1}\dot{\bar{r}}^2,$$

the Euler–Lagrange equation for $\bar{t}$ gives

$$\frac{d}{dt}\left(2A\dot{\bar{t}}\right) = 0,$$

so

$$\dot{\bar{t}} = \frac{C}{A} \tag{4.17}$$

for some positive constant C. Now

$$\dot{\bar{t}} = \frac{d\bar{t}}{dA}\dot{A},$$

where

$$\dot{A} = \frac{2GM}{\bar{r}}\frac{\dot{\bar{r}}}{\bar{r}} = (1 - A)\frac{\dot{R}}{R}. \tag{4.18}$$

Now by (3.10) and (4.13),

$$\frac{\dot{R}}{R} = \frac{\sqrt{k}\sqrt{1-R}}{R^{3/2}},$$

and substituting this into (4.17) yields

$$\frac{d\bar{t}}{dA} = \frac{R^{3/2}}{\sqrt{k}\sqrt{1-R}} \frac{C}{(1-A)A}. \tag{4.19}$$

Since R tends to a finite value as the shock tends to the Schwarzschild radius, i.e., as $A \to 0$, (4.19) implies directly that the $\bar{t}$-coordinate time change tends to infinity as the shock surface position $\bar{r}$ tends to the Schwarzschild radius. (Note that we can integrate (4.18) to obtain

$$\frac{1-A_0}{1-A} = \frac{R_0}{R}, \tag{4.20}$$

which simply reproduces the identity $\bar{r} = rR$.)

Consider next the case $k < 0$. In this case (4.8) becomes

$$\dot{R}^2 = |k| \left(\frac{1}{R} + 1 \right). \tag{4.21}$$

Since (4.21) admits no maximum radius, we restrict ourselves to the outgoing solution that satisfies

$$\dot{R} = |k|^{1/2} \left(\frac{1}{R} + 1 \right)^{1/2} = |k|^{1/2} \frac{\sqrt{R+R^2}}{R}, \tag{4.22}$$

so that

$$\frac{RdR}{\sqrt{R+R^2}} = \sqrt{|k|}dt. \tag{4.23}$$

A calculation gives the explicit formula

$$\int \frac{RdR}{\sqrt{R+R^2}} = \sqrt{R+R^2} - \frac{1}{2} \ln \left(\frac{1}{2} + R + \sqrt{R+R^2} \right). \tag{4.24}$$

Setting the initial condition $R(0) = 0$ so that the *big bang* occurs at $t = 0$, leads to the explicit formula for the solution of (4.21),

$$\sqrt{R+R^2} - \frac{1}{2} \ln \left(1 + 2(R + \sqrt{R+R^2}) \right) = \sqrt{|k|}t. \tag{4.25}$$

We conclude from (4.25) that when $k < 0$, both R and the shock position $\bar{r} = aR$ increase at *linear* rates with respect to the FRW time t, i.e.,

$$\frac{dR}{dt} \to \sqrt{|k|},$$

$$\frac{d\bar{r}}{dt} \to \frac{\sqrt{|k|}}{a},$$

as $t \to \infty$.

Consider, finally, the case $k = 0$, under the assumption that the outer solution is Schwarzschild. In this case, the ODE (3.10) reduces to

$$\dot{R}^2 = \frac{2GM}{a^3} \frac{1}{R}. \tag{4.26}$$

Solutions of (4.26) never have a zero derivative for $R > 0$, so consider the solution for the outgoing shock interface (recall that the solution is time-reversible):

$$\dot{R} = \left(\frac{2GM}{a^3}\right)^{1/2} \frac{1}{R^{1/2}}. \tag{4.27}$$

The solution of (4.27) satisfying $R(0) = 0$ is

$$R(t) = \left(\frac{9\pi GM}{2}\right)^{1/3} t^{2/3}. \tag{4.28}$$

From (4.3) we obtain

$$\rho(t) = \frac{1}{6\pi G}\frac{1}{t^2}. \tag{4.29}$$

In [7] we obtained the following formulas for this same solution:

$$\rho(t) = \frac{1}{\left(\sqrt{6\pi G}(t - t_0) + \frac{1}{\sqrt{\rho_0}}\right)^2}, \tag{4.30}$$

$$R(t) = R_0 \left(\frac{\rho(t)}{\rho_0}\right)^{1/3}. \tag{4.31}$$

Choosing

$$t_0 = \frac{1}{\sqrt{6\pi G\rho_0}},$$

so that $R(0) = 0$, (4.30) reduces to (4.29). Equation (4.31) can then be transformed into (4.28) using the substitution

$$\rho_0 = \frac{3M}{4\pi a^3}\frac{1}{R_0^3},$$

which follows from (4.4). Note that in the $k = 0$ case the growth rate in R and $\bar{r} = aR$ as $t \to \infty$ is like the 2/3 power of t, while it is linear in t for the $k < 0$ case.

5. The general solution when $k = 0$ and $p \neq 0$. In this section we obtain closed form expressions for solutions of the shock equations (3.9), (3.10), in the general case when $k = 0$. Specifically, we obtain the time t at the shock, the scale factor R, and the shock position r in terms of integrals of known functions of $\bar{r}$. In [7] we previously derived closed form expressions for solutions of the FRW equations (3.2), (3.3) in the case $k = 0$, and these were based on integrals of known functions of the variable ρ. We used these formulas to construct the exact solution presented in [7]. So assume that $\bar{\rho}(\bar{r}), \bar{p}(\bar{r}), M(\bar{r})$ correspond to a fixed, known, outer OT solution. Then in the case $k = 0$, the equations (3.9), (3.10) for an outgoing shock reduce to

$$r\dot{R} = {}^{+}\sqrt{1 - A}, \tag{5.1}$$

$$R\dot{r} = {}^{+}\frac{\sqrt{1 - A}}{\gamma A - 1}, \tag{5.2}$$

where

$$A \equiv A(\bar{r}) = 1 - \frac{2GM(\bar{r})}{\bar{r}},$$

$$\gamma \equiv \gamma(\bar{r}) = \frac{\rho + \bar{p}(\bar{r})}{\bar{\rho}(\bar{r}) + \bar{p}(\bar{r})},$$

and ρ is a known function of $\bar{r}$ (which in this section we refer to as $\rho(\bar{r})$), determined by the shock surface equation (3.8) to be

$$\rho \equiv \rho(\bar{r}) = \frac{3}{4\pi G}\frac{M(\bar{r})}{\bar{r}^3}.$$

Now the right-hand sides of (5.1) and (5.2) depend only on known functions of $\bar{r}$. Thus, adding the equations (5.1) and (5.2) yields

$$\dot{\bar{r}} = \frac{\gamma A\sqrt{1-A}}{\gamma A - 1}. \tag{5.3}$$

Direct integration of (5.3) yields a formula for the FRW time at which the shock is at position $\bar{r}$:

$$t - t_0 = \int_{\bar{r}_0}^{\bar{r}} \frac{\gamma(\xi)A(\xi)\sqrt{1-A(\xi)}}{\gamma(\xi)A(\xi) - 1} d\xi. \tag{5.4}$$

To obtain a formula for R, note that by (5.3),

$$\dot{R} = \frac{dR}{d\bar{r}}\dot{\bar{r}} = \frac{dR}{d\bar{r}}\frac{\gamma A\sqrt{1-A}}{\gamma A - 1},$$

so that (5.1) implies

$$\frac{1}{R}\frac{dR}{d\bar{r}} = \frac{\gamma(\bar{r})A(\bar{r}) - 1}{\gamma(\bar{r})A(\bar{r})\bar{r}}. \tag{5.5}$$

Direct integration of (5.5) gives the value of R when the shock is at position $\bar{r}$:

$$R = R_0 \exp\left(\int_{\bar{r}_0}^{\bar{r}} \frac{\gamma(\xi)A(\xi) - 1}{\gamma(\xi)A(\xi)\xi} d\xi\right). \tag{5.6}$$

Finally, to obtain a formula for the shock position r, use (5.3) to write

$$\dot{r} = \frac{dr}{d\bar{r}}\dot{\bar{r}} = \frac{dr}{d\bar{r}}\frac{\gamma A\sqrt{1-A}}{\gamma A - 1},$$

which together with (5.1) yields

$$\frac{dr}{d\bar{r}} = \frac{1}{R\gamma A}. \tag{5.7}$$

This integrates to

$$r = r_0 + \int_{\bar{r}_0}^{\bar{r}} \frac{d\xi}{R(\xi)\gamma(\xi)A(\xi)}, \tag{5.8}$$

where, as above, ξ denotes a dummy variable for $\bar{r}$. The FRW pressure is then given by the following function of the shock position $\bar{r}$ (cf. (3.14)):

$$p = \frac{1-\gamma(\bar{r})A(\bar{r})z(\bar{r})}{\gamma(\bar{r})A(\bar{r})-1}\rho(\bar{r}), \tag{5.9}$$

where

$$z(\bar{r}) = \frac{\bar{\rho}(\bar{r})}{\rho(\bar{r})}. \tag{5.10}$$

We can use (5.4) to write R, ρ, and p as functions of t, and our general theory implies that the resulting FRW metric will solve the Einstein equations (3.2), (3.3).

As an application, we can use our formulas for $k = 0$ to reproduce the exact solution derived in [7]. For the exact solution there, $k = 0$, and the outer OT solution is taken to be the general relativistic version of a static, singular, isothermal sphere:

$$\bar{\rho}(\bar{r}) = \frac{\alpha}{\bar{r}^2}, \tag{5.11}$$

$$M(\bar{r}) = 4\pi\alpha\bar{r}, \tag{5.12}$$

$$\bar{p} = \bar{\sigma}\bar{\rho}, \tag{5.13}$$

$$B = \left(\frac{\bar{r}}{\bar{r}_0}\right)^{\frac{4\bar{\sigma}}{1+\bar{\sigma}}}. \tag{5.14}$$

Here $\bar{\sigma}$, the OT sound speed squared, takes a constant value in $(0, 1)$, and α is the constant given by

$$\alpha = \frac{1}{2\pi G}\left(\frac{\bar{\sigma}}{1+6\bar{\sigma}+\bar{\sigma}^2}\right). \tag{5.15}$$

It is straightforward to verify that (5.11)–(5.15) define an exact solution of the OT equations (3.5), (3.6), and (3.7). For this fixed, outer OT solution, the shock surface equation (3.8) gives

$$\rho = 3\bar{\rho}. \tag{5.16}$$

In this case the formula (5.9) for the FRW pressure is

$$p = \frac{1-\gamma Az}{\gamma A-1}\rho, \tag{5.17}$$

where in this case

$$z = \frac{\bar{\rho}}{\rho} = \frac{1}{3}, \tag{5.18}$$

$$A = 1 - 8\pi G\alpha = \frac{(1+\bar{\sigma})^2}{1+6\bar{\sigma}+\bar{\sigma}^2}, \tag{5.19}$$

and

$$\gamma = \frac{\rho + \bar{p}}{\bar{\rho} + \bar{p}} = \frac{3 + \bar{\sigma}}{1 + \bar{\sigma}}. \tag{5.20}$$

Substituting (5.18)–(5.20) into (5.17) yields

$$p = \sigma \rho, \tag{5.21}$$

where σ takes the constant value

$$\sigma = \frac{\bar{\sigma}(\bar{\sigma} + 7)}{3(1 - \bar{\sigma})}. \tag{5.22}$$

Equation (5.22) agrees with (5.2) of [7]. Using (5.18)–(5.20) in (5.6) yields

$$R = R_0 \left(\frac{\bar{r}}{\bar{r}_0} \right)^{\frac{2(1-\bar{\sigma})}{(3+\bar{\sigma})(1+\bar{\sigma})}}. \tag{5.23}$$

A calculation using (5.22) shows that this is equivalent to

$$R = R_0 \left(\frac{\bar{r}}{\bar{r}_0} \right)^{\frac{2}{3(1+\sigma)}}, \tag{5.24}$$

which is (5.10) of [7]. Substituting (5.18)–(5.20) into (5.8) yields

$$\dot{\bar{r}}(t) = \frac{(1 + \bar{\sigma})(3 + \bar{\sigma})}{(1 - \bar{\sigma})} \sqrt{\frac{\bar{\sigma}}{1 + 6\bar{\sigma} + \bar{\sigma}^2}}. \tag{5.25}$$

A calculation shows that this agrees with

$$\dot{\bar{r}}(t) = 3(1 + \sigma) \sqrt{\frac{\bar{\sigma}}{1 + 6\bar{\sigma} + \bar{\sigma}^2}}, \tag{5.26}$$

which is (5.12) of [7].

In this model, the sound speeds $\sqrt{\bar{\sigma}}$ and $\sqrt{\sigma}$ are constant, and a calculation shows $\sigma > \bar{\sigma}$. Moreover, equation (3.13) gives the shock speed as

$$s = \frac{\sqrt{1 + 6\bar{\sigma} + \bar{\sigma}^2}}{1 - \bar{\sigma}} \sqrt{\bar{\sigma}}, \tag{5.27}$$

which is also constant in this example. The equation of state for a (classical) ideal gas is

$$p = RT\rho,$$

where T is the temperature and R denotes the universal gas constant divided by the effective molecular weight of the particular gas. Thus, when the temperature is constant, the sound speed is proportional to the temperature alone, in which case the higher sound speed in the FRW metric represents an ideal gas at a higher temperature, which accords with the higher density and pressure behind the shock. Alternatively, the case $\sigma = \frac{1}{3}$ gives the equation of state in the extreme relativistic limit of free particles, as well as the equation of state for pure radiation; cf. [5, 7]. As a final comment, note that in this example the total mass inside the shock-wave tends to zero as the shock-wave tends to $\bar{r}=0$ in backward time. (See (5.12), where, as in the zero pressure problems, the total mass inside the shock tends to a delta function source at the origin as $\bar{r}\to 0$.)

The formulas in the above example were especially simple because

$$A \equiv A_0 \in (0,1), \tag{5.28}$$

$$z = \frac{1}{3}, \tag{5.29}$$

$$\rho = \frac{C_\rho}{\bar{r}^2}, \tag{5.30}$$

and

$$\bar{\rho} = \frac{C_{\bar{\rho}}}{\bar{r}^2}, \tag{5.31}$$

where A_0, C_ρ, and $C_{\bar{\rho}}$ are positive constants. We now construct a more general class of equations of state, more general than isothermal, in which the simplifying conditions (5.28)–(5.31) also hold. Note first that (5.29) and (5.30) follow directly from (5.28) and (5.29). Indeed,

$$A \equiv 1 - \frac{2\mathcal{G}M(\bar{r})}{\bar{r}} = A_0 \equiv \text{const.} \tag{5.32}$$

implies that

$$M(\bar{r}) = \frac{1-A_0}{2\mathcal{G}}\bar{r}, \tag{5.33}$$

and since $M'(\bar{r}) = 4\pi\bar{\rho}\bar{r}^2$, it follows from (5.32) that

$$\bar{\rho} = \frac{1}{4\pi}\frac{(1-A_0)}{2\mathcal{G}}\frac{1}{\bar{r}^2}. \tag{5.34}$$

This, together with (5.29), then implies

$$\rho = 3\bar{\rho} = \frac{3}{4\pi}\frac{(1-A_0)}{2\mathcal{G}}\frac{1}{\bar{r}^2}, \tag{5.35}$$

so that

$$C_{\bar{\rho}} = \frac{1}{4\pi}\frac{(1-A_0)}{2\mathcal{G}} \tag{5.36}$$

and

$$C_\rho = \frac{3}{4\pi}\frac{(1-A_0)}{2\mathcal{G}}. \tag{5.37}$$

We now find more general equations of state for which (5.28) and (5.29) hold. To this end, assume that $A = A_0$ is constant, so that $M = \frac{1-A_0}{2\mathcal{G}}\bar{r}$. Substituting these into the OV equation (3.6), and using (5.36) to simplify, yields

$$\frac{d\bar{p}}{d\bar{r}} = -\frac{1-A_0}{2A_0\bar{r}}\bar{\rho}\left(1+\frac{\bar{p}}{\bar{\rho}}\right)^2. \tag{5.38}$$

Using the relation

$$\bar{\rho} = \frac{1 - A_0}{8\pi\mathcal{G}} \frac{1}{\bar{r}^2},$$

we can rewrite (5.38) as the following equation for $\bar{p}$ as a function of $\bar{\rho}$, which is now equivalent to the OV equation:

$$\frac{d\bar{p}}{d\bar{\rho}} = \frac{1 - A_0}{4A_0}\left(1 + \frac{\bar{p}}{\bar{\rho}}\right)^2. \tag{5.39}$$

Thus we conclude that the OT pressure $\bar{p}$ is determined by the choice of A_0, together with an arbitrary additional initial condition for the ODE (5.39). Moreover, if we take $k = 0$, then we can use (5.9) to obtain the following formula for the FRW pressure in terms of the function $\bar{\mu} = \frac{\bar{p}}{\bar{\rho}}$, which is determined by the ODE (5.39):

$$p = \mu\rho, \tag{5.40}$$

where

$$\mu = \frac{(1 - A_0/3)\,\bar{\mu} + (1 - A_0)}{(A_0 - 1)\bar{\mu} + (3A_0 - 1)}, \tag{5.41}$$

The general theory in [8] implies that $p > \bar{p} > 0$ at each value of $\bar{r}$ if and only if $\theta > 1/\gamma$, which translates into the condition

$$\bar{\mu} < \frac{3A_0 - 1}{1 - A_0},$$

when $k = 0$, $\theta = A_0$, and $z = 1/3$. In particular this requires $A_0 > 1/3$ for $\bar{\mu}$ to be positive.

We recover the example of an explosion into a static isothermal sphere [6] by setting $\bar{\sigma} = \bar{\mu} = \text{const.}$, so that $\bar{\sigma} = \frac{\bar{p}}{\bar{\rho}} = \frac{d\bar{p}}{d\bar{\rho}}$. Substituting this into (5.39) then enables us to solve for $\bar{\sigma}$ in terms of A_0 to obtain the relation (5.19) derived above.

We can integrate the ODE (5.39) in general as follows: let

$$w = 1 + \frac{\bar{p}}{\bar{\rho}}$$

and set

$$\alpha = \frac{A_0}{1 - A_0}.$$

Then, clearly, $w > 0$ and $\alpha > 0$. Substituting w for p in (5.39) yields the equivalent ODE:

$$\bar{\rho}\frac{dw}{d\bar{\rho}} = \frac{1}{4\alpha}w^2 - w + 1. \tag{5.42}$$

Now changing to the independent variable $s = \ln(\bar{\rho})$ turns (5.42) into the autonomous ODE

$$\frac{dw}{ds} = \frac{1}{4\alpha}w^2 - w + 1. \tag{5.43}$$

Integrating yields

$$\ln \frac{\bar{\rho}}{\bar{\rho}_0} = \int_{w_0}^{w} \frac{dw}{\frac{1}{4\alpha}w^2 - w + 1}. \tag{5.44}$$

We can integrate the right-hand side by completing the square. It follows that

$$\frac{1}{4\alpha}w^2 - w + 1 = \frac{(w-2\alpha)^2}{4\alpha} + (1-\alpha). \tag{5.45}$$

Thus there are three cases depending on whether $(1-\alpha)$ is positive, negative, or zero. Note that $1-\alpha > 0$ is equivalent to $A_0 < \frac{1}{2}$, $1-\alpha < 0$ is equivalent to $A_0 > \frac{1}{2}$, and $1-\alpha = 0$ is equivalent to $A_0 = \frac{1}{2}$. In the first case ($A_0 < \frac{1}{2}$), making the substitution

$$z = \frac{w - 2\alpha}{2\sqrt{\alpha(1-\alpha)}},$$

integrating, and solving for p in (5.44) gives the formula

$$\frac{\bar{p}}{\bar{\rho}} = \left(\frac{3A_0 - 1}{1 - A_0}\right) + 2\frac{\sqrt{A_0(1-2A_0)}}{1-A_0} \tan\left(\frac{1}{2}\sqrt{\frac{1-2A_0}{A_0}} \ln \bar{\rho} + C\right), \tag{5.46}$$

where C is an arbitrary constant of integration.

For the second case ($A_0 > \frac{1}{2}$), the integration leads to the formula

$$\frac{\bar{p}}{\bar{\rho}} = \left(\frac{3A_0 - 1}{1 - A_0}\right) + \left(2\frac{\sqrt{A_0(2A_0-1)}}{1-A_0}\right) \frac{1 + \left(\frac{\bar{\rho}}{C}\right)^{\sqrt{\frac{2A_0-1}{A_0}}}}{1 - \left(\frac{\bar{\rho}}{C}\right)^{\sqrt{\frac{2A_0-1}{A_0}}}}. \tag{5.47}$$

For the third and final case ($A_0 = \frac{1}{2}$), the integration leads to the formula

$$\frac{\bar{p}}{\bar{\rho}} = 1 + \frac{1}{\ln\left[\bar{\rho}^{-\frac{1}{4}}\right] + C}. \tag{5.48}$$

These exact formulas could be used to approximate realistic equations of state over limited ranges of the variable $\bar{\rho}$.

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