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Applications of Shock-Waves in General Relativity

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Abstract. We discuss recently published work and work in progress on applications of shock-waves in general relativity. We discuss the problem of constructing shock-waves arbitrarily close to the Schwarzschild radius of a star, and the problem of introducing a shock-wave into the standard theory of cosmology.

1. Introduction

In 1915 Albert Einstein introduced the gravitational field equations that now bear his name. These equations provide the foundation for the general theory of relativity. In Einstein's theory, the gravitational field is a Lorentzian metric g, which in a given coordinate system $\mathbf{x} = (x^0, x^1, x^2, x^3)$ on spacetime, has components $g_{ij}(\mathbf{x})$, $i, j = 0, 1, 2, 3$. In this theory, freefall paths through a gravitational field are geodesics of the spacetime metric. For example, the planets follow geodesics of the gravitational metric generated by the Sun, (approximated by the Schwarzschild metric beyond the surface of the Sun, and by the Tolman-Oppenheimer-Volkoff (TOV) metric inside the surface of the Sun), and according to the standard model of cosmology, the galaxies follow geodesics of the Friedmann-Robertson-Walker (FRW) metric. In spherical coordinates $\mathbf{x} = (t, r, \theta, \phi)$, the Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2GM_0}{r}\right) dt^2 + \left(1 - \frac{2GM_0}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{1}$$

the TOV metric is given by

$$ds^2 = -B(r)dt^2 + \left(1 - \frac{2GM(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2}$$

and the FRW metric is given by

$$ds^2 = -dt^2 + R(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right). \tag{3}$$

Here $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$ denotes the standard line element on the 2-sphere, $\mathcal{G}$ denotes Newton's gravitational constant, M_0 denotes the mass of the Sun (or a star), $M(r)$ denotes the total mass inside radius r, (a function that tends smoothly to M_0 at the star surface), and $B(r)$ is a function that tends smoothly to $1 - 2\mathcal{G}M_0/r$ at the star surface. The only unknown in the FRW metric is the cosmological scale factor $R(t)$ from which the Hubble constant H is determined,

$$H = \frac{\dot{R}(t)}{R(t)}. \tag{4}$$

The FRW metric describes the time evolution of a three dimensional space of constant scalar curvature, (the t=constant surfaces), and the sign of the curvature is given by $sign(k)$, a constant that can be rescaled to one of the values $\{-1, 0, 1)$ via a rescaling of the radial coordinate r.

The fundamental tenet of general relativity is the principle that there is no apriori global inertial coordinate system on spacetime. Rather, in general relativity, inertial coordinate systems are *local* properties of spacetime in the sense that they change from point to point. For example, if there were a global Newtonian *absolute space*, then there would exist global coordinate systems in which freefalling objects do not accelerate, and any two such coordinate systems would be related by transformations from the 10 parameter Galilean Group–the set of coordinate transformations that do not introduce accelerations. In special relativity, the existence of absolute space would presume the existence of global coordinate systems related by the transformations of special relativity; that is, in special relativity, the 10 parameter Poincare group replaces the the 10 parameter Galilean Group as the set of transformations that introduce no accelerations. The Poincare Group is obtained from the Galilean group by essentially replacing Euclidean translation in time by Lorentz transformations, and this accounts for time dilation. The spacetime metric can then be viewed as a book-keeping device for keeping track of the location of the local inertial reference frames as they vary from point to point in a given coordinate system. Because the metric components transform like a bilinear form, the metric locates the local inertial frames at a given point as those coordinate systems that diagonalize the metric at that point, $g_{ij} = diag(-1, 1, 1, 1)$, such that the derivatives of the metric components also vanish at the point. Thus, the earth moves "unaccelerated" in each local inertial frame, but these frames change from point to point, thus producing apparent accelerations in a global coordinate system in which the metric is not everywhere diagonal. The fact that the earth moves in a periodic orbit around the Sun is proof that there is no coordinate system that globally diagonalizes the metric, and this is an expression of the fact that gravitational fields produce non-zero spacetime curvature. Indeed, in an inertial coordinate frame, when a gravitational field is present, one cannot in general eliminate the second derivatives of the metric components at a point by any coordinate transformation, and the nonzero second derivatives of the metric that cannot be eliminated, represent the gravitational field. These second derivatives are measured by the Riemann Curvature Tensor associated with the Riemannian metric g. Riemann introduced the curvature tensor in his inaugural lecture of 1854, in which he solved the longstanding open problem of describing curvature in surfaces of dimension higher than two. Although the curvature tensor was first developed for positive definite "spatial" metrics, Einstein accounted for time dilation by letting Lorentz transformations play the role of rotations in Riemann's theory, and except for this, Riemann's theory carries over essentially unchanged. The Riemann Curvature Tensor $R^i_{jkl}(\mathbf{x})$ is a quantity that involves second derivatives of $g_{ij}(\mathbf{x})$, but which transforms like a tensor under coordinate transformation; that is, the components transform like a four component version of a vector field, even though a vector field is constructed essentially from first derivatives. The connection between general relativity and geometry can be summarized in the statement that the Riemann Curvature Tensor associated with the metric g gives an invariant description of gravitational accelerations.

Once one makes the leap to the idea that the inertial coordinate frames change from point to point in spacetime, one is immediately stuck with the idea that, since our non-rotating inertial frames here on earth are also non-rotating with respect to the fixed stars, the stars must have had something to do with the determination of our non- accelerating reference frames here on earth, (Mach's Principle). Indeed, not every Lorentzian metric can describe a gravitational field, which means that gravitational metrics must satisfy a constraint that describes how inertial frames at different points of spacetime interact. In Einstein's theory of gravity, this constraint is given by the Einstein Equations, which, in a given coordinate system $\mathbf{x}$, can be written in the compact form

$$G_{ij}(\mathbf{x}) = \frac{8\pi\mathcal{G}}{c^4}T_{ij}(\mathbf{x}). \tag{5}$$

Here c denotes the speed of light, $G_{ij} = R^{\sigma}_{i\sigma j} - \frac{1}{2}R^{kl}_{kl}g_{ij}$ denote the components of the Einstein curvature tensor, and T_{ij} the components of the stress energy tensor, the source of the gravitational field. We assume the summation convention. The components of the stress energy tensor give the densities of energy and momentum together with their fluxes. When the sources are modeled by a perfect fluid, T is given by

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij}, \tag{6}$$

where $\mathbf{u}$ denotes the unit 4-velocity of the fluid, ρ denotes the energy density, (as measured in the inertial frame moving with the fluid), and p denotes the fluid pressure. The Einstein equations play the role in general relativity that the Poisson equation $-\Delta\phi = 4\pi\mathcal{G}\rho$ plays in the Newtonian theory of gravity, except for an important difference: the Poisson equation describes the time evolution of the (scalar) gravitational potential ϕ, but this must be augmented by some system of conservation laws in order to describe the time evolution of the density ρ as well. In Einstein's theory, the time evolution of the gravitational metric is determined simultaneously with the time evolution of the sources through system (5). This principle is the basis for the discovery of the Einstein equations. Indeed, note that conservation of energy in curved spacetime reduces to the statement

$$Div(T) = 0, \tag{7}$$

where the divergence is taken as the covariant divergence for the metric g so that it agrees with the ordinary divergence in each local inertial coordinate frame. In this way equations (7) reduce to the relativistic compressible Euler equations in flat Minkowski space. Since the covariant derivative depends on the metric components, the conservation equation (7) is essentially coupled to the equation for the gravitational field g. But the stress tensor T is symmetric, and so the tensor on the LHS of (5) must also be symmetric, and thus there are ten equations in the ten independent unknown functions among the components of the symmetric matrix g_{ij} together with the four independent functions among ρ and the unit vector field $\mathbf{u}$. (Here p is assumed to be determined by an equation of state.) But (6) assumes no coordinate system, and thus in principle we are free to give four further relations that tie the components of G and T to the coordinate system. This leaves ten equations in ten unknowns, and thus there are no further constraints allowable to couple system (6) to the conservation laws (7). It follows that (7) must follow as an identity from (5), and this determines the LHS of (5) as the simplest tensor constructable from R^i_{jkl} such that (7) follows identically from the Bianchi identities $R^i_{j[kl,m]=0}$, [12]. In a specified system of coordinates, (5) determines a hyperbolic system of equations that describes the time evolution and interaction of local inertial coordinate frames. Putting the metric ansatz (3) into (5) gives a system of two ODE's for $R(t)$ and $\rho(t)$, and this has a unique solution when we assign the present value of the Hubble constant and the present density of the universe as initial conditions. This solution is the basis for the standard model of cosmology. In the sense that the metric is consistent with equation (5), it "explains" why our local inertial frames are non-rotating relative to the stars.

Since the relativistic Euler equations (7) form a subsystem of the Einstein equations (5), we know that shock-waves must be an important aspect of solutions of (5). The mathematical theory of shock-wave solutions of (5) is rudimentary. For example, we know of no proof or examples demonstrating that shock-waves form from smooth solutions of (5) as they do in solutions of classical Euler equations. There is no local existence theorem, no "Glimm's Theorem", for solutions with shocks, even for 1-dimensional problems, cf [12]. There are no known solutions that represent rarefaction waves in general relativity. In [7], the authors constructed the first explicit examples of shock-wave solutions of the Einstein equations obtained by matching metrics of form (3) to metrics of form (2) across a shock-wave interface. In these solutions the expanding universe of FRW represents the expansion behind a shock-wave that blasts into an ambient spacetime modeled by a TOV metric of form (2). This models a spherically symmetric blast wave in which a fluid dynamical shock-wave carries with it a wave in the metrical properties of spacetime. In these examples, the spacetime metric is only $C^{1,1}$ at the shock, (one Lipschitz continuous derivative at the shock), but exact formulas are made possible in "singular" coordinate systems in which the metric is only Lipschitz continuous at the shock. A general theory for matching spacetime metrics was presented in [8]. Since this time, the authors have studied the problems of placing a shock-wave near the Schwarzschild radius of a star, ($r \approx 2\mathcal{G}M(r)$ in (2)), and the problem of modeling the expansion of the universe as coming from a classical explosion into an ambient spacetime.

2. Recent results

The results in [2] describe surface layers arbitrarily close to a black hole; that is, solutions of (2) near the event horizon of a black-hole where $r = 2\mathcal{G}M(r)$. In a sort of "counterexample" to the Buchdahl stability theorem [9], we show in [1] that a shock-wave can supply the pressure required to hold up a highly collapsed surface layer, and the time it takes the shock-wave to reach the surface tends to infinity as the surface layer tends to the Schwarzschild radius. The developmental work in [2] provides a general theory of solutions of the TOV system starting from initial data at a star surface. Interesting conclusions in [2,3] include a proof that black-holes never form in solutions of the TOV equations, as well as a quantitative description of solutions of the TOV equations when the star surface lies inside the critical radius of 9/8'ths the Schwarzschild radius, (the Buchdahl stability limit for stars, [9]). We show that the density and pressure profiles always remain smooth, positive, and bounded, and tend to zero together with their derivatives at $r = 0$, the center of the star. Thus the star is "hollow". This is a bit surprising because when $r > \frac{9}{8}2GM(r)$, there are examples of solutions of the TOV equations in which the pressure tends to infinity as $r \to 0$, [13]. We show that this happens because $M(r)$ goes negative before $r = 0$. This produces a naked singularity at $r = 0$ which can be interpreted as a negative delta function source of mass. The singularity then supplies the repulsive effect at the center of the star in these solutions. We show that these solutions can match up to empty space only at radii at which $M(r) > 0$, so such negative masses would never be noticed by an observer in the far field. Since the total mass $M(r)$ enters only as a metric component in (2), negative mass is not apriori unphysical in Einstein's theory as it is in Newton's theory. Indeed, it is the local quantities like density and pressure that are well defined in general relativity. Since there are no global inertial reference frames, the "total mass" M in a region of spacetime is not well defined as a global quantity in Einstein's theory of gravity, except in special solutions. Thus an additional principle is required to rule out negative mass in solutions of the Einstein equations. If a positive density, singularity free solution of the Einstein equations could replace a neighborhood of the singularity at $r = 0$ in these solutions, then this would demonstrate that gravity can have a repulsive effect, while if not, then there is an interesting geometrical property that is constraining things. This is an open problem.

We now discuss work in progress on the problem of putting a shock-wave into the standard model of cosmology. The question we ask is this: Could the expanding universe have arisen from a great explosion into an ambient spacetime? Hubble's law correlates the red-shift in galaxies with distance, and this supports the belief that the galaxies are receding from us due to an average uniform expansion of the universe. The idea is that the universe is expanding uniformly in three dimensions analogous to the two dimensional uniform expansion of the surface of a balloon as it expands. This uniform expansion is expressed by metric (3) where the uniform expansion of the three dimensional constant curvature surface $t = constant$ is described by the scale factor $R(t)$. When the ansatz (3) is put into the Einstein equations (5), the resulting ODE's imply that $R(t_*) = 0$ at some time $t = t_*$ in the past, and this represents the *Big Bang*, the moment in which the entire space $t = constant$ burst from a single point. The *Big Bang* is forced on you once you accept that the *entire* universe is undergoing expansion. One consequence of the singularity theorems of Hawking and Penrose is that any perturbation of the FRW metric that remains everywhere expanding, must also have a singularity in backward time, and so the *Big Bang* is not special to FRW metrics. We are exploring the possibility that the expansion of the universe could be due to a limited expansion into an ambient spacetime, such that the leading edge is modeled by a shock-wave. This would be the case if the observed expansion of the universe actually resulted from something more like a classical explosion than from the esoteric *Big Bang* singularity in the FRW metric. Moreover, in the standard cosmological model, the expansion is assumed to be isentropic, and thus it makes physical sense to time reverse the solution all the way back to 10^{-34} seconds after the *Big Bang*, (the onset of inflation), and before. In contrast to this, since shock-waves introduce increase of entropy and loss of information, the existence of a shock-wave would imply that many solutions could have decayed to the present expanding universe, and thus the shock-wave model implies that there is a fundamental limit on our ability to re-construct the details of the initial explosion from the continuum model alone. The work in [6] demonstrates that these shock-wave models of cosmology do not meet the assumptions of the first relevant singularity theorem of Hawking, [10].

In work in progress, and to start, we consider the problem of estimating the present position of a shock-wave under the assumption that the universe is modeled by the standard FRW metric of cosmology on the inside, and a TOV metric on the outside, such that the interface in between is an exact shock-wave solution of (5). The assumption that outside the shock-wave is a time-independent spherically symmetric solution is not so unreasonable if you imagine that the spacetime before the explosion took a long time getting into the pre-explosion configuration. In [7] we we have formulas for such a shock-wave assuming an equation of state $p = \frac{1}{3}\rho$, the equation of state for the very early universe in the standard model. (The equation of state $p = \frac{1}{3}\rho$ represents the equation of state for a stress tensor that is trace free, and this applies to pure radiation, and for free particles in the extreme relativistic limit, [9,12,13].) The equation of state in the TOV metric beyond the shock-wave is $p = \bar{\sigma}\rho$ where $\bar{\sigma} = \sqrt{17} - 4 \approx .1231$. In this model $k = 0$, (deviation from $k = 0$ has not been observed, although most estimates place the density of the universe below the critical density, [11]), and one can calculate the shock position at the time the universe is at the critical density $Q_{crit} = H_0^2$ where $H_0 = h_0 \times 100 km/sMpc$ is the present value of the Hubble constant, giving $Q_0 \approx 1.87 \times 10^{-}29h_0^2\, g\, cm^{-3}$. Taking $h_0 \approx .55$ for the Hubble constant puts the position of the shock at an intriguing 11 billion light years away. Until recently, this was roughly where it was thought that the superclusters of galaxies end and the quasars begin.

To go further, we assume the equation of state of the universe since the uncoupling of radiation from thermal equilibrium with matter. This occurred at approximately $4000^o K$ at about $t_* = 300.000$ years after the *Big Bang* in the standard model. To analyze this situation, we use the theory in [5,8] to derive equations for the time evolution of the shock position and the outer TOV solution given the FRW metric that applies to the equation of state during this epoch in the standard FRW cosmology. For this we must re-work our prior theory which was based on fixing the outer TOV metric and solving the ODE for the shock position and FRW metric. We have obtained this system, and have found that in the case $k = 0$, this system can be essentially completely analyzed by transform to variables in which the cosmological scale factor R is taken as the independent variable instead of the FRW time t. We show that a physically interesting shock-wave model emerges. What we find most interesting is that the condition that the outer density in the TOV metric be positive places a constraint on the possible shock positions. Our analysis shows that this condition implies that the shock position at time $t = t_*$ must be closer than about 10^5 light-years from the center of the explosion, (on the order of the size of galaxies). Once this condition is met, we show that the density and pressure profiles are reasonable from that time onward. Starting with this initial condition at $t = t_*$, our calculations then place the shock-wave at present time in this model at about 270 million light-years from the center of the explosion. The constraint on the shock position based on our positive energy density condition applied at *present* time yields that the outer limit of the shock position is about twelve billion light-years from the center. What we find most interesting about this work is that there are unexpected constraints, and not all shock positions are possible. The authors are now completing the case for $k \neq 0$ where different estimates are possible. Details will appear in the authors' forthcoming paper.

In conclusion, we ask whether a shock-wave cosmology gives a more robust explanation for why the non-rotating inertial frames here on earth are non-rotating relative to the stars, or why the universe is so close to critical expansion. If we evolved from the center of a great explosion, it also makes one wonder whether some of the far away objects that we observe in the night-time sky are possibly due to similar explosions that originated at other locations in spacetime. We now know that the scale of supernovae is not the largest scale on which classical explosions have occurred in the universe. Indeed, we just received word, (as reported by George Djorgovski from Caltech in the recent issue of Nature), that on May 7, 1998, a gamma ray explosion emanating from a faint galaxy known as GRB971214 erupted, and for two seconds the burst was more luminous than the rest of the universe combined. This is the largest explosion ever recorded, redshifts placed it about 12 billion light-years away, and conditions at the explosion were judged to be equivalent to those one millisecond after the Big Bang in the standard model. "... this is such an extreme phenomenon that it is possible that we are dealing with something completely unanticipated..." Could explosions such as this be similar to the one that gave rise to our own "expanding universe", but in a region of spacetime beyond the expansion of our own universe, (that is, beyond the shock-wave that marks the edge of our own expansion)? At least we believe that these examples will provide an interesting alternative scenario for the Big Bang, and mathematically will provide an interesting inroad for the study of shock-waves in Einstein's theory of gravity.

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