A Locally Inertial Glimm Scheme for General Relativity

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October 15, 2001

Abstract

In 1915, Albert Einstein wrote down his famous field equations of general relativity, and in 1965, James Glimm gave his theory of wave interactions. Here we discuss the issues in our forthcoming paper *Shock wave solutions of the Einstein equations: existence and consistency for the initial value problem*, in which we put these two theories together by introducing a locally inertial Glimm scheme for spherically symmetric spacetimes. The result produces the first analysis of weak solutions of the Einstein equations for a perfect fluid, (in the PDE sense of the word "analysis").

1 Introduction

In Einstein's theory of General Relativity [2], all properties of the gravitational field are determined by the *gravitational metric tensor* g, a Lorentzian metric that describes a continuous field of symmetric bilinear forms of signature $(-1, 1, 1, 1)$, defined at each point of a four dimensional manifold M called "spacetime." Freefall paths through the gravitational field are the geodesics of the metric; the non-rotating vectors carried by an observer in free fall are those vectors that are parallel transported by the metric connection determined by g; spatial lengths of objects correspond to the lengths of the spacelike curves that define their shape—length measured by the metric g; and time changes for an observer are determined by the length of the observer's timelike curve through spacetime, as measured by the metric g.

In a given coordinate system x on spacetime, the length of a spacetime curve can be computed by integrating the element of arclength

*Supported by the Institute of Theoretical Dynamics, UC-Davis

†Supported by NSF grant DMS 0102493, by the Institute of Theoretical Dynamics, UC-Davis, and by CNPq, Brazil.

$$ds^2 = g_{ij}dx^i dx^j, \tag{1}$$

where g_{ij} are the x-components of the spacetime metric. Here all indexed objects are assumed to tranform like tensors, and we adopt the Einstein summation convention whereby repeated up-down indices are assumed summed from 0 to 3. The metric components g_{ij} are smoothly varying, and transform like a symmetric bilinear form under coordinate transformation. It follows that, in a neighborhood of any point, there exist *locally inertial*, or *locally Lorentzian* coordinates. That is, coordinates such that $g_{ij} = diag(-1,1,1,1)$, and $g_{ij,k} = 0$ at the point, $i,j,k = 0,...,3$. The notion of geodesic motion and parallel translation have a very natural physical interpretation in General Relativity in terms of the locally inertial coordinate frames. Indeed, General Relativity makes contact with (the flat spacetime theory of) Special Relativity by identifying the locally Lorentzian frames at a point as the "locally non-rotating" inertial coordinate systems in which spacetime behaves as if it were locally flat. Physically, the non-rotating vector fields carried by an observer in free fall are the vector fields that are *locally constant in the locally inertial coordinate frames defined at each point along the curve.* (Here, *locally constant* means constant to within higher order errors whose sum tends to zero under refinement of the coordinate charts.) In fact, a vector field is parallel translated along a curve (in the sense that $\nabla_X Y = 0$ along a curve, c.f. [1, 21, 14, 22]), if and only if its components are (locally) constant in the locally inertial coordinate frames defined at each point along the curve. Thus, the non-rotating vector fields carried by an observer in free fall are exactly the vectors that are parallel transported by the unique symmetric connection determined by the gravitational metric g. Similarly, the geodesics of the metric g are just the curves that are "locally straight lines" in the locally inertial coordinate frames, c.f. [21, 22].

The fundamental tenet of General Relativity is the principle that there is no apriori global inertial coordinate system on spacetime. Rather, in General Relativity, inertial coordinate systems are *local* properties of spacetime in the sense that they change from point to point. For example, if there were a global Newtonian *absolute space*, then there would exist global coordinate systems in which free falling objects do not accelerate, and any two such coordinate systems would be related by transformations from the 10 parameter Galilean Group–the set of coordinate transformations that do not introduce accelerations. Thus, the spacetime metric can then be viewed as a book-keeping device for keeping track of the location of the local inertial reference frames as they vary from point to point in a given coordinate system–diagonalize the metric to find the local inertial frames. Thus, the earth moves "unaccelerated" in each local inertial frame, but these frames change from point to point, thus producing apparent accelerations in a global coordinate system in which the metric is not everywhere diagonal. The fact that the earth moves in a periodic orbit through the time independent gravitational field of the Sun gives a topological proof that there is no coordinate system that globally diagonalizes the metric, and this is an expression of the fact that gravitational fields produce non-zero spacetime curvature. Indeed, in an inertial coordinate frame, when a gravitational field is present, one cannot in general eliminate the second derivatives of the metric components at a point by any coordinate transformation, and the nonzero second derivatives of the metric that cannot be eliminated, represent the gravitational field. These second derivatives are measured by the Riemann Curvature Tensor associated with the Riemannian metric g, [22]. Our analysis of the initial value problem by Glimm's method exploits the locally flat nature of spacetime by approximating spacetime by a Minkowski flat metric in each grid rectangle, the connection between neighboring coordinates being accounted for by discontinuities at the grid boundaries. Thus our goal is to construct and analyze a locally inertial Glimm Scheme, [3].

Now not every metric can be a gravitational field. In 1915, Einstein introduced his gravitational field equations, which can be written in the compact form

$$G = \kappa T. \tag{2}$$

The Einstein equations (2) describe the time evolution of the spacetime metric g, and provide the constraint that a gravitational metric must meet in order to be a physical gravitational field. Here G, (a 2-tensor constructed from the Riemann curvature tensor), is the Einstein curvature tensor, T is the stress energy tensor, (the source of the gravitational field), and $\kappa = \frac{8\pi\mathcal{G}}{3c^4}$, is a universal constant that ensures correspondence with the Newtonian theory, $\mathcal{G} =$ Newton's constant. In a given coordinate system x, the field equations (2) take the component form

$$G^{ij}(x) = \kappa T^{ij}(x), \tag{3}$$

where G^{ij} denote the x-components of the Einstein curvature tensor, and T^{ij} denote the x-components of the stress energy tensor, as a function of the coordinate position x. The components of the stress energy tensor give the energy density and i-momentum densities and their fluxes at each point of spacetime, $i = 0, ..., 3$. When the sources are modeled by a perfect fluid, T is given by

$$T^{ij} = (\rho c^2 + p)w^i w^j + pg^{ij}, \tag{4}$$

where $\mathbf{w}$ denotes the unit 4-velocity vector of the fluid, (the tangent vector to the world line of the fluid particle), ρc^2 denotes the energy density, (as measured in the inertial frame moving with the fluid, c=speed of light), and p denotes the fluid pressure, [22]. It follows that in Einstein's theory of gravity, the time evolution of the gravitational metric is determined simultaneously with the time evolution of the sources through system (2), and all of the components of the stress tensor directly influence the components of the gravitational field g_{ij}.

Since the 0-column of the stress-energy tensor (4) gives the energy and momentum densities, and the i-column gives the corresponding i-fluxes, (in the relativistic sense), it follows that conservation of energy-momentum in curved spacetime reduces to the statement

$$Div(T) = 0, \tag{5}$$

where (capital) Div denotes the covariant divergence for the metric g, so that it agrees with the ordinary divergence in each local inertial coordinate frame, c.f. [22]. In this way equations (5) reduce to the relativistic compressible Euler equations in flat Minkowski spacetime. In Einstein's theory, (5) follows as an identity from (2), because the Einstein tensor G_{ij} is chosen to satisfy $DivG = 0$ as a consequence of the Bianchi identities of Riemannian geometry, c.f., [22]).

In a given coordinate system, the Einstein equations (2) determine a hyperbolic system of equations that simultaneously describe the time evolution of the gravitational metric, as well as the time evolution of the fluid according to (5). Since GR is coordinate independent, we can always view the time evolution (2) in local inertial coordinates at any point in spacetime, in which case (5) reduces to the classical relativistic Euler equations at the point, [12]. This tells us that, heuristically, shock-waves must form in the time evolution of (2) because one could in principle drive a solution into a shock while in a neighborhood where the equations remained a small perturbation of the classical Euler equations. It follows that shock-waves are as fundamental to the time evolution of solutions of the Einstein equations for a perfect fluid, as they are for the time evolution of the classical compressible Euler equations. At a shock wave, the fluid variables ρ, $\mathbf{w}$ and p are discontinuous. Notice that (2) implies that the Einstein curvature tensor G will be discontinuous at any point where T is discontinuous. Since G involves second derivatives of the metric tensor g, the only way (2) can hold in the classical pointwise a.e. sense at the shock is if the component functions g_{ij} are continuously differentiable at the shock, with bounded derivatives on either side, that is, if $g_{ij} \in C^{1,1}$. Thus we expect from (2) that the spacetime metric g should be $C^{1,1}$ at shock waves. However, we now show that for a spherically symmetric metric in standard Schwarzschild coordinates, the best one can expect is that $g \in C^{0,1}$.

A spacetime metric g is spherically symmetric if it takes the general form

$$ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + 2D(r,t)drdt + C(r,t)d\Omega^2, \tag{6}$$

where A, B, C, D are arbitrary, smooth, positive functions of spherical coordinates (t, r, θ, ϕ), and $d\Omega^2 = d\theta^2 + sin^2(\theta)d\phi^2$ denotes the standard line element on the unit 2-sphere. Now the planets follow geodesics of the gravitational metric generated by the Sun, (approximated by the Schwarzschild metric outside the surface of the Sun, and by the Tolman-Oppenheimer-Volkoff (TOV) metric inside the surface of the Sun), and according to the standard theory of cosmology, the galaxies follow geodesics of the Friedmann-Robertson-Walker (FRW) metric. The Schwarzschild line element is given by,

$$ds^2 = -\left(1 - \frac{2GM_0}{r}\right)dt^2 + \left(1 - \frac{2GM_0}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{7}$$

the FRW line element is given by

$$ds^2 = -B(r)dt^2 + \left(1 - \frac{2GM(r)}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{8}$$

and the TOV line element is given by,

$$ds^2 = -dt^2 + R(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right). \tag{9}$$

(Here M_0 denotes the mass of the Sun (or a star), $M(r)$ denotes the total mass inside radius r, (a function that tends smoothly to M_0 at the star surface), $B(r)$ is a function that tends smoothly to $1 - 2\mathcal{G}M_0/r$ at the star surface, $H = \frac{\dot{R}(t)}{R(t)}$ is the Hubble "constant", and $sign(k)$ determines the sign of the curvature of 3-space in the cosmological FRW universe, [22].) Each of the metrics (7)-(9) is a special case of a general spherically symmetric spacetime metric of the form (6). We now discuss the authors' results in [6] in which we give the first general existence theorem for shock wave solutions in spherically symmetric spacetimes.

2 Our Results

It is well known that for a general time dependent spherically symmetric metric of form (6), there exists a coordinate transformation to standard Schwarzschild coordinates in which the metric takes the simpler form

$$ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2 d\Omega^2, \tag{10}$$

c.f. [22, 20]. Thus we always assume a metric of form (10). Using MAPLE to put the metric ansatz (10) into the Einstein equations (2) produces the following system of four coupled partial differential equations, (c.f. (3.20)-(3.23) of [5]),

$$\begin{aligned}
\frac{A}{r^2 B}\left\{r\frac{B'}{B} + B - 1\right\} &= \kappa A^2 T^{00} && (11)\\
-\frac{B_t}{rB} &= \kappa ABT^{01} && (12)\\
\frac{1}{r^2}\left\{r\frac{A'}{A} - (B-1)\right\} &= \kappa B^2 T^{11} && (13)\\
-\frac{1}{rAB^2}\left\{B_{tt} - A'' + \Phi\right\} &= \frac{2\kappa r}{B}T^{22}, && (14)
\end{aligned}$$

where the quantity Φ in the last equation is given by,

$$\Phi = -\frac{BA_tB_t}{2AB} - \frac{B}{2}\left(\frac{B_t}{B}\right)^2 - \frac{A'}{r} + \frac{AB'}{rB} + \frac{A}{2}\left(\frac{A'}{A}\right)^2 + \frac{A}{2}\frac{A'}{A}\frac{B'}{B}.$$

Here "prime" denotes $\partial/\partial r$, "dot" denotes $\partial/\partial t$, $\kappa = \frac{8\pi\mathcal{G}}{c^4}$ is the coupling constant, $\mathcal{G}$ is Newton's gravitational constant, c is the speed of light, T^{ij}, $i,j = 0,...,3$ are the components of the stress energy tensor, and $A \equiv A(r,t)$, $B \equiv B(r,t)$ denote the components of the gravitational metric tensor in standard Schwarzschild coordinates $\mathbf{x} = (x^0, x^1, x^2, x^3) \equiv (t, r, \theta, \phi)$. The mass function M is defined through the identity

$$B = \left(1 - \frac{2M}{r}\right)^{-1}, \tag{15}$$

and $M \equiv M(r,t)$ is interpreted as the mass inside radius r at time t. In terms of the variable M, equations (11) and (12) are equivalent to

$$M' = \tfrac{1}{2}\kappa r^2 A T^{00}, \tag{16}$$

and

$$\dot{M} = -\tfrac{1}{2}\kappa r^2 A T^{01}, \tag{17}$$

respectively. Equations (11)-(14) are obtained by plugging metric ansatz (10) into the Einstein equations (3), assuming a perfect fluid (4). Using (10) in (4), the components T^{ij} satisfy

$$T^{00} = \frac{1}{A} T_M^{00}, \tag{18}$$

$$T^{01} = \frac{1}{\sqrt{AB}} T_M^{01}, \tag{19}$$

$$T^{11} = \frac{1}{B} T_M^{11}, \tag{20}$$

where T_M^{ij} denote the components of T in flat Minkowski spacetime. To keep things as simple as possible, we assume the equation of state

$$p = \sigma^2 \rho, \quad 0 < \sigma < c, \tag{21}$$

where σ, the sound speed, is assumed to be constant.[3] When $p = \sigma^2 \rho$, the components of T_M are given by

$$T_M^{00} = \frac{c^4 + \sigma^2 v^2}{c^2 - v^2} \rho, \tag{22}$$

$$T_M^{01} = \frac{c^2 + \sigma^2}{c^2 - v^2} c v \rho, \tag{23}$$

$$T_M^{11} = \frac{v^2 + \sigma^2}{c^2 - v^2} \rho c^2, \tag{24}$$

c.f., [18, 5, 4]. Here v, taken in place of $\mathbf{w}$, denotes the fluid velocity as measured by an observer fixed with respect to the radial coordinate r. It follows from (16) together with (22)-(24) that, if $r \geq r_0 > 0$, then

$$M(r,t) = M(r_0,t) + \frac{\kappa}{2} \int_{r_0}^{r} T_M^{00}(r,t) r^2 \, dr; \tag{25}$$

it follows from (3) together with (22)-(24) that the scalar curvature R is proportional to the density,

$$R = (c^2 - 3\sigma^2)\rho; \tag{26}$$

and it follows directly form (22)-(24) that

$$|T_M^{01}| < T_M^{00}, \tag{27}$$

$$\tfrac{\sigma^2}{c^2+\sigma^2} T_M^{00} < T_M^{11} < T_M^{00}. \tag{28}$$

This defines the simplest possible setting for shock wave propagation in the Einstein equations.

We prove that shock-wave solutions of (11)-(14), (4) and (21), defined outside a ball of fixed total mass, exist up until some positive time $T > 0$, and

[3] This simplifying assumption, as well as insuring that wave speeds are bounded by the speed of light for arbitrarily strong shock waves, also prevents the formation of vacuum states, and allows us to exploit special properties of the relativistic compressible Euler equations derived in [15, 16, 18, 12].

we prove that the total mass $M_\infty = \lim_{r\to\infty} M(r,t)$ is constant throughout the time interval $[0,T)$. A local existence theorem is all that we can expect for system (11)-(14) in general because black holes are singularities in standard Schwarzschild coordinates, $B = \frac{1}{1-\frac{2M}{r}} \to \infty$ at a black hole, and black holes can form in finite time. For these solutions, the fluid variables ρ, p and $\mathbf{w}$, and the components of the stress tensor T^{ij}, are *discontinuous*, and the metric components A and B are *Lipschitz continuous*, at the shock waves, c.f. (11) and (13). Since (14) involves second derivatives of A and B, it follows that these solutions satisfy (11)-(14) only in the weak sense of the theory of distributions. Thus our theorem establishes the consistency of the initial value problem for the Einstein equations at the weaker level of shock-waves.

To be precise, assume the initial boundary conditions

$$\begin{aligned} &\rho(r,0) = \rho_0(r), \qquad v(r,0) = v_0(r), \quad for\ r > r_0, \\ &M(r_0,t) = M_{r_0}, \qquad v(r_0,t) = 0, \qquad for\ t \geq 0, \end{aligned} \tag{29}$$

where r_0 and M_{r_0} are positive constants, and assume the no black hole and finite total mass conditions,

$$\frac{2M(r,t)}{r} < 1, \qquad \lim_{r\to\infty} M(r,t) = M_\infty < \infty, \tag{30}$$

hold at $t = 0$. For convenience, assume further that

$$\lim_{r\to\infty} r^2 T_M^{00}(r,t) = 0, \tag{31}$$

holds at $t = 0$, c.f., (25), (30). Our main result in [6] can be stated as follows:

Theorem 1 *Assume that the initial boundary data satisfy (29)-(31), and assume that there exist positive constants L, V and $\bar{v}$ such that the initial velocity and density profiles $v_0(r)$ and $\rho_0(r)$ satisfy*

$$TV_{[r,r+L]} \ln \rho_0(\cdot) < V, \quad TV_{[r,r+L]} \ln \left(\frac{c+v_0(\cdot)}{c-v_0(\cdot)}\right) < V, \quad |v_0(r)| < \bar{v} < c, \tag{32}$$

for all $r_0 \leq r < \infty$, where $TV_{[a,b]}f(\cdot)$ denotes the total variation of the function f over the interval $[a,b]$. Then a bounded weak (shock wave) solution of (11)-(14), satisfying (29) and (30), exists up to some positive time $T > 0$. Moreover, *the metric functions A and B are Lipschitz continuous functions of (r,t), and (32) continues to hold for $t < T$ with adjusted values for V and $\bar{v}$ that are determined from the analysis.*

Note that the theorem allows for arbitrary numbers of interacting shock waves, of arbitrary strength. Note that by (11), (13), the metric components A and B will be no smoother than Lipschitz continuous when shocks are present, and thus since (14) is second order in the metric, it follows that (14) is only satisfied in the weak sense of the theory of distributions. Note finally that $\lim_{r\to\infty} M(r,t) = M_\infty$ is a *non-local* condition.

2.1 The Proof Strategy

In previous work [5], the authors show that when the metric components A and B are Lipschitz continuous, and T is bounded in L^{∞}, (when viewed as functions of the coordinate variables (t, r, θ, ϕ)), system (11)-(14) is weakly equivalent to the following system of equations obtained by replacing (12) and (14) with the 0- and 1-components of (covariant) $DivT = 0$,

$$\left\{T_M^{00}\right\}_{,0} + \left\{\sqrt{\frac{A}{B}}T_M^{01}\right\}_{,1} = -\frac{2}{x}\sqrt{\frac{A}{B}}T_M^{01}, \tag{33}$$

$$\left\{T_M^{01}\right\}_{,0} + \left\{\sqrt{\frac{A}{B}}T_M^{11}\right\}_{,1} = -\frac{1}{2}\sqrt{\frac{A}{B}}\left\{\frac{4}{x}T_M^{11} + \frac{(B-1)}{x}(T_M^{00} - T_M^{11}) + 2\kappa x B(T_M^{00}T_M^{11} - (T_M^{01})^2) - 4xT^{22}\right\}, \tag{34}$$

$$\frac{B'}{B} = -\frac{(B-1)}{x} + \kappa x B T_M^{00}, \tag{35}$$

$$\frac{A'}{A} = \frac{(B-1)}{x} + \kappa x B T_M^{11}. \tag{36}$$

This is the system of equations that we work with here. (Cf. (4.67), (4.68) together with (3.20), (3.22) of [5].) Here, "$,i$" denotes $\partial/\partial x^i$, and T_M is defined in (22)-(24).

System (33),(34),(35),(36) forms a system of conservation laws with source terms which we write in the compact form, (c.f. [5]),

$$u_t + f(\mathbf{A}, u)_x = g(\mathbf{A}, u, x), \tag{37}$$

$$\mathbf{A}' = h(\mathbf{A}, u, x), \tag{38}$$

where

$$u = (T_M^{00}, T_M^{01}) \equiv (u^0, u^1), \tag{39}$$

$$\mathbf{A} = (A, B), \tag{40}$$

$$f(\mathbf{A}, u) = \sqrt{\frac{A}{B}}\left(T_M^{01}, T_M^{11}\right), \tag{41}$$

and

$$g(\mathbf{A},u,x) = \left(g^0(\mathbf{A},u,x), g^1(\mathbf{A},u,x)\right), \tag{42}$$

$$h(\mathbf{A},u,x) = \left(h^0(\mathbf{A},u,x), h^1(\mathbf{A},u,x)\right), \tag{43}$$

where

$$\begin{aligned} g^0(\mathbf{A},u,x) &= -\frac{2}{x}\sqrt{\frac{A}{B}}T_M^{01}, & (44)\\ g^1(\mathbf{A},u,x) &= -\frac{1}{2}\sqrt{\frac{A}{B}}\left\{\frac{4}{x}T_M^{11} + \frac{(B-1)}{x}(T_M^{00}-T_M^{11})\right. & (45)\\ &\qquad \left. +2\kappa x B(T_M^{00}T_M^{11} - (T_M^{01})^2) - 4xT^{22}\right\}, \end{aligned}$$

and

$$\begin{aligned} h^0(\mathbf{A},u,x) &= \frac{(B-1)A}{x} + \kappa x A B T_M^{11}, & (46)\\ h^1(\mathbf{A},u,x) &= -\frac{(B-1)B}{x} + \kappa x B^2 T_M^{00}. & (47) \end{aligned}$$

The vector $h(\mathbf{A},u,x)$ is just obtained by solving (11) and (13) for A' and B'. Note that we have set $x \equiv x^1 \equiv r$, and we use x in place of r in the analysis to follow since this is standard notation in the literature on hyperbolic conservation laws. Note also that we write t when we really mean ct, in the sense that t must be replaced by ct whenever we put dimensions of time, i.e., factors of c, into our formulas. We interpret this as taking $c = 1$ when convenient.

A new twist in formulation (37), (38) is that the conserved quantities are taken to be the the energy and momentum densities $u = (u^0, u^1) = (T_M^{00}, T_M^{01})$ of the relativistic compressible Euler equations in flat Minkowski spacetime–quantities that, unlike the entries of T, are independent of the metric. Note that, (remarkably), all time derivatives of metric components cancel out from the equations when this change of variables is made, c.f. [5]. We take advantage of this formulation in the numerical method that we introduce here for the study of the initial value problem.

Our proof of Theorem 1 is based on a new fractional step Glimm scheme, c.f. [13, 11, 12]. The fractional step method employs a Riemann problem step[4] that simulates the source free conservation law $u_t + f(\mathbf{A}, u)_x = 0$, ($\mathbf{A} \equiv Const$), followed by an ODE step that accounts for the sources present in both f and g. Our idea for the numerical scheme is to stagger discontinuities in the metric with discontinuities in the fluid variables so that the conservation law step, as well as the ODE step of the method, are both generated in grid rectangles on which the metric components $\mathbf{A} = (A, B)$, (as well as x), are constant. At the end of each time step, we solve $A' = h(\mathbf{A}, u, x)$ and re-discretize, to update the metric sources. Part of our proof involves showing that the ODE step $u_t = g(\mathbf{A}, u, x) - \nabla_{\mathbf{A}} f \cdot \mathbf{A}'$, with h substituted for $\mathbf{A}'$, accounts for both the source term g, as well as the *effective sources* that are due to the discontinuities in the metric components at the boundaries of the grid rectangles.

By our formulation (37), (38), only the flux f in the conservation law step, depends on $\mathbf{A}$. From this we conclude that the only effect of the metric on the Riemann problem step of the method is to change the wave speeds, but not the states of the waves that solve the Riemann problem. Thus, on the Riemann problem step, when we assume $p = \sigma^2 \rho$, we can apply the estimates obtained in [19], which were originally derived for flat Minkowski spacetime $\mathbf{A} = (1, 1)$. Applying these results, it follows that the Riemann problem is *globally* solvable in each grid cell, and the total variation in $\ln \rho$ is non-increasing in time on the Riemann problem step of our fractional step scheme, [19]. Thus we need only estimate the increase in total variation of $\ln \rho$ for the ODE step of the method, in order to obtain a local total variation bound, and hence compactness of the numerical approximations up to some time $T > 0$.

One nice feature of our method is that the ODE that accomplishes the ODE step of the method, turns out to have surprisingly nice properties. Indeed, a phase portrait analysis shows that $\rho > 0$, $|v| < c$ is an invariant region for solution trajectories. (Since x and $\mathbf{A}$ are taken to be constant on the ODE step, the ODE's form an autonomous system at each grid cell.) We also show that even though the ODE's are quadratic in ρ, solutions of the ODE's do not blow up, but in fact remain bounded for all time. It follows that the fractional step

[4]The Riemann problem is the initial value problem when the initial data is a pair of constant states centered by a jump discontinuity. For a pure conservation law of the form $u_t + f(u)_x = 0$, the solution, which typically only exists for constant states in restricted regions of u-space, consists of elementary waves, c.f. [10, 17].

scheme is defined and bounded so long as the Courant-Freidrichs-Levy (CFL) condition is maintained, [17]. We show that the CFL bound depends only on the supnorm of the metric component $\|B\|_\infty$, together with the supnorm $\|S\|_\infty$, where $S \equiv S(x,t) = x\rho(x,t)$. We go on to prove that all norms in the problem are bounded by a function that depends only on $\|B\|_\infty$ $\|S\|_\infty$, and $\|TV_L \ln\rho(\cdot,t)\|_\infty$, where the latter denotes the sup of the total variation over intervals of L. By this we show that the solution can be extended up until the first time at which one of these three norms tends to infinity. (Our analysis rules out the possibility that $v \to c$ before one of these norms blows up.) The condition $B \to \infty$ corresponds to the formation of a black hole, and $\rho \to \infty$ corresponds to the formation of a naked singularity, (because the scalar curvature satisfies $R = \{c^2 - 3\sigma^2\}\rho$). It is known that black holes can form in solutions of the Einstein equations, and it is an open problem whether or not naked singularities can form (in the time evolution of a perfect fluid), or whether we can have $\|S\|_\infty \to \infty$, or $\|TV_L \ln\rho(\cdot,t)\|_\infty \to \infty$, in some other way.

The main technical problem is to prove that the total mass $M_\infty = \frac{\kappa}{2}\int_{r_0}^\infty \rho r^2\,dr$ is bounded. The problem is that, in our estimates, the growth of ρ depends on M and the growth of M depends on ρ, and M is defined by a *non-local* integral. Thus, an error estimate of order Δx for $\Delta\rho$ after one time step, is not sufficient to bound the total mass M_∞ after one time step.

As a final comment, we note that we can view this fractional step method as a *locally inertial* version of Glimm's method in the sense that it exploits the locally flat character of spacetime. That is, the Riemann Problem step solves the equations $u_t + f(\mathbf{A},u)_x = 0$ inside grid rectangles $\mathcal{R}_{ij}$. Now each grid rectangle is an "inertial reference frame" because $\mathbf{A} \equiv Const.$ implies the metric is flat in $\mathcal{R}_{ij}$. The boundaries between these local inertial reference frames are the discontinuities that appear along the top, bottom and both sides of the grid rectangles. The term $-\nabla_{\mathbf{A}} f \cdot \mathbf{A}'$ on the RHS of the ODE step $u_t = g(\mathbf{A},u,x) - \nabla_{\mathbf{A}} f \cdot \mathbf{A}'$, accounts for the discontinuities in $\mathbf{A}$ along the sides of the grid rectangles $\mathcal{R}_{ij}$, and the term g in the ODE step, together with the imposition of the constraint $A' = h(\mathbf{A},u,x)$ at the end of each time step, account for the discontinuities in $\mathbf{A}$ at the top and bottom of each $\mathcal{R}_{ij}$. It follows that once the convergence of an approximate solution is established, one can just as well replace the true approximate solution by the solution of the Riemann problem in each grid rectangle $\mathcal{R}_{ij}$–the two differ by only order Δx. The resulting approximation scheme converges to a weak solution of the

Einstein equations, and has the property that it solves the compressible Euler equations exactly in local inertial coordinate frames, (grid rectangles), and the transformations between neighboring coordinate frames are accounted for by discontinuities at the coordinate boundaries. In this sense, the fractional step Glimm method is a locally inertial numerical method. (Note that we could not have a locally inertial method without incorporating shock waves, because the Riemann problem essentially contains shock waves.)

2.2 The Smoothness Class of the Metric

The RHS of the Einstein equations $G^{ij} = \kappa T^{ij}$ involve the fluid variables ρ, p and $\mathbf{w}$, thus it follows that when shock waves are present, T is discontinuous. Since the Einstein curvature tensor G on the LHS of (3) involves second derivative of the metric g_{ij}, one expects that, in general, the metric components $g_{ij}(x)$ should be at least $C^{1,1}$ functions of the coordinates, (that is, in the smoothness class of continuous functions with Lipschitz continuous first order derivatives), in order that the LHS of (3) be free of delta function sources, c.f. [9, 19]. However, the metric components A and B in the solutions of (11)-(14) constructed here, are only Lipschitz continuous. We know that these solutions are in fact "free of delta function sources" as a consequence of the fact that they are genuine weak solutions of (3). It remains an open problem whether or not there exist coordinate transformations that smooth the metric components of these solutions from the smoothness class $C^{0,1}$ up to the class $C^{1,1}$. In such coordinates, (3) would hold in the pointwise sense at shock waves, and hence, such a transformation would map weak solutions of the Einstein equations to strong solutions. It was pointed out in [5], (see also [19]), that the transformation that takes an arbitrary spherically symmetric metric over to a metric of form (10), necessarily involves derivatives of the metric components, and so the existence of such $C^{1,1}$ coordinates would be consistent with the fact that the A and B that solve (11)-(14) are only Lipschitz continuous at shock waves. Moreover, in [19, 9] it was shown that for a general smooth shock surface in four dimensional spacetime, such a coordinate transformation always exists, and can be taken to be the Gaussian normal coordinates at the shock surface. But the solutions constructed in [6] can contain arbitrary numbers of interacting shock-waves, of arbitrary strength–and the Gaussian normal coordinate systems break down at points where shock waves interact. With this in mind, we pose the following open question: *Given a weak solution of the Einstein equations for which the metric components are only $C^{0,1}$ functions of the coordinate variables, does there always exist a coordinate transformation that improves the regularity of the metric components to $C^{1,1}$ when the components are viewed as functions of the transformed coordinate variables?* In particular, we ask if this statement is true for the $C^{0,1}$ solutions that we have constructed in [6]?

We believe that this question goes to the heart of the issue of the regularity of solutions of the Einstein equations. Indeed, the Einstein equations are inherently hyperbolic in character; that is, there is finite speed of propagation because all wave speeds are bounded by the speed of light. It follows that, unlike Navier Stokes type parabolic regularizations of the classical compressible Euler equations, incorporating the effects of viscosity and dissipation into Einstein's theory of gravity, cannot alter the fundamental hyperbolic character of the Einstein equations themselves. Thus, even when dissipative effects are accounted for, it is not clear apriori that the corresponding solutions of the Einstein equations will in general be more regular than the solutions that we have constructed in [6]. We also note that the singularity theorems in [8] presume that metrics are in the smoothness class $C^{1,1}$, one degree *smoother* than the solutions we have constructed, c.f. [8], page 284.

In summary, if a transformation exists that improves the regularity of solutions of the Einstein equations from the class $C^{0,1}$ up to the class $C^{1,1}$, then it defines a mapping that takes weak solutions of the Einstein equations to strong solutions. It then follows that in general relativity, the theory of distributions and the Rankine Hugoniot jump conditions for shock waves need not be imposed on the compressible Euler equations as extra conditions on solutions, but rather must follow as a logical consequence of the strong formulation of the Einstein equations by themselves. If such a transformation does not always exist, then solutions of the Einstein equations are one degree less regular than previously assumed.

For details we refer the reader to [5, 6], and to the authors' forthcoming book [7].

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