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A BOUND ON THE TOTAL VARIATION OF THE CONSERVED QUANTITIES FOR SOLUTIONS OF A GENERAL RESONANT NONLINEAR BALANCE LAW*

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Abstract. We introduce a new potential interaction functional and use it to define a new Glimm-type functional that bounds the total variation of the conserved quantities at time $t > 0$ by the total variation at time $t = 0+$ in Glimm approximate solutions of a general resonant nonlinear balance law.

Key words. shock waves, resonance, Glimm scheme, balance laws

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1. Introduction.

In [13], Isaacson and Temple introduced the 2×2 system

$$\begin{aligned} &a_t = 0, \\ &u_t + f(a,u)_x = a' g(a,u) \end{aligned} \tag{1.1}$$

as a general nonlinear balance law that models resonance between a nonlinear wave field and a stationary source (cf. [5, 7, 8, 9, 10, 11, 14, 17, 19, 20, 21, 22, 23, 25, 26]). Here a and u are assumed to be scalar valued, and resonance occurs at states $U_* = (a_*, u_*)$, where the nonlinear wave speed $\lambda = f_u$ vanishes. Assume further that f and g are smooth functions and that the following conditions are satisfied at the state U_*:

$$f_u(U_*) = 0, \tag{1.2}$$

$$g(U_*) - f_a(U_*) \neq 0 \quad \text{(w.l.o.g. assume } g(U_*) - f_a(U_*) > 0), \tag{1.3}$$

$$f_{uu}(U_*) \neq 0 \quad \text{(w.l.o.g. assume } f_{uu}(U_*) < 0), \tag{1.4}$$

and

$$g_u(U_*) \neq 0. \tag{1.5}$$

It was shown in [13] that the generic conditions (1.2)–(1.5) imply that the structure of elementary wave curves (shock waves, rarefaction waves, and standing waves) and the solution of the Riemann problem (the initial value problem when the initial data consists of constant states U_L, U_R, separated by a discontinuity) are canonical[1] in a neighborhood Ω of the state U_*; cf. [16, 13, 24]. (The cases $g_u > 0$ and $g_u < 0$ are

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[1]Here we show that the condition $f_a \neq 0$, stated as a condition for genericity in [13], is not required, except by (1.3) in the case $g = 0$; cf. the construction of the zero speed shock curve below.

qualitatively different.) Here $a' \equiv a_x \equiv \frac{da}{dx}$, and $a = a(x)$ is an inhomogeneous term that is treated as a variable so that (1.1) takes the form of a system of two equations that expresses the dependence of the solution on the source a.

In this paper we introduce a new potential interaction functional and use it to construct a nonlinear Glimm functional that is positive decreasing on solutions of (1.1) and bounds the total variation of the conserved quantity u in terms of the initial data for all time $t > 0$. We show that the functional is always locally finite at time $t = 0+$ of the random choice method, and so the limit solution will be of bounded total variation for all time so long as this functional is bounded uniformly at $t = 0+$ as the mesh length $\Delta x \to 0$. This then gives a condition on the initial data that guarantees the solution will be of bounded total variation in u for all time. Moreover, the potential interaction estimate can be interpreted as the best possible estimate for the increase in total variation in u that can occur due to the interaction of an initial set of waves, taking no account of the initial distances between the waves or the times at which pairs of waves will interact. As part of our proof, we show that the only potential for increase of total variation is due to the interaction of rarefaction waves and standing waves. An immediate consequence of this is a proof that *the total variation of u at any $t > 0$ will be uniformly bounded by a constant times the total variation of u at $t = 0+$ in any weak solution of* (1.1) *generated by the generalized Glimm method, which initially consists entirely of shock waves and standing waves.*

The lack of a total variation estimate in the conserved quantities is the main obstacle to extending the results in [25, 13] to systems of equations (that is, when u is a vector instead of a scalar), and this is the primary motivation for our work. An important example of a system of form (1.1) is given by the equations for compressible Euler flow in a variable area duct:

$$\begin{aligned}
&a_t = 0,\\
&\rho_t + (\rho u)_x = -\frac{a'}{a}\rho u,\\
&(\rho u)_t + (\rho u^2 + p)_x = -\frac{a'}{a}\rho u^2,\\
&(\rho E)_t + (\rho E u + pu)_x = -\frac{a'}{a}(\rho E u + pu),
\end{aligned} \tag{1.6}$$

where ρ is the density, p is the pressure, E is the energy density, and $a(x)$ is the diameter of the duct at position x [2]. It is a mathematical open problem to show that wave strengths remain bounded in the time evolution of solutions of (1.6) in a neighborhood of a point of resonance U_* when the flow is transonic; cf. [1]. The main thrust of this paper is thus to establish total variation estimates for (1.1) that can be extended to a general class of systems of form (1.1), which includes (1.6). Now the total variation in the conserved quantity u at time $t > 0$ in a solution of (1.1) is not in general bounded by any uniform constant times the total variation of u at time zero in the presence of resonance. In fact, solutions of the linearization of (1.1) about $U = U_*$ grow unboundedly as $t \to \infty$ [13]. In [25, 13] a time independent bound on the supnorm and global existence of weak solutions is demonstrated based on obtaining a time independent total variation estimate for solutions in the coordinate system of Riemann invariants,[2] which is related to the conserved variables $U = (a, u)$ by a *singular* coordinate transformation. These estimates do not carry over naturally to

[2]The fact that solutions are bounded at all is thus a purely nonlinear effect.

systems like (1.6), in which u is a vector. Indeed, Glimm's method indicates that a time independent bound on the total variation of the conserved quantities is needed to extend the analysis to systems. To establish a bound on the total variation of the conserved quantity U, we introduce a singular transformation *of the coordinate system of Riemann invariants* and give essentially the best possible bound on the total variation at time $t > 0$ in terms of the initial data in these coordinates, which are regular with respect to the coordinates of conserved quantities. Our method of analysis is then to adapt the linear functional introduced in [25, 13] over to these new coordinates (which requires a correction term for the wave strengths of certain standing waves in order to make the linear part of the functional continuous) and then to add a potential interaction term for rarefaction wave-standing wave interactions to account for the fact that the functional is not contractive (decreasing in time) in these new coordinates. The total variation bounds on the solutions imply supnorm bounds, and these bounds help explain why, as waves interact due to the nonlinearity of wave speeds, solutions of the nonlinear problem (1.1) do not blow up like the resonant linear equation but rather decay to time asymptotic wave patterns given by the solutions of the Riemann problem.

We use the notation $U = (a, u)$, $\mathcal{F} = (0, f)$, $G = (0, a'g)$ so that the initial value problem for (1.1) is a special case of the general initial value problem,

$$\begin{aligned} U_t + \mathcal{F}(U)_x &= G(U), \\ U(x, 0) &= U_0(x). \end{aligned} \tag{1.7}$$

The advantage of treating systems in the form (1.1) instead of general systems of form (1.7) is that for system (1.1) we can define a generalized Riemann problem and analyze solutions by Glimm-type methods that can be applied, in principle, to systems of equations. The point of incorporating the a' term in front of g on the right-hand side of (1.1) is that it ensures that standing waves can be rescaled into discontinuities [13, 6]. It was shown in [6] that in the strictly hyperbolic regime, general source terms can be treated like contact discontinuities in such a way that the Riemann problem of Lax, and the random choice method of Glimm, both extend *virtually unchanged* to systems of the form (1.1)—that is, general systems with sources can be treated numerically just as the source-free equations. Of course, since the right-hand side of (1.1) involves the derivative a', there is no classical weak formulation of (1.1) when a is discontinuous—you cannot multiply a delta function by a discontinuous function in the classical theory of distributions; cf. [3]. Thus, the generalized Riemann problems used to construct the Glimm approximates are *weaker than weak* solutions of the equations; cf. [6]. To justify the method, it is important to show that the limits of approximate solutions of the generalized Glimm method are veritable weak solutions of (1.1) *when system* (1.1) *has a weak formulation*, namely, when $a(x)$ is Lipschitz continuous. This is accomplished in [6].[3] The interesting point to make here is that because the Riemann problems are based on approximating $a(x)$ by piecewise constant states, it follows that the Glimm scheme approximates can give only a C^0 and not a C^1 approximation of $a(x)$, and thus a' is not well approximated in L^1. Even so, Hong showed in [6] that for any test function ϕ, the residual and, in particular, $\int_{t\geq 0} a'g(a, u)\phi(x, t)dxdt$ converges not by L^1 convergence (as in Glimm's original results) but weakly, by *oscillation*, when a is Lipschitz continuous; cf. [21]. This argument, appropriately modified for the resonant

[3] Note also that every a of bounded variation can be approximated by a Lipschitz continuous function.

case considered here, is presented in section 6 below. Interestingly, three mollification parameters are needed to conclude the proof of convergence of the residual in section 6.

In section 2 we review the results in [13]; we define the regular transformation $(a,u) \to (a,w)$ and the linear functional $L_w(J)$ and compare these to the singular transformation $(a,u) \to (a,z)$ and linear functional $L_z(J)$ defined in [25, 13].[4] We then review the solution of the Riemann problem and construct the admissible solution $[U_L, U_R]$ based on an L_w minimization principle that is finer than the L_z minimization principle introduced in [13]. The L_w minimization is required for the subsequent analysis. The nonuniqueness of solutions of the Riemann problem even in the presence of the classical entropy condition for the nonlinear waves reflects an interesting instability in the time asymptotics of solutions of (1.1).

In section 3 we construct the approximate solutions $U_{\Delta x}$ by the generalized Glimm method. For a given approximate solution, the functionals $L_w(J)$ and $L_z(J)$ both sum the strengths of waves that cross an I-curve J with weight factors according to whether the wave is a nonlinear wave, a *weak* standing wave, or a *strong* standing wave, respectively; cf. [25].[5] The purpose of the weight factors is to make $L_z([U_L, U_R])$ and $L_w([U_L, U_R])$ continuous functions of U_L and U_R for the admissible solution of the Riemann problem $[U_L, U_R]$ (cf. [16, 24] and (2.1) below). Now it was shown in [13] that the weight factors $1, 2,$ and 4 on nonlinear waves, weak standing waves, and strong standing waves, respectively, suffice to make L_z continuous (these weights were introduced in [25]). We show here that in the case $g = 0$, the weight factors $1, 2, 4$ also suffice to make $L_w([U_L, U_R])$ continuous functions of U_L and U_R. However, when $g_u \neq 0$, we must adjust the definition of strength for the standing waves in order to preserve continuity when the standing wave curves diverge from the zero speed shock curves; cf. [13]. It was shown in [13] that the functional L_z is positive and nondecreasing across interaction diamonds Δ that lie between successive I-curves J_1 and J_2 in an approximate Glimm scheme solution, and $L_z(J)$ bounds the total variation in (a,z) of the solution along J [24, 4]. On the other hand, $L_w(J)$ bounds the total variation in (a,w) (and hence also the total variation in (a,u)) along an I-curve J but does not decrease across interaction diamonds.

In section 4 we define the interaction potential $d(\gamma_0, \gamma_r)$ between a rarefaction wave γ_r and a standing wave γ_0, and in section 5 we define the nonlocal Glimm functional, $P(J) = \sum_{(\alpha,\beta)\in App(J)} d(\gamma_0^\alpha, \gamma_r^\beta)$, and prove that the functional $F(J) = L_w(J) + P(J)$ decreases across interaction diamonds Δ, where the sum is taken over all approaching waves that cross J in a Glimm approximate solution. From this we establish the total variation bound for the generalized Glimm approximates and thus conclude the main total variation bound in the conserved variables (a,u) for solutions of the resonant nonlinear system (1.1). It is fortunate that at the transitions between regions where the structure of the admissible solution Riemann problem changes, the Riemann problem *never* involves rarefaction waves. Moreover, rarefaction waves are never created by interaction, and thus, since the potential interaction functional P only requires the potential for rarefaction waves to interact with standing waves, it follows that the continuity of both P and F is also maintained as states cross

[4]The functionals L_z were labeled "F" in [25] and [13], but we refer to these as L_z here because they contain no potential interaction term and are therefore linear on sequences of elementary waves; cf. [4].

[5]A standing wave is *strong* if the jump in u across the wave has the same sign as the jump in u across a shock wave and *weak* if the jump has the same sign as the jump in u across a rarefaction wave; cf. [13].

transitional boundaries between different regions of the Riemann problem.

In section 6 we modify the argument in [6] and prove the convergence of the residual when a is Lipschitz continuous.

2. Review of the Riemann problem. The Riemann problem is the initial value problem with initial data given at $t = 0$ by the jump discontinuity

$$U_0(x) = \begin{cases} U_L = (a_L, u_L) & \text{if } x < 0, \\ U_R = (a_L, u_L) & \text{if } x > 0. \end{cases} \tag{2.1}$$

The solution of the Riemann problem for (1.1), assuming (1.2)–(1.5), was first described in [13]. The solutions that minimize L_z were constructed within the class of shock waves, rarefaction waves, and standing waves, and the solution was thereby shown to have a canonical structure for pairs of states U_L and U_R in a sufficiently small neighborhood of U_*. In this section we review the solution of the Riemann problem and define the functionals L_z and L_w.

To motivate this, we note that by [13], near a point of resonance U_* of system (1.1), solutions of (2.1) have an interesting multiplicity of solutions even when the standard entropy condition for shocks is imposed on the nonlinear waves. An additional admissibility condition is required to fix a unique solution. For system (1.1) in the case $g = 0$, uniqueness is implied by the Lax entropy condition for shocks, together with the condition that the wave curves for the waves that solve the Riemann problem should lie between the values of a on the left and right; cf. [13]. This is a natural condition if one views the discretization of a as approximating a smooth duct—the time asymptotic wave pattern will depend on the interior structure of the duct as well as the left and right most diameters. However, when $g \neq 0$, system (1.1) has a more interesting and nontrivial multiplicity of solutions of the Riemann problem: in certain cases, there is a multiplicity of three distinct solutions of the Riemann problem that preserve the bounds in a from the left and right, and these reduce to two possible solutions at boundary cases. The main purpose of this section is to define the functional L_w and show that the following admissibility condition is sufficient to pick out a unique solution of the Riemann problem (except of course for a dual ambiguity at the boundary regions where the qualitative wave structure changes).

DEFINITION 2.1. *A solution of the Riemann problem* (2.1) *is called* admissible *if it minimizes L_w among all other solutions of the Riemann problem that preserve the bounds in a and contain only Lax entropy shocks.*[6]

In contrast, the admissibility criterion in [13], which requires that L_z be minimized, still leaves some ambiguity in cases where there are three solutions. We let $[U_L, U_R]$ denote the admissible solution of the Riemann problem, and we will show that $[U_L, U_R]$ always consists of three elementary waves: a negative speed nonlinear wave followed by a single standing wave followed by a positive speed nonlinear wave. However, in two cases diagrammed in Figures 15 and 17, the standing wave must be taken to be what we call a *triple composite standing wave*, a wave that consists of a standing wave followed by a zero speed shock wave followed by a second standing wave.

To start, let γ denote an arbitrary elementary wave, and let subscripts $q = 0, r, s$ identify the wave as a standing wave, rarefaction wave, or shock wave, respectively.

[6] Unlike L_z, L_w is *not* minimized on solutions among all connected sequences of elementary waves that take U_L to U_R, and if it were, $F(J) = L_w(J)$ would decrease on solutions, and no potential interaction term would be required in our analysis; cf. [13].

A wave γ is determined by its left and right states, and we say that $\gamma_a \cdots \gamma_b$ is a *connected* sequence of elementary waves that takes U_L to U_R if the right state of any wave in the list is equal to the left state of its successor in the list, and U_L, U_R is the left, right state of the first, last wave in the list, respectively. Thus the admissible solution of the Riemann problem $[U_L, U_R]$ is just a particular connected sequence of elementary waves that takes U_L to U_R. Two connected sequences of elementary waves $\gamma_a \cdots \gamma_b$ and $\bar{\gamma}_a \cdots \bar{\gamma}_b$ are said to be *similar* if they both take U_L to U_R, in which case we write $\gamma_a \cdots \gamma_b \sim \bar{\gamma}_a \cdots \bar{\gamma}_b$. For a nonlinear wave γ that takes U_L to U_R, we say that $\gamma \sim \gamma^a \gamma^b$ is a *partition* of the wave γ if the state U_M, the right state of γ^a and the left state of γ^b, lies strictly between U_L and U_R in the (a,u)-plane. (We allow both rarefaction waves and shock waves to be partitioned.)

To begin the review of the Riemann problem, we first remind the reader that system (1.1) has standing wave solutions that can be rescaled into discontinuities so that the standing waves can be treated like a family of contact discontinuities in the theory of hyperbolic conservation laws [13, 6]. Indeed, let $(a(x), u(x))$ be a standing wave (i.e., time independent) solution of (1.1). Then

$$\frac{d}{dx} f = a' g,$$

which is equivalent to

$$f_a da + f_u du = g da.$$

We rewrite this as

$$(f_a - g) da + f_u du = 0. \tag{2.2}$$

The nondegeneracy assumption (1.3) implies that $f_a - g \neq 0$ in a neighborhood of U_*, and therefore (2.2) is equivalent to the autonomous ODE

$$\frac{da}{du} = \frac{f_u}{g - f_a}. \tag{2.3}$$

This equation has a unique solution through each point in a neighborhood of U_* in the (a,u)-plane. Thus, for any solution $a = a_s(u)$ of (2.3) and any smooth function $\varphi(x)$, the curve $u = \varphi(x), a = a_s(\varphi(x))$ is a standing wave solution of (1.1). Moreover, if $a_L = a_s(u_L)$ and $a_R = a_s(u_R)$, then the standing wave discontinuity

$$U(x,t) = \begin{cases} (a_L, u_L) & \text{if } x < 0, \\ (a_R, u_R) & \text{if } x > 0 \end{cases} \tag{2.4}$$

is obtained as a limit of smooth solutions; specifically, if $\varphi_\epsilon(x) \to \varphi_0(x)$, where

$$\varphi_0(x) = \begin{cases} u_L & \text{if } x < 0, \\ \\ u_R & \text{if } x > 0, \end{cases}$$

then $U_\epsilon = (a_s(\varphi_\epsilon(x)), \varphi_\epsilon(x)) \to U(x,t)$. Thus we can view the standing wave discontinuities defined in (2.4) as a family of elementary waves for system (1.1), similar to a family of contact discontinuities.

The standing wave curves define solutions of (2.2). Note that for a standing wave,

$$\frac{da}{du} = 0 \ \text{ if and only if } \ f_u = 0. \tag{2.5}$$

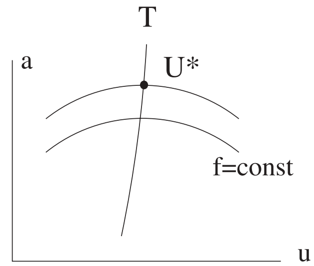

FIG. 1.

Moreover, if $da/du = 0$, then

$$\frac{d^2a}{du^2} = \frac{f_{uu}}{g - f_a} < 0. \tag{2.6}$$

Thus, $d^2a/du^2 < 0$ in a neighborhood of U_*.

DEFINITION 2.2. *The* transition curve $\mathcal{T}$ *associated with system* (1.1) *is the set*

$$\mathcal{T} = \{(a, u) : \lambda \equiv f_u = 0\}. \tag{2.7}$$

Since $f_{uu} \neq 0$, the implicit function theorem implies that (in a neighborhood of U_*) $\mathcal{T}$ is a smooth curve passing through U_*, which we denote by

$$u = u_{\mathcal{T}}(a). \tag{2.8}$$

The curve $\mathcal{T}$ comprises the states near U_* for which the nonlinear wave speed $\lambda \equiv f_u$ is zero. By (2.5) and (2.6), the standing wave curves $u \mapsto (a_s(u), u)$ are convex down, cross $\mathcal{T}$ transversally, and maximize a on $\mathcal{T}$ in some neighborhood of U_*. (The notation comes from [7]. See Figure 1.)

We now define the zero speed shock curve corresponding to a given standing wave curve. By our choice of signs ($f_{uu} < 0$ and $g - f_a > 0$), the entropy shock waves (see [24]) for the nonlinear scalar conservation law $u_t + f(a, u)_x = 0$ jump always from left to right in the (x, t)-plane and (a, u)-plane simultaneously; thus, by the Rankine–Hugoniot jump relation for shocks,

$$s[u] = [f],$$

the zero speed shocks ($s = 0$) cross $\mathcal{T}$ from left to right at a constant value of f.

Now, for a given standing wave $a = a_s(u)$ and a given state (a, u) on this standing wave, define $\bar{u}$ to be the value of u such that the state $(a, \bar{u})$ lies on the opposite side of $\mathcal{T}$ at the same a-level and on the same standing wave curve as the given state (a, u). If the state $U = (a, u)$ lies on the left-hand side of $\mathcal{T}$ (we write $U < \mathcal{T}$), then define $\tilde{u}$ to be the value of u such that the state $(a, \tilde{u})$ lies on the right-hand side of $\mathcal{T}$ and at the same level a, but on the same constant f curve as the given state (a, u). That is, for $U < \mathcal{T}$, $\bar{u}$ satisfies

$$a_s(\bar{u}) = a_s(u), \tag{2.9}$$

and $\tilde{u}$ satisfies

$$f(a, \tilde{u}) = f(a, u) \tag{2.10}$$

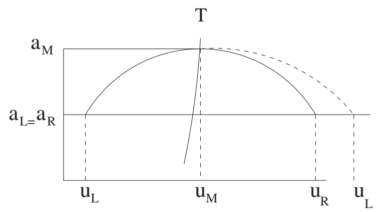

FIG. 2.

(see Figure 1).

DEFINITION 2.3. *Let $a = a_s(u)$ be a standing wave curve. Then, (assuming $f_{uu} < 0$) the zero speed shock curve corresponding to standing wave curve a_s is the curve (lying to the right of $\mathcal{T}$) defined by*

$$\{\tilde{u} : f(a, \tilde{u}) = f(a, u) \text{ where } u \leq u_{\mathcal{T}}(a) \text{ and } \tilde{u} \geq u_{\mathcal{T}}(a)\}.$$

(When $f_{uu} > 0$, we change to $u \geq u_{\mathcal{T}}(a)$ and $\tilde{u} \leq u_{\mathcal{T}}(a)$.)

LEMMA 2.4. *If $g_u < 0$, then for each standing wave curve $a = a_s(u)$, the corresponding zero speed shock curve lies to the right of the standing wave curve in the (a, u)-plane. That is, if (a, u) satisfies $a = a_s(u)$ with $u < u_{\mathcal{T}}(a)$, then*

$$f(a, \bar{u}) < f(a, \tilde{u}) = f(a, u). \tag{2.11}$$

If $g_u > 0$, then the corresponding zero speed shock curve lies to the left of the standing wave curve in the (a, u)-plane. That is,

$$f(a, \bar{u}) > f(a, \tilde{u}) = f(a, u). \tag{2.12}$$

For example, in the case $g_u < 0$, Lemma 2.4 implies that the zero speed shock curve lies above and to the right of the standing wave curve $a_s(u)$ (see Figure 2). (For a proof of Lemma 2.4, see [13, Lemma 2.4, p. 13], and note that the condition $f_a \neq 0$ was not required.)

We now define the nonsingular coordinate w and functional L_w and formulate the L_w minimization principle to select a unique admissible solution of the Riemann problem. To construct L_w, we first construct w and a functional L_w^* that is analogous to the construction of the singular coordinate z and functional L_z defined in [25, 13], and then we obtain L_z by modifying L_w^* so that $L_w^*[U_L, U_R]$ depends continuously on U_L and U_R. To start, we first review the construction of z and L_z for system (1.1).

The coordinate z is based on the singular coordinate system of nonlinear hyperbolic wave curves (a =constant) and standing wave curves ($a = a_s(u)$) as observed in the (a, u)-plane and is defined as follows. For each point (a, u), let $(a_{\mathcal{T}}, u_{\mathcal{T}})$ denote the unique point where the standing wave curve through (a, u) crosses $\mathcal{T}$, and set

$$z(a, u) = \operatorname{sgn}(u - u_{\mathcal{T}})|a - a_{\mathcal{T}}|.$$

Using this, define the strength $|\gamma|_z$ of an elementary wave γ by

$$|\gamma|_z = \begin{cases} |z(U_R) - z(U_L)| & \text{if } \gamma \text{ is a nonlinear wave,} \\ 2|z(U_R) - z(U_L)| & \text{if } \gamma \text{ is a weak standing wave,} \\ 4|z(U_R) - z(U_L)| & \text{if } \gamma \text{ is a strong standing wave.} \end{cases} \tag{2.13}$$

Here a standing wave is *weak* if the jump in u across the wave is in the direction of a rarefaction wave ($u_R < u_L$ since we assume $f_{uu} < 0$) and is *strong* if the jump in u across the wave is in the direction of a shock wave ($u_R > u_L$ when $f_{uu} < 0$); cf. [25, 21]. For a sequence of elementary waves $\gamma_1, \ldots, \gamma_n$, define

$$L_z[\gamma_1, \ldots, \gamma_n] = \sum_{i=1}^{n} |\gamma_i|_z. \tag{2.14}$$

Analogously, define the nonsingular coordinate w by

$$w(a,u) = \begin{cases} u - u_{\mathcal{T}} & \text{if } u < \mathcal{T}, \\ u_{\mathcal{T}} - \bar{u} & \text{if } u > \mathcal{T} \end{cases}$$

and the strength $|\gamma|_w$ of an elementary wave γ by

$$|\gamma|_w^* = \begin{cases} |w(U_R) - w(U_L)| & \text{if } \gamma \text{ is a nonlinear wave}, \\ 2|w(U_R) - w(U_L)| & \text{if } \gamma \text{ is a weak standing wave}, \\ 4|w(U_R) - w(U_L)| & \text{if } \gamma \text{ is a strong standing wave}. \end{cases} \tag{2.15}$$

For a sequence of elementary waves $\gamma_1, \ldots, \gamma_n$, define

$$L_w^*[\gamma_1, \ldots, \gamma_n] = \sum_{i=1}^{n} |\gamma_i|_w^*. \tag{2.16}$$

We next show that the change in w across an elementary wave bounds the change in u across the wave in any neighborhood Ω of U_* that is sufficiently small.[7] This is guaranteed by the simpler condition stated in the following lemma. (Since the change in u across a nonlinear wave is equal to the change in w across the wave, the only issue is with the standing waves.)

LEMMA 2.5. *Let γ_0 denote a standing wave with left state U_L and right state U_R, both states lying on one side of the transition curve. Then for Ω sufficiently small, there exists a constant $c > 1$ such that the condition $U_L, U_R \in \Omega$ implies that the absolute change in u across γ_0 between a_L and a_R is always that constant times larger than the absolute change in u along the transition curve $\mathcal{T}$ between a_L and a_R.*

Proof. We verify the lemma in the case diagrammed in Figure 3 (other cases are similar). Thus we show that for Ω sufficiently small, there exists $c > 1$ such that if $U_L, U_R \in \Omega$, then $|DF| > c|GG'|$. (We use the notation that $|DF|$ denotes the absolute change in u between states D and F.) But $|DF| = |DC|$ is the change in u across the wave γ_0. Thus, by construction of the standing wave curves, we know that

$$\frac{du}{da} = \frac{g - f_a}{f_u}$$

along a standing wave curve, so by the mean value theorem

$$|DC| = \frac{g - f_a}{f_u} |a_R - a_L|, \tag{2.17}$$

[7] We treat the local problem here because it demonstrates that the analysis is generic in a neighborhood of any state U_*, but all of this can be globalized to apply to any neighborhood Ω where the solution of the Riemann problem has the canonical structure described in section 2 such that Lemma 2.5 applies.

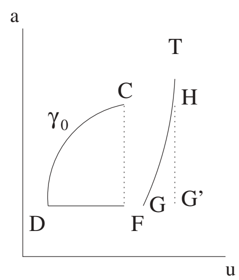

FIG. 3.

where it is understood that $\frac{g-f_a}{f_u}$ is evaluated at some point in Ω. Also $|GG'| = |GH|$ is the change in u along $\mathcal{T}$ between a_L and a_R. Since G and H lie on $\mathcal{T}$, we have $f_u(H) = f_u(G) = 0$, and so differentiating and using the mean value theorem we obtain that

$$|GH| = \frac{f_{ua}}{f_{uu}}|a_R - a_L|. \tag{2.18}$$

Since f_u can be taken arbitrarily small in a neighborhood of $\mathcal{T}$, it follows from (2.17) and (2.18) that there exists a constant $c > 1$ such that $|DC| > c|GH|$ so long as $U_L, U_R \in \Omega$. $\square$

COROLLARY 2.6. *Assume that Ω is sufficiently small so that Lemma 2.5 holds. Then there exists a constant $c > 0$ such that, if $U_L, U_R \in \Omega$, then*

$$c^{-1}|u_L - u_R| < |\gamma|_* < c|u_L - u_R|. \tag{2.19}$$

Proof. The second inequality in (2.19) is clear by construction. We verify the first inequality in (2.19) in the case of a standing wave $|\gamma_0|$ diagrammed in Figure 3. (Again, there is no issue for nonlinear waves, and the cases for other standing waves are similar, because we always assume that standing waves do not cross $\mathcal{T}$). In the case of Figure 3, $|\gamma_0| = 4\{|w(C) - w(D)|\}$. However,

$$\begin{aligned}\frac{1}{4}|\gamma_0| = |w(C) - w(D)| &= ||CH| - |DG|| = ||FG'| - |DG|| = ||DF| - |GG'|| \\ &\geq \big||DF| - c^{-1}|DF|\big| = \left(1 - \frac{1}{c}\right)|DF| = \left(1 - \frac{1}{c}\right)|u(F) - u(D)|\end{aligned}$$

for the $c > 1$ of Lemma 2.5. It follows that

$$|u_L - u_R| \leq \frac{1}{4}\left(1 - \frac{1}{c}\right)^{-1}|\gamma_0|,$$

which proves the corollary. $\square$

From here on out, we always assume that all states lie in a region Ω where lemma 2.5 and Corollary 2.6 apply.

In order to deduce the solution of the Riemann problem from a minimization principle, we will use the following property of the functional L_w^*.

LEMMA 2.7. *Let points A, B, C, D denote the vertices of a region in U-space bounded on either side by standing wave curves and above and below by nonlinear wave curves such that the region lies entirely on one side of the transition curve.*

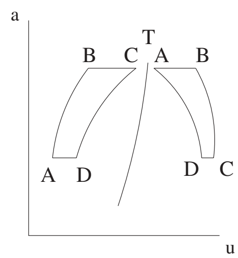

FIG. 4.

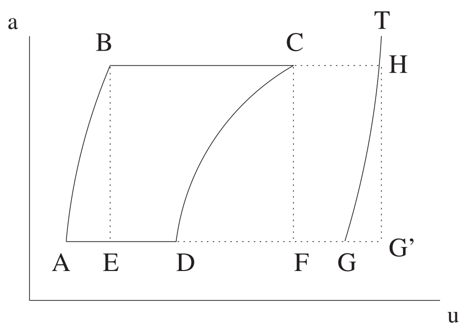

FIG. 5.

Assume that the vertices of the two possible such regions of this type are labeled with the orientation shown in Figure 4. *Then*

$$L_w^*(A \to B \to C) \leq L_w^*(A \to D \to C), \tag{2.20}$$

$$L_w^*(D \to A \to B) \leq L_w^*(D \to C \to B). \tag{2.21}$$

(Again, we use the convention that an elementary wave can be denoted by the left and right states of the wave separated by an arrow.)

Proof. We verify (2.20) in the case diagrammed in Figure 5, which is similar to Figure 3 of Lemma 2.5. (The other cases are similar.) Referring to Figure 5, we can estimate

$$\begin{aligned} & L_w^*(A \to D \to C) - L_w^*(A \to B \to C) \\ &\qquad = |AD| + 4|w(C) - w(D)| - |EF| - 4|w(B) - w(A)| \\ &\qquad = |AE| - |DF| + 4||CH| - |DG|| - 4||BH| - |AG||. \end{aligned} \tag{2.22}$$

But by Lemma 2.5,

$$\begin{aligned} ||CH| - |DG|| &= |DF| - |GG'|, \\ ||BH| - |AG|| &= |AE| - |GG'|. \end{aligned}$$

Substituting these into (2.22) gives

$$\begin{aligned} & L_w^*(A \to D \to C) - L_w^*(A \to B \to C) \\ &\qquad = 3|DF| - 4|GG'| + |AE| - 4|AE| + 4|GG'| \\ &\qquad = 3|DF| - 3|AE| \geq 0, \end{aligned}$$

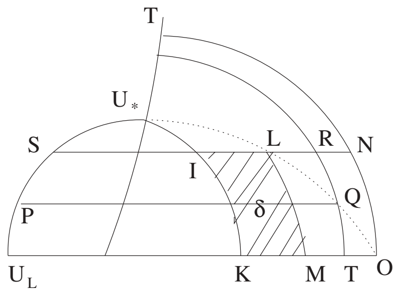

FIG. 6.

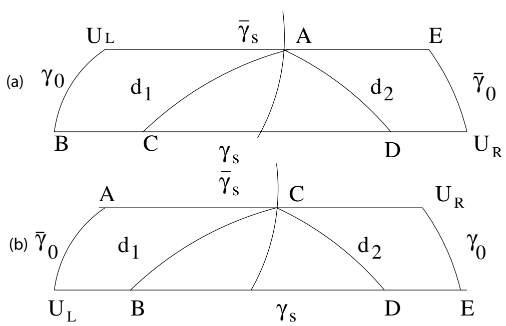

FIG. 7.

where we have used $|AE| \geq |GG'|$ by Lemma 2.5 to conclude the last line. □

The following is a simple corollary of Lemma 2.7.

COROLLARY 2.8. *If γ_a and γ_b are two standing waves on the same side of $\mathcal{T}$ that pass between the same values of a, then $L_w^*(\gamma_a) > L_w^*(\gamma_b)$ if γ_a is the wave closer to the transition curve $\mathcal{T}$.*

The next lemma provides an important continuity property of the functional L_w^* for waves that cross the transition curve.

LEMMA 2.9. *Consider the interaction $\bar{\gamma}_0 + \bar{\gamma}_s \to \gamma_s + \gamma_0$ diagrammed in Figure 7(a). Then, referring to the points referenced in that diagram, we have*

$$L_w^*(U_L \to A \to U_R) = L_w^*(U_L \to E \to U_R) \tag{2.23}$$

and

$$\begin{aligned} d_1 &\equiv L_w^*(U_L \to B \to C) - L_w^*(U_L \to A \to C) \\ &= L_w^*(D \to C \to U_R) - L_w^*(D \to E \to U_R) \equiv d_2. \end{aligned} \tag{2.24}$$

Moreover, statement (2.23) *also holds for the analogous points diagrammed in Figure* 7(b)*, together with*

$$\begin{aligned} d_1 &\equiv L_w^*(U_L \to A \to C) - L_w^*(U_L \to B \to C) \\ &= L_w^*(A \to D \to U_R) - L_w^*(A \to E \to U_R) \equiv d_2. \end{aligned} \tag{2.25}$$

Proof. We verify (2.23) and (2.25). For (2.23), let F and G denote the points such that $L_w^*(A \to U_R) = L_w^*(F \to G)$. Then by the $1, 2, 4$ weightings on wave strengths, it follows that $|\bar{\gamma}_0|_* = L_w^*(U_L \to A) = L_w^*(U_L \to F) + L_w^*(G \to E) + L_w^*(E \to U_R)$. This is enough to verify (2.23). For (2.25), note that $L_w^*(U_L \to A \to U_R) = d_1 + L_w^*(U_L \to B \to C \to U_R) = d_1 + L_w^*(U_L \to D \to C \to U_R) = d_1 - d_2 + L_w^*(U_L \to E \to U_R)$, so by (2.23), $d_1 = d_2$. ▯

We now define L_w in terms of L_w^*. To this end, note that because of Lemma 2.7, the functional $L_w^*([U_L, U_R])$ will be a continuous function of U_L and U_R on the admissible solution of the Riemann problem *only in the case* when $g_u \equiv 0$, and in this case, we can take $L_w \equiv L_w^*$. However, when $g_u \neq 0$, we show below that the functional $L_w([U_L, U_R])$ will not be continuous everywhere (for any choice of admissible solution of the Riemann problem) due to the divergence of the zero speed shock curve from the standing wave curves when $g_u \neq 0$. Moreover, we must modify the definition of wave strength for the triple composite standing waves, (described by the wave $U_L \to P \to Q \to R$ in Figure 15 and Figure 17, when $g_u < 0$ and $g_u > 0$, respectively) in order to insure that L_w is minimized on a triple composite standing wave. The idea is to first modify the strength of a triple composite standing wave to be equal to the strength of the two waves (a positive speed shock wave followed by a standing wave on the right when $g_u < 0$, and a standing wave on the left followed by a negative speed shock wave when $g_u > 0$) that *would* solve the same Riemann problem in the case $g_u = 0$. We call these two waves the *projection* of the triple composite wave γ_0, and label it $P(\gamma_0)$. By so changing the wave strength, we introduce a new discontinuity in the functional L_w^* that must be corrected for. Thus, to modify L_w^* into a continuous functional L_w, we must further add a compensating term $\delta(\gamma_0)$ to each standing wave γ_0 on the right, left when $g_u < 0$, $g_u > 0$, respectively. (We label a triple composite standing wave as being on the left, right of $\mathcal{T}$ according to the side of $\mathcal{T}$ on which the standing wave in $P(\gamma_0)$ falls. Thus, triple composite standing waves lie on the right, left of $\mathcal{T}$ when $g_u < 0$, $g_u > 0$, respectively.) Thus, the strategy for modifying L_w^* into a continuous functional L_w at triple composite standing waves is to redefine the strength of a triple composite standing wave $|\gamma_0| = |P(\gamma_0)|_* + \delta(\gamma_0)$, where $P(\gamma_0)$ and $\delta(\gamma_0)$ are appropriately defined below.

So assume first that $g_u < 0$. We first show that L_w^* is discontinuous under perturbation of a zero speed shock wave followed by a strong standing wave on the right of $\mathcal{T}$; cf. Figure 8. Indeed, referring to Figure 8, the elementary waves defined by $U_L \to U_M \to U_R$ and $U_L \to E \to U_R$ both must serve as admissible solutions of the Riemann problem, but $L_w^*(U_L \to U_M \to U_R) \neq L_w^*(U_L \to I \to K \to U_R) = L_w^*(U_L \to E \to U_R)$. We correct for this in the case $g_u < 0$ by modifying the definition of wave strength for strong standing waves ($u_L < u_R$) on the *right* of $\mathcal{T}$ by exactly the amount required to make L_w^* continuous.

To make this precise, let U_L and U_R denote the left and right states of a strong standing wave γ_0 on the right of $\mathcal{T}$. Let $f(a, u) = f(a_L, u_R)$ define the unique zero speed shock curve that passes through the state U_L, and for our purposes here, let U_* denote the unique point where this zero speed shock curve intersects the transition curve $\mathcal{T}$. The state $U_* = (a_*, u_*)$ is determined by the conditions $f(a_*, u_*) = f(a_L, u_L)$ and $u_* = u_{\mathcal{T}}(a_*)$; cf. Figure 9. Let $a_s(u)$ denote the unique standing wave curve that emanates from the point U_*. The curve a_s lies to the left of the standing shock curve emanating from U_* because $g_u < 0$. Now define the points I and K that lie on the standing wave curve a_s to the right of $\mathcal{T}$, at levels a_L and a_R, respectively (again see Figure 9). Since I and K are determined by γ_0 alone, we can define

$$\delta(\gamma_0) = L_w^*(I \to K \to U_R) - L_w^*(I \to U_L \to U_R) \tag{2.26}$$

for any strong standing wave γ_0 lying to the right of $\mathcal{T}$ in the case $g_u < 0$. (Note

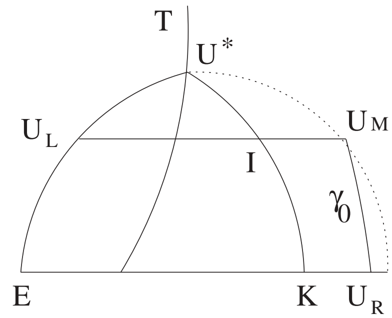

FIG. 8.

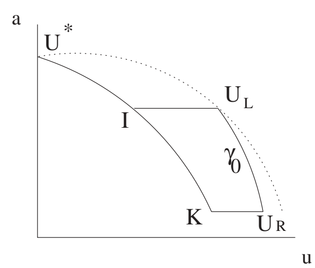

FIG. 9.

that $\delta(\gamma_0)$ depends only on U_L and U_R across the standing wave and is exactly the deficit between $L_w^*(U_L \to U_M \to U_R)$ and $L_w^*(U_L \to E \to U_R)$ encountered in Figure 8. Note also that $\delta(\gamma_0) = 0$ when $U_L \in \mathcal{T}$, because $|\gamma_0|$ reduces to $|\gamma_0|_*$ in this limit.) Thus, in the case $g_u < 0$, we define the modified strength $|\gamma_0|$ of a *strong standing wave on the right of* $\mathcal{T}$ by the rule

$$|\gamma_0|_w = |\gamma_0|_w^* + \delta(\gamma_0), \tag{2.27}$$

where $d(\gamma_0)$ is defined in (2.26).

Consider next the triple composite standing waves in the case $g_u < 0$. The main examples are given by $\gamma_0 \equiv U_L \to P \to Q \to R$ in Figures 15 and 6, the general case isolated in Figure 6. In both diagrams, $R = U_R$ denotes the right state of the triple composite standing wave γ_0. In these cases, the projection $P(\gamma_0)$ is given by $P(\gamma_0) = U_L \to T \to R$. We now show that the value of $L_w^*(P(\gamma_0))$ is discontinuous as $U_L = R$ varies from R to I along the line segment SN in Figure 6. Indeed, as $P(\gamma_0)$ varies from $U_L \to T \to R$ to $U_L \to M \to L$, the solution of the Riemann problem changes to $U_L \to S \to L$ and then to $U_L \to K \to I$. Thus for continuity, we require that $L_w(U_L \to S \to L) = L_w(U_L \to M \to L)$. But $L_w^*(U_L \to S \to L) = L_w^*(U_L \to K \to I \to L) = L_w^*(U_L \to M \to L) + \delta$, where $\delta = \delta(\gamma_0) = L_w^*(K \to I \to L) - L_w^*(K \to M \to L)$. Thus, for the general weak standing wave $U_L \to U_R$ on the right of $\mathcal{T}$ when $g_u < 0$, diagrammed in Figure 10, define

$$\delta = \delta(\gamma_0) = L_w^*(K \to I \to U_R) - L_w^*(K \to U_L \to U_R). \tag{2.28}$$

We take this as defining $\delta(\gamma_0)$ for any weak standing wave on the right of $\mathcal{T}$ that takes U_L to U_R, where for triple composite waves, (2.28) is assumed to apply to the weak standing wave on the right in $P(\gamma_0)$. (Note that the points K and I in Figure

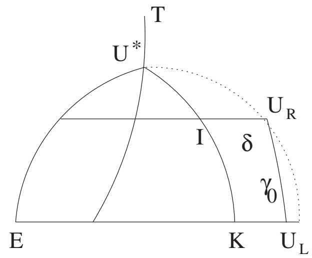

FIG. 10.

10 are determined by U_L and U_R alone.) To put this all together, let $P(\gamma_0) = \gamma_0$ for any standing wave that is not triple composite, and let $\delta(\gamma_0)$ be defined in (2.26) and (2.28) for strong and weak standing waves on the right of $\mathcal{T}$. Then we define the modified strength $|\gamma|$ of an elementary wave γ in the case $g_u < 0$ by

$$\text{(2.29)} \quad |\gamma|_w = \begin{cases} |P(\gamma)|_w^* + \delta(\gamma) & \text{if } \gamma \text{ is a standing wave on the right of } \mathcal{T}, \\ |\gamma|_w^* & \text{otherwise,} \end{cases}$$

where $d(\gamma)$ is defined in (2.26) and (2.28). For a sequence of elementary waves $\gamma_1, \ldots, \gamma_n$, we define the modified linear functional

$$\text{(2.30)} \qquad L_w[\gamma_1, \ldots, \gamma_n] = \sum_{i=1}^{n} |\gamma_i|_w.$$

This completes the definition of L_w in the case $g_u < 0$. We now define the modified linear functional L_w in the case $g_u > 0$.

So assume now that $g_u > 0$. We show first that L_w^* is discontinuous under perturbation of a strong standing wave on the left of $\mathcal{T}$ followed by a zero speed shock wave; cf. Figure 11. Referring to Figure 11, we see that both $U_L \to U_M \to U_R$ and $U_L \to E \to U_R$ both solve the Riemann problem, but $L_w^*(U_L \to U_M \to U_R) = L_w^*(U_L \to K \to I \to U_R) \neq L_w^*(U_L \to E \to U_R)$. To correct for this in the case $g_u > 0$, we modify the definition of wave strength for strong standing waves on the *left* of $\mathcal{T}$ by exactly the amount required to make L_w^* continuous.

To make this precise, let U_L and U_R denote the left and right states of a strong ($u_L < u_R$) standing wave γ_0 on the left of $\mathcal{T}$. In this case, let $a_s(u)$ denote the unique standing wave curve that passes through the states U_L and U_R, and let $U_* = (a_*, u_*)$ denote the unique point at which this standing curve a_s intersects the transition curve $\mathcal{T}$. Let $f(a, u) = f(a_*, u_*)$ define the unique zero speed shock curve that passes through the state U_*, defined to the right of $\mathcal{T}$, and let $I = (a_\#, u_\#)$ denote the state on this zero speed shock curve at level a_R; cf. Figure 12. Thus, I is determined by the condition that $I > \mathcal{T}$, together with $a_\# = a_R$, and $f(a_*, u_*) = f(a_R, u_\#)$. (Note that the zero speed shock curve emanating from U_* lies to the left of the standing wave curve emanating from U_* because $g_u > 0$.) Now define the state K to be the state at level a_L on the standing wave curve through I lying on the right-hand side of the transition curve $\mathcal{T}$ on the opposite side from U_L (see Figure 12). Since I and K are determined by γ_0 alone, we can define

$$\text{(2.31)} \qquad \delta(\gamma_0) = L_w^*(\bar{U}_L \to K \to I) - L_w^*(\bar{U}_L \to \bar{U}_R \to I),$$

which is defined for any strong standing wave γ_0 lying to the left of $\mathcal{T}$ in the case $g_u > 0$. (Note that this is exactly the deficit between $L_w^*(U_L \to K \to I)$ and $L_w^*(U_L \to$

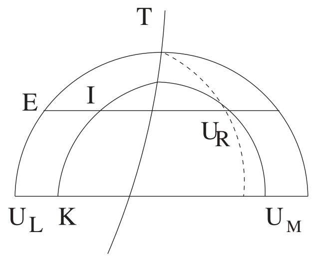

Fig. 11.

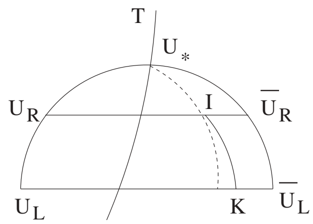

Fig. 12.

$U_R \to I$) in Figure 12. Note also that as before, $\delta(\gamma_0) = 0$ when $U_R \in \mathcal{T}$, $|\gamma_0|$ reduces to $|\gamma_0|_*$ in this limit.) Thus, in the case $g_u > 0$, we define the modified strength $|\gamma_0|$ of a strong standing wave on the left of $\mathcal{T}$ by the rule $|\gamma_0|_w = |\gamma_0|_w^* + \delta(\gamma_0)$.

Consider finally the triple composite standing wave $\gamma_0 \equiv U_L \to P \to Q \to R$ in Figure 17, isolated in Figure 13, for the case $g_u > 0$. In both diagrams, $R = U_R$ denotes the right state of the triple composite standing wave γ_0. In this case, the projection $P(\gamma_0)$ is given by $P(\gamma_0) = U_L \to T \to R$. We now show that the value of $L_w^*(P(\gamma_0))$ is discontinuous as $U_L = R$ varies from R to I along the line segment SN in Figure 13. Indeed, as $P(\gamma_0)$ varies from $U_L \to T \to R$ to $U_L \to M \to L$, the solution of the Riemann problem changes to $U_L \to M \to L$ and then to $U_L \to K \to I$. Thus, for continuity, we require that $L_w(U_L \to M \to L) = L_w(U_L \to T \to L)$. But $L_w^*(U_L \to M \to L) + \delta = L_w^*(U_L \to T \to L)$, where $\delta = \delta(\gamma_0) = L_w^*(M \to K \to I) - L_w^*(M \to L \to I)$. Thus, for the general weak standing wave $U_L \to U_R$ on the left of $\mathcal{T}$ when $g_u > 0$, diagrammed in Figure 14, define

$$\delta = \delta(\gamma_0) = L_w^*(I \to K \to \bar{U}_R) - L_w^*(I \to \bar{U}_L \to \bar{U}_R). \tag{2.32}$$

We take this as defining $\delta(\gamma_0)$ for any weak standing wave on the left of $\mathcal{T}$ that takes U_L to U_R, where for triple composite waves, (2.32) is assumed to apply to the weak standing wave on the left in $P(\gamma_0)$. (Again, note that the points K and I in Figure 14 are determined by U_L and U_R alone.) To put this together, let $P(\gamma_0) = \gamma_0$ for any standing wave that is not triple composite, and let $\delta(\gamma_0)$ be defined in (2.31) and (2.32) for strong and weak standing waves on the right of $\mathcal{T}$. Then we define the modified strength $|\gamma|$ of an elementary wave γ in the case $g_u > 0$ by

$$|\gamma|_w = \begin{cases} |P(\gamma)|_w^* + \delta(\gamma) & \text{if } \gamma \text{ is a standing wave on the left of } \mathcal{T}, \\ |\gamma|_w^* & \text{otherwise,} \end{cases} \tag{2.33}$$

where $d(\gamma)$ is defined in (2.31) and (2.32). For a sequence of elementary waves

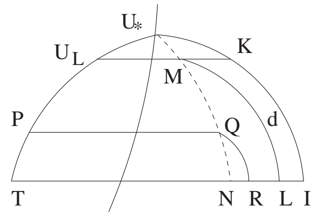

FIG. 13.

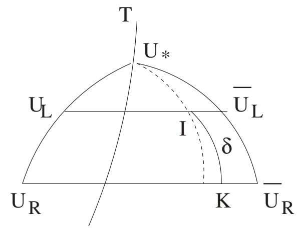

FIG. 14.

$\gamma_1, \ldots, \gamma_n$, again define the modified linear functional

$$L_w[\gamma_1, \ldots, \gamma_n] = \sum_{i=1}^{n} |\gamma_i|_w. \tag{2.34}$$

This completes the definition of L_w for the case $g_u > 0$ and so completes the definition of L_w in general.

We can now present in detail the admissible solution of the Riemann problem based on the L_w minimization principle. The solutions $[U_L, U_R]$ that are admissible by Definition 2.1 are diagrammed in Figures 15–18 for the cases $g_u < 0, g_u > 0$ and U_L to the left of $\mathcal{T}$, U_L to the right of $\mathcal{T}$. The solutions that minimize L_z are pointed out for comparison.[8] The cases $g_u < 0$ and $g_u > 0$ are qualitatively different because of the location of the zero speed shock curve. To read the diagrams, start at U_L and follow the arrows to an arbitrary state U_R. The wave curves traversed then give the elementary waves in the solution of the Riemann problem going from left to right in the (x, t)-plane. In the limit as g tends to zero, these diagrams reduce to those for the resonant homogeneous system $u_t + f(a, u)_x = 0$ [10, 12].

In Figures 15–18, the solid convex down curves denote standing wave curves, and the dotted curve to the right of $\mathcal{T}$ denotes the zero speed shock curve corresponding to the standing wave curve through U_L. In Figures 15 and 16, the dotted line falls to the right of the standing wave curve through U_L because $g_u < 0$. Similarly, in Figures 17 and 18, it falls to the left because $g_u > 0$. We discuss the multiplicity of solutions in Figures 15–17 below. In Figure 18, solutions are unique.

In each of Figures 15–17, there is a region of right states U_R for which there are multiple solutions of the Riemann problem that minimize the total variation in a.

[8]In [13] it was shown that the solutions of the Riemann problem that minimize L_z actually minimize L_z over all sequences of connected elementary waves that connect U_L to U_R. This essentially implies that L_z is nonincreasing on solutions. On the other hand, this is not the case for the solutions of the Riemann problem that minimize L_w, and this explains why a potential interaction term is required to construct a decreasing functional that incorporates L_w.

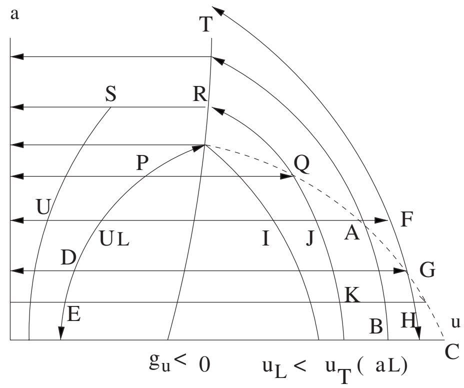

FIG. 15.

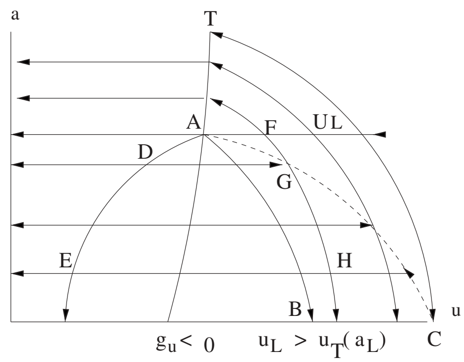

FIG. 16.

In the region of multiple solutions, there is always a multiplicity of three solution in the interior of the region, but this reduces to a multiplicity of two on the boundary of the region. The L_w minimization principle rules out every four wave solution, *except for* the two special cases labeled by $U_L \to P \to Q \to R$ in Figures 15 and 17. However, in both cases, the zero speed wave in the solution of the Riemann problem always consists of a standing wave followed by a zero speed shock wave followed by another standing wave (all zero speed) and the monotonicity in a is preserved across triple composite standing waves. From the point of view of wave interactions, such composite waves interact like a single wave, and so in our discussion below, we will treat triple composite standing waves as *a single standing wave.* With this convention, (and allowing waves to have zero strength), the admissible solution of the Riemann problem always consists of three elementary waves: a negative speed nonlinear wave followed by a single standing wave, followed by a positive speed nonlinear wave.

Discussion of Figure 15 [$\boldsymbol{g_u < 0; U_L}$ to the left of $\boldsymbol{\mathcal{T}}$]. A multiplicity of solutions occurs when U_R lies within the interior of the region ABC, e.g., $U_R = H$. The three solutions are: $U_L \to F \to H$, $U_L \to D \to G \to H$, and $U_L \to E \to H$. (Here, e.g., $U_L \to F$ denotes the elementary shock wave taking U_L on the left to F on the right. Since F lies to the right of the zero speed shock curve (the dotted line), and since $f_{uu} < 0$, $U_L \to F$ is a shock wave of negative speed.) All of these solutions have the same L_z-value, but only the solution $U_L \to F \to H$ minimizes

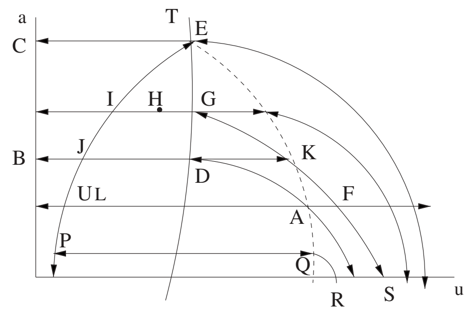

FIG. 17.

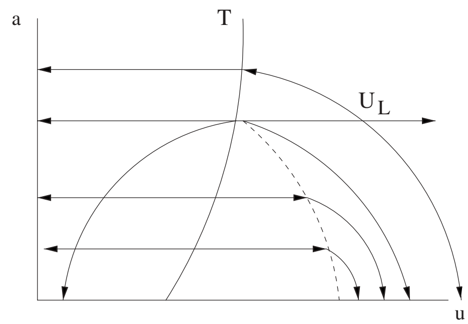

FIG. 18.

L_w. Indeed, consider region U_L, F, H, E of Figure 15, which is described in Figure 19. $L_w(U_L \to F \to H) - L_w(U_L \to E \to H) = L_w^*(U_L \to F \to H) + \delta(F \to H) - L_w^*(U_L \to E \to H) = L_w^*(U_L \to F \to H) + \delta(F \to H) - L_w^*(U_L \to I \to K \to H) = L_w^*(I \to F \to H) - L_w^*(I \to K \to H) + \delta(F \to H) = L_w^*(I \to F \to H) - L_w^*(M \to F \to H) + L_w^*(M \to N \to H) - L_w^*(I \to K \to H) = L_w^*(I \to M \to N) - L_w^*(I \to K \to N) < 0$ by Lemma 2.7. Similarly we can show that $L_w(U_L \to F \to H) - L_w(U_L \to D \to G \to H) < 0$. It follows that $U_L \to F \to H$ is the unique solution of the Riemann problem selected by the L_w minimization principle.

Discussion of Figure 16 [$g_u < 0; U_L$ to the right of $\mathcal{T}$]. A multiplicity of solutions that minimize the total variation in a (but do not necessarily minimize L_z) occurs when U_R lies within the interior of the region ABC, e.g., $U_R = H$. The three solutions are $U_L \to F \to H$, $U_L \to A \to E \to H$, and $U_L \to A \to D \to G \to H$. The L_z-value is minimized only on the first of these, and thus in this case the L_z minimization principle selects a unique admissible solution. The functional L_w is also minimized on the solution $U_L \to F \to H$. For example, referring to Figure 20, $L_w(U_L \to F \to H) - L_w(U_L \to A \to E \to H) = L_w^*(F \to H) + \delta(F \to H) - L_w^*(F \to A \to B \to H)$. But $\delta(F \to H) = L_w^*(I \to K) + L_w^*(K \to H) - L_w^*(F \to H) - L_w^*(F \to I)$, $L_w^*(K \to H) - L_w^*(B \to H) = -L_w^*(B \to K)$, and $L_w^*(I \to K) \le L_w^*(A \to B)$ (by the Corollary to Lemma 2.7). Substituting these as inequalities into the previous line gives $L_w(U_L \to F \to H) - L_w(U_L \to A \to E \to H) \le -[L_w^*(F \to I) + L_w^*(F \to A) + L_w^*(B \to K)] < 0$. The case for $L_w(U_L \to F \to H) - L_w(U_L \to A \to D \to G \to H) < 0$ is similar.

Discussion of Figure 17 [$g_u > 0; U_L$ to the left of $\mathcal{T}$]. A multiplicity of solutions that minimize the total variation in a occurs when U_R lies within the

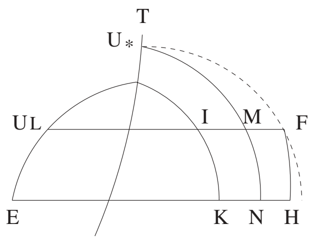

Fig. 19.

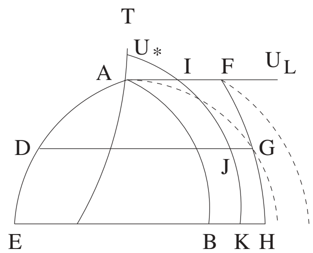

Fig. 20.

interior of the region CEADB, e.g., $U_R = H$. The three solutions are $U_L \to I \to H$, $U_L \to F \to G \to H$, and $U_L \to J \to K \to G \to H$. In this case the L_z and L_w minimization principles both pick out the unique solution $U_L \to I \to H$. Note that on the boundary, say $U_R = M$, where the wave structure changes, there is a multiplicity of *two* distinct solutions, $U_L \to I \to M$ and $U_L \to N \to M$, and at this boundary, the L_z and L_w values of both solutions are equal, a requirement for the continuity of the functionals with respect to U_L and U_R. (cf. the paragraph preceding (2.29)). As an example, we verify that $L_w(U_L \to I \to H) - L_w(U_L \to F \to G \to H) < 0$, (see Figure 21). To this end, write $L_w(U_L \to I \to H) - L_w(U_L \to F \to G \to H) = L^*_w(U_L \to I \to H) + \delta(U_L \to I) - L^*_w(U_L \to L \to G \to H) = L^*_w(U_L \to I) + L^*_w(I \to H) + [L^*_w(U_L \to P) + L^*_w(P \to Q) - L^*_w(U_L \to I) - L^*_w(I \to Q)] - L^*_w(U_L \to L) - L^*_w(L \to G) - L^*_w(G \to H) = L^*_w(P \to Q \to H) - L^*_w(P \to L \to G) - L^*_w(G \to H) < 0$ by Lemma 2.7.

Discussion of Figure 18 [$\boldsymbol{g_u > 0; U_L}$ to the right of $\boldsymbol{\mathcal{T}}$]. In this case the solution that minimizes the total variation in a is unique.

We now summarize the main results regarding the solution of the Riemann problem.

Proposition 2.10. *The admissible solution of the Riemann problem* $[U_L, U_R]$ *always consists of a sequence of three connected waves, a negative speed nonlinear wave* γ_1 *followed by a standing wave* γ_0 *followed by a positive speed nonlinear wave* γ_2*, where we allow* $\gamma_i = 0$*, and we treat the composite zero speed waves of type* $U_L \to P \to Q \to R$ *in Figure* 15 *as a single wave* γ_0*. We write*

$$[U_L, U_R] = \gamma_1\gamma_0\gamma_2. \tag{2.35}$$

Proposition 2.11. *The functional* $L_w([U_L, U_R])$ *is a continuous function of* U_L *and* U_R *throughout the domain* Ω.

Proposition 2.12. *The convex side, (i.e., lower side when* $f_{uu} < 0$*), of each standing wave curve is an invariant region for admissible solutions of the Riemann*

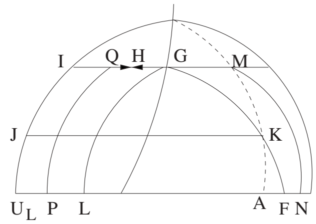

FIG. 21.

problem.

Proof. Proposition 2.10 is clear by construction. The proof of Propositions 2.11 has been indicated above and follows directly via a case by case inspection of the Riemann problem. Note that the continuity of L_w follows upon verifying that the only places where $L_w^*([U_L, U_R])$ is discontinuous were identified in the discussion following Corollary 2.8, and the deficit was accounted for by the correction term $\delta(\gamma)$. Proposition 2.12 follows immediately from the fact that all intermediate states in the admissible solution of the Riemann problem lie on the convex side of the outer of the two standing wave curves through U_L and U_R. $\square$

Proposition 2.12 implies an L^∞ bound on Glimm approximate solutions generated by the admissible solution of the Riemann problem.

3. The generalized Glimm method. In this section we construct the approximate Glimm scheme solutions $U_{\Delta x}(x,t)$ and prove the compactness of approximate solutions under the assumption that the initial data is of bounded total variation in (a,z) (which implies that the initial data is of bounded total variation in (a,w)). We call this a generalized Glimm scheme because the standing waves are generalized weak solutions of system (1.1) due to the presence of the a' in the term $a'g$ on the right-hand side of (1.1). The proof of convergence of the residual must be modified because a piecewise constant approximation of $a(x)$ does not give an L^∞ approximation of $a'g$, and so the residual does not converge strongly, but rather weakly. This argument was first given in the strictly hyperbolic case in [6], and for completeness, we include the argument adapted to the problem here (in section 6).

To begin, assume that the initial data $U_0(x)$ takes values in a neighborhood Ω which lies below a standing wave curve and above a curve $a = \text{const}$ contained within a neighborhood of U_*, where the unique solution of the Riemann problem exists as constructed in the previous section, and small enough so that Lemmas 2.4–2.7 and Propositions 2.10–2.12 hold throughout Ω. Since Ω is an invariant region for Riemann problems, it follows that Ω is also an invariant region for the Glimm scheme, which is therefore defined for all time for any mesh length. To construct the approximate solutions, first discretize $R \times [0,\infty)$ by spatial mesh length Δx and time mesh length Δt such that

$$\frac{\Delta x}{\Delta t} = \lambda, \tag{3.1}$$

where

$$\lambda \equiv 2 \sup_{(a,u)\in\Omega} \left\{ \left| \frac{\partial f}{\partial u} \right| \right\}. \tag{3.2}$$

We let $x_i = i\Delta x$, $t_j = j\Delta t$ so that (x_i, t_j) denote the mesh points of the approximate solution. Define

$$S_j = \{(x,t) : t_j \le t < t_{j+1}\}.$$

The approximate solution $U_{\Delta x}$ generated by the Glimm scheme is defined as follows. First, fix a sample sequence $\theta = \{\theta_{ij}\} \in \Theta$, where Θ denotes the infinite product of intervals $(0,1)$ indexed by mesh points (with Lebesgue measure) so that $\Theta = \Pi(0,1)_{ij}$ and $\theta_{ij} \in (0,1)$, $-\infty < i < \infty$, $j \ge 0$ [4, 24]. (We randomize in space and time to facilitate the proof of convergence of the residual; cf. [25]). To initiate the scheme at $j = 0$, approximate the initial data by piecewise constant states by setting

$$U_i^0 = U_{\Delta x}(x,0) = U_0(x_i + \theta_{i0}\Delta x), \quad x_i < x < x_{i+1}. \tag{3.3}$$

Assuming that $U_{\Delta x}(x,t)$ has been constructed for $(x,t) \in \bigcup_{j=0}^{j-1} S_j$, then define $U_{\Delta x}$ in S_j as the solution of (1.1) with the initial values

$$U_i^j = U_{\Delta x}(x, t_j+) = U_{\Delta x}(x_i + \theta_{ij}\Delta x, t_j-), \quad x_i < x < x_{i+1}. \tag{3.4}$$

In other words, at each time t_j, a piecewise constant approximation $U_{\Delta x}(x, t_j+)$ is obtained by sampling the solution $U_{\Delta x}(x, t_j-)$ in each interval of the mesh at time level t_j, so that the solution in S_j can be constructed by solving the Riemann problems $[U_{i-1}^j, U_i^j]$ posed at each point of discontinuity $(x_i, t_j), i \in Z$. The Courant–Friedrichs–Levy restriction (3.1) ensures that the Riemann problem solutions in each S_j do not interact before time t_{j+1} [13].

We need to define the I-curves for the analysis of the nonlocal functional F defined below; cf. [4]. An I-curve J is a continuous space-like piecewise linear curve in the (x,t)-plane that connects adjacent mesh points of type $(x_i + \theta_j \Delta x, t_j)$ to ones of type $(x_i, t_{j+1/2})$, where $(x_i, t_{j+1/2}) = (i\Delta x, (j+1/2)\Delta t)$. Given an I-curve J_1 that extends from $i = -\infty$ to $i = +\infty$, we obtain a successor J_2 of J_1 by lifting the point $(x_i, t_{j-1/2})$ to the point $(x_i, t_{j+1/2})$ when the points $(x_{i-1} + \theta_j \Delta x, t_j)$ and $(x_i + \theta_j \Delta x, t_j)$ both lie on J_1. We call the region $(x_i, t_{j-1/2}), (x_i, t_{j+1/2}), (x_{i-1} + \theta_j \Delta x, t_j), (x_i + \theta_j \Delta x, t_j)$ between J_1 and J_2 the interaction diamond Δ. We let J^j denote the I-curve that contains all of the sample points $(x_i + \theta_j \Delta x, t_j)$ at time level t_j. The I-curve J^j crosses all of the waves in the Riemann problems posed in $U_{\Delta x}$ at time level t_j, and the I-curve J^j can be obtained by a sequence of successive I-curves. (Note that lifting the mesh point $(x_i + \theta_j \Delta x, t_j)$ to $(x_i + \theta_j \Delta x, t_{j+1})$ when mesh points $(x_{i-1}, t_{j+1/2})$ and $(x_i, t_{j+1/2})$ both lie on J, does not change the waves that J crosses, and so we can consider these to be equivalent.) It follows that to show that a functional F satisfies $F(J^j) \le F(J^0)$, it suffices only to prove that $F(J_2) \le F(J_1)$ for any pair of successive mesh curves J_1 and J_2 [4].

We have the following theorem; cf. [13].

THEOREM 3.1. *If the neighborhood Ω containing U_* is chosen to be small enough, then the Glimm approximate solutions $U_{\Delta x}(x,t)$ are defined for all time. Moreover,*

$$L_z(J^{j+1}) \le L_z(J^j) \tag{3.5}$$

for each $j \ge 0$, where J^j identifies the sequence of elementary waves appearing in the approximate solution $U_{\Delta x}$ in the strip S_j, and L_z is defined in (2.14).

Proof. The proof of (3.5) was given in [13]. The supnorm bound on solutions follows from Proposition 2.11 which asserts the existence of convex invariant regions for Riemann problems in a neighborhood of U_*. The main point in the proof of (3.5) is that the solutions of the Riemann problems used in the construction of the Glimm approximate solutions are admissible solutions of the Riemann problem, and so were selected to minimize the L_z-value of the elementary waves among all possible solutions of the Riemann problem. But L_z has the further property of being minimized on the solution of the Riemann problem among all connected sequences of elementary waves that take U_L to U_R. (This was proven in [13].) Using this, estimate (3.5) follows because L_z decreases across any interaction diamond Δ_{ij} lying between the two successive I-curves J_1 and J_2 with interaction diamond centered on (x_i, t_j). Indeed, the Glimm scheme replaces the sequence of waves that take U_{i-1}^j to U_i^j at time level t_j- by the waves that solve the Riemann problem $[U_{i-1}^j, U_i^j]$ at $t = t_j+$. □

Theorem 3.1 leads directly to the following compactness result for approximate solutions generated by the Glimm method.

THEOREM 3.2. *Assume that the initial data $U_0(x) \in \Omega$ satisfies the condition $\mathrm{Var}_z\{U_0(\cdot)\} = V_z < \infty$ and $\mathrm{Var}\{a(\cdot)\} = V_a < \infty$. Then $U_{\Delta x}(x,t) \in \Omega$ for all $x, t \geq 0$, $\mathrm{Var}_z\{U_{\Delta x}(\cdot,t)\} < 4V_z$ for all $t \geq 0$, and a subsequence of $\{U_{\Delta x}\}$ converges boundedly, almost everywhere, to a bounded measurable function $U(x,t)$ as Δx tends to zero.*

Proof. See Theorem 3.2 [18]. □

From here on out we assume that $U_{\Delta x}(x,t)$ is a sequence of Glimm approximate solutions that converges boundedly, pointwise almost everywhere to a function $U(x,t)$, and satisfies the estimate

$$\mathrm{Var}_z\{U_{\Delta x}(\cdot,t)\} < 4V_z. \tag{3.6}$$

In section 6 we conclude this argument by showing that the limit function $U(x,t)$ is a classical weak solution of (1.1) when a' has no delta function singularities.

4. The interaction potential $d(\gamma_0, \gamma_r)$. Assume that $U_{\Delta x}(x,t)$ is an approximate Glimm scheme solution starting from initial data $U_0(x)$ of bounded total variation in (a,u) and hence (a,w) as well. Then the total variation in (a,z) of $U_{\Delta x}(\cdot,0)$ is uniformly bounded, and hence the existence theory of section 3 applies. Thus, without loss of generality, we can assume that $U_{\Delta x} \to U$, where $U(x,t)$ is a weak solution of (1.1) of bounded total variation in z at each fixed time. (The convergence is in L^1_{loc} at each fixed time, uniformly on compact sets.) We now estimate the growth of the total variation in w (and hence in u) in the approximate solutions $U_{\Delta x}(x,t)$.

Our idea is to use the functional L_w to estimate the total variation in w at each time in an approximate solution $U_{\Delta x}(x,t)$. The problem of estimating L_w is more difficult than the problem of estimating L_z because in the case of L_w, it is *not* true that $L_w(J_{j+1}) \leq L_w(J_j)$ across interactions. The point is that L_w is minimized on the admissible solution of the Riemann problem among all solutions of the Riemann problem, but it is *not* minimized on the admissible solution of the Riemann problem among all connected sequences of elementary waves that take U_L to U_R, even if there is just a single standing wave within the sequence. Indeed, if a fast rarefaction wave followed by a slow standing wave interacts to produce a slow standing wave followed by a fast rarefaction wave, then L_w increases across this interaction. This is because rarefaction wave-standing wave interactions, in which incoming and outgoing waves all lie on one side of $\mathcal{T}$, always have the effect of moving the standing wave closer to the transition curve—this increases the L_w because it shifts the total variation in u from the nonlinear waves to the standing waves, which are weighted by the larger factors of 2 and 4 over the weight on the nonlinear waves. We verify this in two examples below. The remarkable fact that the functional L_w increases *only* on rarefaction

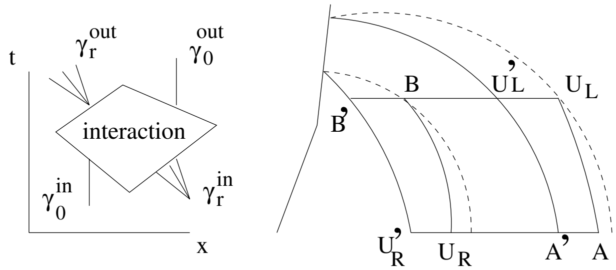

FIG. 22.

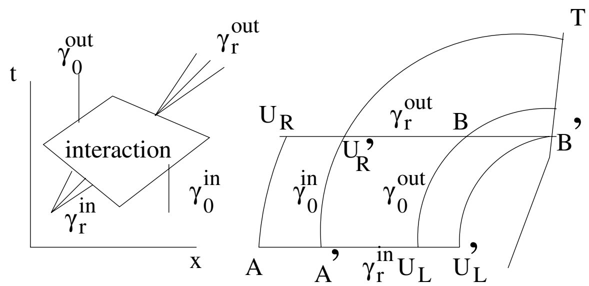

FIG. 23.

wave/standing wave interactions, and is nonincreasing on *all other* interactions, is discussed after the examples. Our strategy is then to define a potential for the increase in L_w due to the interaction of a standing wave and a rarefaction wave and to prove that L_w plus the sum of all potential interaction terms define a nonlocal functional F that bounds the total variation in w and decrease across interactions.

We begin by verifying that L_w increases on rarefaction wave/standing wave interactions in two salient examples: the case when $g_u < 0$ and the standing wave is a strong standing wave on the right of $\mathcal{T}$, and the case when $g_u > 0$ and the standing wave is a strong standing wave on the left of $\mathcal{T}$. These two examples clarify the problem of bounding the increase in L_w on interactions. So consider first the interaction diagrammed in Figure 22, the case when $g_u < 0$, and a standing wave γ_0^{IN} starting to the left of a negative speed rarefaction wave γ_r^{IN} interacts to produce a negative speed rarefaction wave γ_r^{OUT} followed by a standing wave γ_0^{OUT}. (For simplicity we assume in this example that all waves lie to the right of $\mathcal{T}$.) Then $[U_L, U_R] = (\gamma_r^{OUT}, \gamma_0^{OUT})$, but $L_w(\gamma_r^{OUT}, \gamma_0^{OUT}) - L_w(\gamma_r^{IN}, \gamma_0^{IN}) = L_w^*(\gamma_r^{OUT}) + L_w^*(\gamma_0^{OUT}) + \delta(\gamma_0^{OUT}) - L_w^*(\gamma_r^{IN}) - L_w^*(\gamma_0^{IN}) - \delta(\gamma_0^{IN}) = L_w^*(U_L \to B \to U_R) + \delta(\gamma_0^{OUT}) - L_w^*(U_L \to A \to U_R) - \delta(\gamma_0^{IN}) = L_w^*(U_L' \to B' \to U_R') - L_w^*(U_L' \to A' \to U_R') > 0$ by Lemma 2.7.

Consider next the case of the interaction diagrammed in Figure 23, the case when $g_u > 0$, and a positive speed rarefaction wave γ_r^{IN} starts to the left of a standing wave γ_0^{IN} and interacts to produce a standing wave γ_0^{OUT} followed by (that is, to the left of) a positive speed rarefaction wave γ_r^{OUT}. (Again, for simplicity, we assume in this example that all waves lie to the left of $\mathcal{T}$.) Then $[U_L, U_R] = (\gamma_0^{OUT}, \gamma_r^{OUT})$, but $L_w(\gamma_0^{OUT}, \gamma_r^{OUT}) - L_w(\gamma_0^{IN}, \gamma_r^{IN}) = L_w^*(\gamma_0^{OUT}) + \delta(\gamma_0^{OUT}) + L_w^*(\gamma_r^{OUT}) - L_w^*(\gamma_0^{IN}) -$

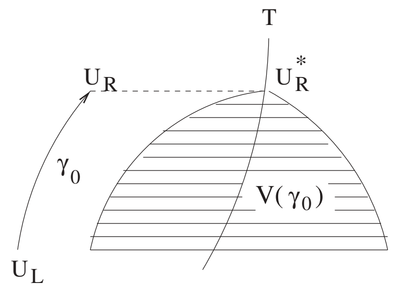

FIG. 24.

$\delta(\gamma_0^{IN}) - L_w^*(\gamma_r^{IN}) = L_w^*(U_L \to B \to U_R) + \delta(\gamma_0^{OUT}) - L_w^*(U_L \to A \to U_R) - \delta(\gamma_0^{IN}) = L_w^*(U_L' \to B' \to U_R') - L_w^*(U_L' \to A' \to U_R') > 0$ by Lemma 2.7. One can verify that L_w is nonincreasing on shock wave-standing wave interactions that lie on one side of $\mathcal{T}$ by similar examples. This concludes the examples.

What is remarkable is that the increase in L_w due to rarefaction wave-standing wave interactions that are not transonic (that is, all waves in the interaction lie entirely on the same side of $\mathcal{T}$) accounts for *all* of the ways L_w can increase, even for complicated transonic wave interactions that carry waves across the transition curve. The proof that we need only a potential interaction term for nontransonic rarefaction wave-standing wave interactions is a consequence of our proof below that the nonlocal functional F is nonincreasing on all interactions, but in the proof it is difficult to see the reason for the decrease in the functional in the complicated case when the interactions are transonic. To motivate the argument, consider a standing wave γ_0 that takes $U_L = (a_L, u_L)$ to $U_R = (a_R, u_R)$. Then this wave lies entirely on one side of $\mathcal{T}$, or else it is a composite wave of type $U_L \to P \to Q \to R$ of Figure 15. Let $a_* = \max\{a_L, a_R\}$, and let $U_* = (a_*, u_*)$ denote the point on $\mathcal{T}$ that lies at level $a = a_*$. Consider now the region $V(\gamma_0)$ that lies below the standing wave curves on the left and right of $\mathcal{T}$ that pass through the state $U = U_*$; cf. Figure 24. The claim then is that any rarefaction wave that lies in the region $V(\gamma_0)$ in an approximate solution that contains the wave γ_0 cannot interact with γ_0 in such a way as to produce an increase in L_w. For example, one can verify that when the connected sequence of waves $\gamma_r\gamma_0$ or $\gamma_0\gamma_r$ interact to produce the waves in the Riemann problem $[U_L, U_R]$, L_w will be nonincreasing and the wave γ_r will be eliminated by the interaction when γ_r in $V(\gamma_0)$. This helps explain why we needn't include such portions of the rarefaction wave in the definition of the interaction potential $d(\gamma_0, \gamma_r)$ below.

We now define $d(\gamma_0, \gamma_r)$, the potential for the increase in L_w due to the interaction of a standing wave γ_0 that *approaches* a rarefaction wave γ_r; cf. [4]. (Although there is an ordering of the waves in the (x,t)-plane implied by the condition that two waves approach, we assume no ordering in d, so that $d(\gamma_0, \gamma_r) \equiv d(\gamma_r, \gamma_0)$.) So assume that γ_0 and γ_r are waves that cross the same I-curve J in an approximate Glimm scheme solution $U_{\Delta x}$. We say that γ_0 and γ_r *approach* on J if the faster of the two waves is positioned to the left of the slower wave on J in the (x,t)-plane. Any two such waves will interact at a later time in the approximate solution $U_{\Delta x}$. Note that standing waves always have zero speed, and to make the definition of approaching unambiguous, assume that all rarefaction waves have purely positive or negative speed by treating any rarefaction wave that crosses $\mathcal{T}$ as two separate waves by partitioning such a rarefaction wave into its positive and negative speed parts. (In this case, the wave

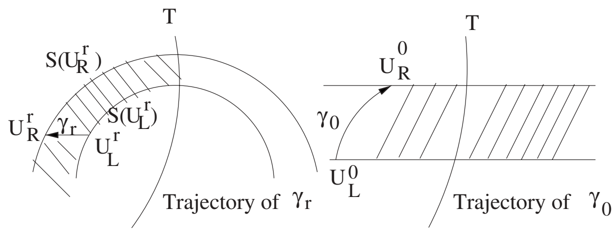

FIG. 25.

will be partitioned at the point where the wave crosses $\mathcal{T}$ since this is the curve of zero characteristic speed.) If α and β are indices that identify two waves that cross J, then we write $(\alpha, \beta) \in App(J)$ if γ^α approaches γ^β on J.

In order to define $d(\gamma_0, \gamma_r)$ for two approaching waves γ_0 and γ_r, we first define what we call the interaction region $\Delta(\gamma_0, \gamma_r)$, the region in U-space where the interaction of γ_r and γ_0 will take place (assuming the rarefaction wave is not canceled out before the interaction occurs). To this end, we first define what we call the *trajectory* of the waves γ_r and γ_0. If the waves interact, then the interaction will occur within the region determined by the intersection of the two trajectories. Since the standing wave curves and nonlinear wave curves act like Riemann invariants for the system (1.1), it follows that when a rarefaction wave interacts with a standing wave, the standing wave is just translated along the nonlinear wave curves and the rarefaction wave is translated along the standing wave curves. Thus let $\gamma_0 = [U_L^0, U_R^0]$ and $\gamma_r = [U_L^r, U_R^r]$ denote a standing wave and a rarefaction wave, respectively. In the case when the standing wave γ_0 is a composite wave of type $U_L \to P \to Q \to R$ of Figure 15, we define

$$d(\gamma_0, \gamma_r) = d(\gamma_0', \gamma_r), \tag{4.1}$$

where γ_0' denotes the standing wave in the projection $P(\gamma_0)$ (e.g., $\gamma_0' = \mathcal{T} \to R$ in Figure 15). Thus to define $d(\gamma_0, \gamma_r)$, it suffices to assume that the standing wave γ_0 lies entirely on one side of $\mathcal{T}$ (admissible, noncomposite standing waves do not cross the transition curve), and we can assume that the rarefaction wave γ_r lies entirely on one side of $\mathcal{T}$ because rarefaction waves are partitioned so as to have unambiguous positive or negative speed. For the rarefaction wave γ_r let $\mathcal{S}(U_L^r), \mathcal{S}(U_R^r)$ denote the standing wave curves that pass through states U_L^r, U_R^r, respectively. We can now define the *trajectory* of a rarefaction wave γ_r and a standing wave γ_0; cf. Figure 25.

DEFINITION 4.1. *Let $\gamma_r = [U_L^r, U_R^r]$ denote a rarefaction wave that lies entirely one side of the transition curve, say $\gamma_r \le \mathcal{T}$ or $\gamma_r \ge \mathcal{T}$. Then the* trajectory $Traj(\gamma_r)$ *of γ_r is the region in U-space between the two standing wave curves $\mathcal{S}(U_L^r)$ and $\mathcal{S}(U_R^r)$, intersected with $u \le \mathcal{T}$ or $u \ge \mathcal{T}$, according to whether $\gamma_r \le \mathcal{T}$ or $\gamma_r \ge \mathcal{T}$, respectively.*

DEFINITION 4.2. *The trajectory $Traj(\gamma_0)$ of a standing wave γ_0 is the region between the curves $a = a_L^0$ and $a = a_R^0$, i.e., the region between the two nonlinear wave curves through U_L^0 and U_R^0, respectively.*

We note that $Traj(\gamma_r)$ includes only the region on the side of $\mathcal{T}$ that contains the wave γ_r because a rarefaction wave cannot cross $\mathcal{T}$ without being canceled out by a shock wave, but $Traj(\gamma_0)$ contains the region on both sides of $\mathcal{T}$ because a standing wave can cross $\mathcal{T}$ as a result of interaction. It follows that the interaction of γ_0 and γ_r can only take place on the side of $\mathcal{T}$ that contains γ_r.

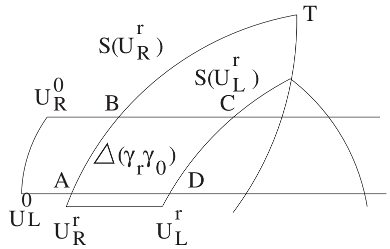

FIG. 26.

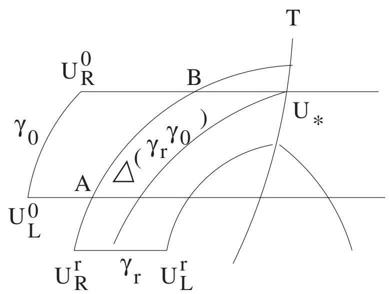

FIG. 27.

We now define the interaction region $\Delta(\gamma_0, \gamma_r)$. To this end, consider the region equal to the intersection between $Traj(\gamma_r)$ and $Traj(\gamma_0)$, which lies entirely on the same side of $\mathcal{T}$ as the rarefaction wave γ_r. If there exists a full set of four intersection points between the curves $\mathcal{S}(U_L^r), \mathcal{S}(U_R^r)$ and $a = a_L^0$, $a = a_R^0$ that all lie on the same side of $\mathcal{T}$ as the wave γ_r (diagrammed A, B, C, D in Figure 26), then define the interaction region $\Delta(\gamma_0, \gamma_r)$ to be the region $ABCD$, which is exactly equal to the intersection of the trajectory of γ_0 and the trajectory of γ_r. If the curves $\mathcal{S}(U_L^r), \mathcal{S}(U_R^r)$ and $a = a_L^0$, $a = a_R^0$ do not intersect in four distinct points on the same side of $\mathcal{T}$ as γ_r, we must modify the definition of $\Delta(\gamma_0, \gamma_r)$ to account for the fact that portions of the rarefaction wave γ_r will be canceled out before γ_0 can interact with the standing wave γ_0. To this end, let U_* denote the highest point on $\mathcal{T}$ where the trajectory of γ_0 intersects $\mathcal{T}$, i.e., let $U_* = (a_{max}, u_{\mathcal{T}}(a_{max}))$, where $a_{max} = \max\{a_L^0, a_R^0\}$; see Figure 27. Consider then the standing wave $\mathcal{S}(U_*)$ that passes through the point U_*, and ask whether $\mathcal{S}(U_*)$ lies within the trajectory of γ_r. If it does not (which means the trajectory of γ_r lies below U_*), then we say that the interaction region $\Delta(\gamma_0, \gamma_r) = \phi$, the empty set; that is, there is no potential for interaction of the waves γ_r and γ_0. If $\mathcal{S}(U_*)$ does lie within the trajectory of γ_r, then let $\Delta(\gamma_0, \gamma_r)$ denote the intersection of the trajectory of γ_0 with the trajectory of γ_r and *take away* all points U that lie below the standing wave curve $\mathcal{S}(U_*)$. In this case, $\Delta(\gamma_0, \gamma_r) = ABU_*D$, as diagrammed in Figure 27. This completes the definition of $\Delta(\gamma_0, \gamma_r)$. Note that in every case, $\Delta(\gamma_0, \gamma_r)$ consists of a region on the side of the transition curve that contains γ_r, bounded on the right and left by standing wave curves and above and below by nonlinear wave curves, determined by four vertices, which we label $ABCD$ as in Figure 28.

Now for any approaching waves γ_r and γ_0 (assuming rarefaction waves are partitioned at points where they cross $\mathcal{T}$), define $d(\gamma_0, \gamma_r)$ in terms of $\Delta(\gamma_0, \gamma_r)$ as follows.

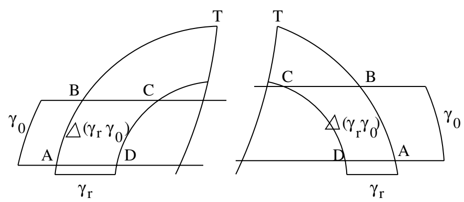

FIG. 28.

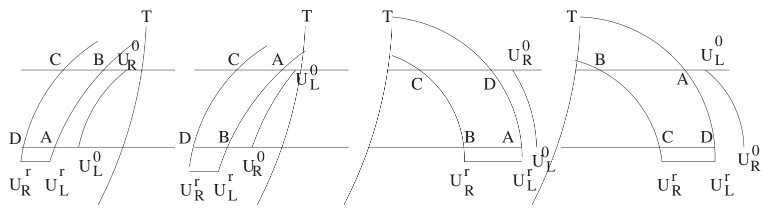

FIG. 29.

The interaction potential $d(\gamma_0, \gamma_r)$ is equal to the change in L_w between the waves that enter and the waves that leave the interaction region $\Delta(\gamma_0, \gamma_r)$, as determined by the orientation of the original waves γ_0 and γ_r. That is, there is only one way to project the waves γ_r and γ_0 to incoming waves on the boundary of $\Delta(\gamma_0, \gamma_r)$ so that γ_r is projected to a rarefaction wave, γ_0 is projected to a standing wave that preserves the increasing/decreasing of a across the wave, and the projected waves define a connected sequence of waves that preserve the left/right orientation of the original waves γ_r and γ_0. Thus, there are four cases in which γ_r and γ_0 can approach, labeled in Figure 29. These are determined by whether a increases or decreases across the standing wave γ_0 and whether the wave γ_r lies to the left or right of $\mathcal{T}$. In the four cases (1)–(4) labeled in Figure 29, $d(\gamma_0, \gamma_r)$ in each case is defined by

$$d(\gamma_0, \gamma_r) = L_w(A \to B \to C) - L_w(A \to D \to C). \tag{4.2}$$

Therefore, assuming that all rarefaction waves have been partitioned at points on $\mathcal{T}$, equation (4.2) defines $d(\gamma_0, \gamma_r)$ for any pair of approaching waves γ_r and γ_0, and we set $d(\gamma_0, \gamma_r) = 0$ for any pair of nonapproaching waves. For our arguments below, we wish to index the waves in an approximate Glimm scheme solution as they are given in the solution of the Riemann problems themselves, without further partitioning. Thus for a rarefaction wave γ_r that crosses $\mathcal{T}$ and is partitioned into $\gamma_r = \gamma_r^a \gamma_r^b$ at the point where it crosses $\mathcal{T}$, we say γ_r approaches a standing wave γ_0 if γ_r^a approaches γ_0 or γ_r^b approaches γ_0, and we define $d(\gamma_0, \gamma_r) = d(\gamma_0, \gamma_r^a) + d(\gamma_0, \gamma_r^b)$. It follows that for *any partitioning* of a rarefaction wave $\gamma_r = \gamma_r^a \cdots \gamma_r^b$, we have that $d(\gamma_0, \gamma_r) = d(\gamma_0, \gamma_r^a) + \cdots + d(\gamma_0, \gamma_r^b)$. This completes the definition of $d(\gamma_0, \gamma_r)$ for any pair of waves γ_0, γ_r that crosses an I-curve J in an approximate Glimm scheme solution of (1.1).

We note that the potential $d(\gamma_0, \gamma_r)$ is symmetric, $d(\gamma_0, \gamma_r) = d(\gamma_r, \gamma_0)$, and is constructed so that if a standing wave γ_0 is displaced to $\hat{\gamma}_0$ by interaction with a nonlinear wave and a rarefaction wave γ_r is displaced to $\hat{\gamma}_r$ by interaction with a standing wave, then (assuming no cancellation of shock and rarefaction waves) $d(\gamma_0, \gamma_r) = d(\hat{\gamma}_0, \hat{\gamma}_r)$. Thus, $d(\gamma_0, \gamma_r)$ is invariant under such interactions even though $|\gamma_0|_w \neq |\hat{\gamma}_0|_w$ and

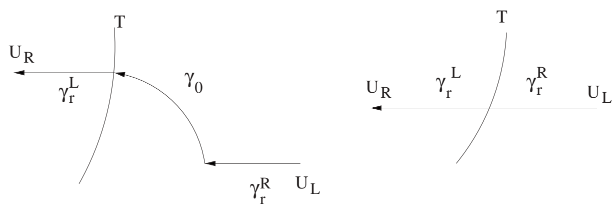

FIG. 30.

$|\gamma_r|_w \neq |\hat{\gamma}_r|_w$. Remarkably, this statement holds even when γ_0 is a composite wave of form $U_L \to P \to Q \to R$ of Figure 15. Therefore, even though wave strengths change as waves evolve in the solution, the potential interaction between waves is constructed so as to be an invariant of interactions (assuming no cancellation of rarefaction waves by shock waves).

The following proposition gives the main property that tells how rarefaction wave trajectories change when waves interact. To state the proposition, note that the rarefaction waves in any admissible solution of the Riemann problem $[U_L, U_R]$ can always be partitioned into a positive speed rarefaction wave γ_r^L on the left of $\mathcal{T}$ and a negative speed rarefaction wave γ_r^L on the right of $\mathcal{T}$. The fact that one can always uniquely identify exactly two such waves for every choice of U_L and U_R (allowing for one or both of the waves to be zero) can be verified directly in Figures 15–18. (Recall that a rarefaction wave cannot cross $\mathcal{T}$ in any admissible solution of the Riemann problem unless the standing wave is zero, in which case the solution is a single rarefaction wave γ_r that can be partitioned into $\gamma_r \sim \gamma_r^R \gamma_r^L$; cf. Figure 30.)

PROPOSITION 4.3. *Let γ_r^L and γ_r^R denote the left and right rarefaction waves in the solution of the Riemann problem $[U_L, U_R]$, and let $\bar{\gamma}_1 \bar{\gamma}_0 \bar{\gamma}_2$ be any connected sequence of elementary waves that takes U_L to U_R such that $\bar{\gamma}_1, \bar{\gamma}_2$ are nonlinear waves and $\bar{\gamma}_0$ is a standing wave. Then $Traj(\gamma_r^L) \subseteq Traj(\bar{\gamma}_r^L)$ and $Traj(\gamma_r^R) \subseteq Traj(\bar{\gamma}_r^R)$, where $\bar{\gamma}_r^L$ and $\bar{\gamma}_r^R$ denote the union of all left and right rarefaction waves, respectively, among $\bar{\gamma}_i$, i=1, 2.*

Proof. The proof of Proposition 4.3, which can be verified case by case from the admissible solution of the Riemann problem, is postponed until the appendix. □

5. The nonlocal functional. In this section we define the nonlocal functional $F(J)$ that bounds the total variation in w for the waves that cross an I-curve J in an approximate Glimm scheme solution. We then prove that F is nonincreasing on approximate solutions. To start, let J denote a fixed I-curve, and for notational convenience let Λ be an index set such that γ_q^α, $\alpha \in \Lambda$, $q \in \{0, r, s\}$, lists all of waves that cross J. Here $q = 0, r, s$ means that the wave is a standing wave, rarefaction wave, or shock wave, respectively, so that, for example, $\{\gamma_0^\alpha\}_{\alpha\in\Lambda}$ denotes the set of all standing waves that cross J, etc. (To achieve such an indexing just allow for arbitrarily many zero waves.) Thus the local functional $L_w(J)$ is defined by

$$L_w(J) = \sum_{\alpha\in\Lambda, q\in\{0,r,s\}} |\gamma_q^\alpha|_w. \tag{5.1}$$

Define the functional $F(J)$ by

$$F(J) = L_w(J) + P(J), \tag{5.2}$$

where $P(J)$ is the nonlocal potential interaction functional defined by

$$P(J) = \sum_{(\alpha,\beta)\in App(J)} d(\gamma_0^\alpha, \gamma_r^\beta), \tag{5.3}$$

where we use the notation $(\alpha, \beta) \in App(J)$ if and only if γ_0^α approaches γ_r^β on J. Since $F(J)$ is determined by the connected sequence of waves that cross J, we can similarly define $F(\gamma_a \cdots \gamma_b)$ for any connected sequence of elementary waves. (Two waves in the sequence *approach* if the left wave is faster than the right wave in the sequence, etc.) Then, for example, $F([U_L, U_R]) = L_w([U_L, U_R])$ because the solution of the Riemann problem contains no pairs of approaching waves. We now prove the following theorem.

THEOREM 5.1. *If J_2 is a successor of J_1 in an approximate Glimm scheme solution of system* (1.1)*, then*

$$F(J_2) - F(J_1) \leq 0. \tag{5.4}$$

The proof of Theorem 5.1 is a consequence of the following lemma. (We think of bar, tilde, and hat as identifying incoming waves and unbarred waves as representing outgoing waves, and $[U_L, U_R]$ denotes the admissible solution of the Riemann problem, where the strength of each zero speed composite wave γ_0 has a strength equal to the strength of the waves in its projection $P(\gamma_0)$.)

PROPOSITION 5.2. *Let $\bar\gamma_1\bar\gamma_0\bar\gamma_2$ denote any connected sequence of three elementary waves that takes U_L to U_R such that $\bar\gamma_1, \bar\gamma_2$ are nonlinear waves, and $\bar\gamma_0$ is a standing wave.*

$$F(\gamma_1\gamma_0\gamma_2) \leq F(\bar\gamma_1\bar\gamma_0\bar\gamma_2), \tag{5.5}$$

where $\gamma_1\gamma_0\gamma_2 = [U_L, U_R]$.

The proofs of Propositions 4.3 and 5.2 involve a case by case study of the Riemann problem and will be dealt with together in the appendix. Assuming Propositions 4.3 and 5.2, we now give the following proof.

Proof of Theorem 5.1. Assume that Propositions 4.3 and 5.2 hold, and assume that J_2 is an immediate successor of J_1 in the approximate Glimm scheme solution $U_{\Delta x}$ of system (1.1). We show that $F(J_2) \leq F(J_1)$. Let Δ denote the interaction diamond between J_1 and J_2, let J_1', J_2' denote the restriction of J_1, J_2 to the region Δ, respectively, and let J_0 denote the restrictions of J_1 and J_2 to the region *outside* Δ; cf. [4, 24]. Thus we write $J_1 = J_0 \cup J_1'$ and $J_2 = J_0 \cup J_2'$. Note that since we use an unstaggered grid, the states U_L and U_R that lie at the right and left vertices of Δ are consecutive sample points at some time level t_j in the approximate solution $U_{\Delta x}$, and thus there is at most one standing wave between U_L and U_R on both J_1' and J_2'. It follows that there are at most five incoming waves that cross J_1', i.e., at most two nonlinear waves $\bar\gamma_1^a$ and $\bar\gamma_1^b$, followed by a standing wave $\bar\gamma_0$, followed by at most two nonlinear waves $\bar\gamma_2^a\bar\gamma_2^b$. (Subscripts $1, 2$ denote nonlinear waves, and subscript 0 denotes a standing wave.) Thus let $\bar\gamma_1^a\bar\gamma_1^b\bar\gamma_0\bar\gamma_2^a\bar\gamma_2^b$ denote the connected sequence of elementary waves that take U_L to U_R and cross the curve J_1', the incoming waves for the interaction diamond Δ. The waves that leave the interaction diamond Δ cross J_2' and hence solve the Riemann problem $[U_L, U_R]$.

Now first let $\bar\gamma_1$ and $\bar\gamma_2$ denote the nonlinear waves such that $\bar\gamma_1 \sim \bar\gamma_1^a\bar\gamma_1^b$ and $\bar\gamma_2 \sim \bar\gamma_2^a\bar\gamma_2^b$. Define $\bar J_1'$ to be the I-curve obtained by replacing the waves on J_1' by the waves $\bar{\gamma}_1\bar{\gamma}_0\bar{\gamma}_2$, and set $\bar{J}_1 = J_0 \cup \bar{J}_1'$. Then the proof of Theorem 5.1 is complete once we prove the following claim.

CLAIM. *The following inequalities hold:*

$$F(J_2) \le F(\bar{J}_1) \le F(J_1). \tag{5.6}$$

Proof of claim. The second inequality holds because in replacing the nonlinear waves $\bar{\gamma}_i^a\bar{\gamma}_i^b$ by $\bar{\gamma}_i$, $i = 1, 2$, there can be no increase in wave strength—only a cancellation of wave strength can occur, this happening when one of $\bar{\gamma}_i^a, \bar{\gamma}_i^b$ is a shock wave and the other is a rarefaction wave. Thus $L_w(\bar{J}_1) - L_w(J_1) \le 0$. Moreover, since the potential $d(\gamma_0, \gamma_r)$ is in general independent of the partitioning of the nonlinear wave γ_r, it follows that $d(\gamma_0, \bar{\gamma}_i) \le d(\gamma_0, \bar{\gamma}_i^a) + d(\gamma_0, \bar{\gamma}_i^b)$ for any standing wave γ_0 on J_0, $i = 1, 2$. From this it readily follows that $P(\bar{J}_1) - P(J_1) \le 0$, and hence $F(\bar{J}_1) - F(J_1) = L_w(\bar{J}_1) - L_w(J_1) + P(\bar{J}_1) - P(J_1) \le 0$.

To verify that $F(J_2) \le F(\bar{J}_1)$, write

$$F(J_2) - F(\bar{J}_1) = F(J_2') - F(\bar{J}_1') + P(J_2', J_0) - P(\bar{J}_1', J_0) + P(J_2', J_2') - P(\bar{J}_1', \bar{J}_1'). \tag{5.7}$$

(Here we use the notation that if $J_a, J_b \subset J$, then $P(J_a, J_b) = \sum d(\gamma^\alpha, \gamma^\beta)$, where the sum is taken over all approaching waves on J such that $\gamma^\alpha \in J_a$, $\gamma^\beta \in J_b$.) But $P(J_2', J_2') = 0$ because the solution of the Riemann problem contains no approaching waves, and by Proposition 5.2, $F(J_2') - F(\bar{J}_1') \le 0$. Moreover, $P(J_2', J_0) \le P(\bar{J}_1', J_0)$ because, by Proposition 4.3, the trajectories of the rarefaction waves on $\bar{J}_1'$ contains the trajectories of the rarefactions waves on J_2'; hence there will be an interaction potential between rarefaction waves in $\bar{J}_1'$ and standing waves in J_0 that cancels any interaction potential between rarefaction waves in J_2' and standing waves in J_0. Thus (5.7), $F(J_2) - F(\bar{J}_1) \le 0$, and the proof of the claim is complete. ◻

The final theorem follows directly from Theorem 5.1.

THEOREM 5.3. *If the initial I-curve $J_{t=0}$ satisfies $F(J_{t=0}) < \infty$ in a Glimm approximate solution $U_{\Delta x}$, then the total variation of $U_{\Delta x}(\cdot, t) <$ const.$F(J_{t=0})$ for all $t > 0$.*

6. Convergence of the residual. In this section we give the proof of convergence of the residual for the approximate Glimm scheme solution constructed in section 3. The residual for system (1.1) is defined by

$$R(a, u, \varphi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{u\varphi_t + f\varphi_x + a'g\varphi\}\ dx\,dt + \int_{-\infty}^{+\infty} u_0(x)\varphi(x, 0)\ dx. \tag{6.1}$$

Then (a, u) is a *weak solution* of (1.1) if and only if $R(a, u, \varphi) = 0$ for all compactly supported smooth test functions $\varphi = \varphi(x, t)$. Assume that $U_{\Delta x}$ is a sequence of Glimm approximate solutions that satisfy

$$\mathrm{Var}_z U_{\Delta x}(\cdot, t) < V_z \tag{6.2}$$

for some constant V_z independent of Δx (cf. (3.6)), and assume $U_{\Delta x}(x, t) = (a_{\Delta x}(x), u_{\Delta x}(x, t)) \to U(x, t) = (a(x), u(x, t))$ piecewise a.e. and in L^1_{loc} at each fixed time, uniformly on compact sets (the conclusion of the Oleinik compactness argument; cf. [25]). Note that for fixed initial data, $U_{\Delta x}$ is a function of both Δx and the sample sequence $\theta = \{\theta_{ij}\} \in \Theta$. Assume that $a(x)$ is Lipschitz continuous, so that there exists a constant M such that

$$|a(x) - a(y)| \le M|x - y| \text{ for all } x, y \in \mathbf{R}, \tag{6.3}$$

$$|a(x) - a_{\Delta x}(x)| \le M\Delta x \text{ for all } x \in \mathbf{R}. \tag{6.4}$$

For an approximate solution $U_{\Delta x}$, define

$$R_{\Delta x}(a_{\Delta x}, u_{\Delta x}, \varphi) \equiv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{u_{\Delta x}\varphi_t + f(a_{\Delta x}, u_{\Delta x})\varphi_x + a' g(a_{\Delta x}, u_{\Delta x})\varphi\}\, dx\, dt + \int_{-\infty}^{+\infty} u_{\Delta x}(x,0)\varphi(x,0)\, dx \tag{6.5}$$

(obtained by replacing U by $U_{\Delta x}$ in (6.1) everywhere except at a'). We prove the following theorem; cf. [4].

THEOREM 6.1. *There exists a set $\mathcal{N}$ of measure zero in Θ such that, if $\theta \in \Theta/\mathcal{N}$, then*

$$R(a, u, \varphi) = \lim_{\Delta x \to 0} R_{\Delta x}(a_{\Delta x}, u_{\Delta x}, \varphi) = 0 \tag{6.6}$$

for all test functions φ of compact support in $-\infty < x < \infty$, $t \geq 0$. Thus, in particular, passing the limit through the integral sign, we conclude that $U(x,t)$ is a weak solution of (1.7).

Proof of Theorem 6.1. To start, let γ_{ij}^1 and γ_{ij}^2 denote the negative and positive speed waves positioned at mesh point (x_i, t_j) in the approximate solution $U_{\Delta x}$. Let $U_{ij}(x,t)$ denote the approximate solution $U_{\Delta x}$ restricted to the mesh rectangle $\mathcal{R}_{ij} = [x_i, x_{i+1}) \times [t_j, t_{j+1})$, and let $\mathrm{Var}_z U_{ij}$ and $\mathrm{Var}_u U_{ij}$ denote the total variation of U_{ij} in x at fixed time $t \in (t_j, t_{j+1})$, $x_i \leq x < x_{i+1}$. For the proof of Theorem 6.1, we introduce three regularization parameters $\epsilon, \hat{\epsilon}$, and δ, whose values will be chosen at the end: ϵ is a regularization parameter for the standing waves described below; $\hat{\epsilon}$ measures distance to the transition curve so that

$$S(\hat{\epsilon}) \equiv \{U : |U - \mathcal{T}| \leq \hat{\epsilon}\};$$

and δ is a mollification parameter for $g_{\Delta x}$ (so that we can integrate the source term in (6.6) by parts),

$$(g \cdot U_{\Delta x})_\delta = (g \cdot U_{\Delta x}) * \psi_\delta,$$

where $\psi_\delta = (\frac{1}{\delta^2})\psi(\frac{x}{\delta}, \frac{t}{\delta})$ denotes the standard convolution kernel supported on $|(x,t)| \leq \delta$.

For the mollification of the standing waves, let $U_{\Delta x}^\epsilon(x,t) \equiv (a_\epsilon(x), u_{\Delta x}^\epsilon(x,t))$ denote the regularization of $U_{\Delta x}$ obtained by translating γ_{ij}^1 (respectively, γ_{ij}^2) $\epsilon\Delta x$ units to the left (respectively, right) at each mesh point (x_i, t_j) and then replacing each standing wave discontinuity γ_{ij}^0 at (x_i, t_j) by the smoothed out standing wave on the interval $x_i - \epsilon\Delta x < x < x_i + \epsilon\Delta x$, as described in the discussion after (2.4). Thus, states on the smoothed out standing wave $\gamma_{ij}^{0,\epsilon}$ lie on the standing wave curve between the same left and right states as γ_{ij}^0 so that $\mathrm{Var}_z U_{ij}^\epsilon = \mathrm{Var}_z U_{ij}$ and $\mathrm{Var}_u U_{ij}^\epsilon = \mathrm{Var}_u U_{ij}$. (Indeed, recall that in Section 2, standing wave discontinuities were constructed as limits of smooth standing waves under rescaling into discontinuities; cf. [6].) Since U^ϵ satisfies the same total variation bounds as $U_{\Delta x}$, by taking appropriate subsequences, we can assume that at each $\epsilon > 0$, $\lim_{\Delta x \to 0} U_{\Delta x}^\epsilon = U^\epsilon$, where convergence is in the same sense as $U_{\Delta x} \to U$. (We are forced to introduce $U_{\Delta x}^\epsilon$ because our approximate solutions are constructed to (formally) meet (6.6) with $a'_{\Delta x}$, not a'.)

We use the following lemmas.

LEMMA 6.2. *There exists a constant $C_0 > 0$ and a function $\hat{K}(\hat{\epsilon})$ independent of Δx such that*

$$\mathrm{Var}_u U_{\Delta x}(\cdot, t) \leq C_0 \hat{\epsilon} + \hat{K}(\hat{\epsilon}) \mathrm{Var}_z U_{\Delta x}(\cdot, t), \tag{6.7}$$

$$\int_E |U^\epsilon_{\Delta x} - U_{\Delta x}| \, dx \, dt \leq C_0 |E| \epsilon, \tag{6.8}$$

$$\left| \frac{\partial}{\partial x} (g \cdot U^\epsilon_{\Delta x})_\delta \right| \leq \frac{C_0}{\delta}. \tag{6.9}$$

By (6.8) we know that $\int_E |U^\epsilon - U| \, dxdt < O(1)\epsilon$ for each compact set E.[9]

Proof. Estimate (6.7) follows from the fact that the mapping $(a, z) \to (a, u)$ is one-to-one and regular except at the transition curve $\mathcal{T}$ and any wave that lies entirely within $S(\hat{\epsilon})$ has amplitude order $\hat{\epsilon}$; cf. [25]. For (6.8), observe that $meas\{(x, t) \in E : U^\epsilon_{\Delta x} \neq U_{\Delta x}\} = O(1)|E|\epsilon$, where $|E|$ denotes the measure of the set E. Estimate (6.9) follows directly from the definition of convolution. □

LEMMA 6.3. *For every compact set E in $-\infty < x < \infty$, $t \geq 0$, there exists a function $K(\epsilon)$ independent of δ such that*

$$\int_E |(g \cdot U^\epsilon_{\Delta x})_\delta - (g \cdot U^\epsilon_{\Delta x})| \, dxdt = \int_E |(g \cdot U^\epsilon)_\delta - (g \cdot U^\epsilon)| \, dxdt + o(\Delta x) K(\epsilon), \tag{6.10}$$

$$\int_E |(g \cdot U^\epsilon_{\Delta x})_\delta - (g \cdot U_{\Delta x})| \, dxdt = \int_E |(g \cdot U^\epsilon)_\delta - (g \cdot U)| \, dxdt + o(\Delta x) K(\epsilon). \tag{6.11}$$

Here we mean that $o(\Delta x)$ is independent of $\epsilon, \hat{\epsilon}$, and δ, and $\lim_{\Delta x \to 0} o(\Delta x) = 0$.

Proof. Both (6.10) and (6.11) follow directly from the convergence of $U^\epsilon_{\Delta x} \to U^\epsilon$ and $U_{\Delta x} \to U$. □

The next lemma is the main step in the proof of Theorem 6.1.

LEMMA 6.4. *Let*

$$R^\epsilon_\phi \equiv R(a_\epsilon, u^\epsilon_{\Delta x}, \varphi) = \int \int_{t \geq 0} U^\epsilon_{\Delta x} \phi_t + f(U^\epsilon_{\Delta x}) \phi_x + a'_\epsilon g(U^\epsilon_{\Delta x}) \phi dxdt + \int_{-\infty}^{+\infty} U_0(x) \phi(x, 0) dx, \tag{6.12}$$

and write $R^\epsilon_\phi \equiv R^\epsilon_\phi(\theta)$ to express the dependence on $\theta \in \Theta$ when Δx and ϕ are fixed. Then there exists a constant C_1 such that

$$\int_\Theta \left(R^\epsilon_\phi\right)^2 d\theta \leq O(1) \left\{ \hat{\epsilon} + \hat{K}(\hat{\epsilon}) \Delta x + \epsilon (C_0 \hat{\epsilon} + \hat{K}(\hat{\epsilon}))^2 \right\}. \tag{6.13}$$

Proof of Lemma 6.4. Since $U^\epsilon_{\Delta x}$ is an exact solution in each strip $t_j < t < t_{j+1}$, integrating (6.13) over each mesh rectangle $\mathcal{R}_{ij}$ gives

$$R^\epsilon_\phi = \sum_{i,j} D^\epsilon_{ij}(\theta, \Delta x, \phi), \tag{6.14}$$

[9] We use the notation that C_0, C_1 denote constants that can depend on the equations and the initial data but are independent of $\epsilon, \hat{\epsilon}, \delta, \Delta x$, and the test function ϕ, while $O(1)$ denotes a constant that is independent of $\epsilon, \hat{\epsilon}, \delta$, and Δx, the convergence parameters.

where for $j > 0$,

$$D^{\epsilon}_{ij}(\theta, \Delta x, \phi) = \int_{x_i}^{x_{i+1}} \left\{ U^{\epsilon}_{\theta,\Delta x}(x, t_j+) - U^{\epsilon}_{\theta,\Delta x}(x, t_j-) \right\} \phi(x, t_j-) dx$$

$$\equiv \int_{x_i}^{x_{i+1}} [U^{\epsilon}_{ij}] \phi| dx \tag{6.15}$$

and

$$D^{\epsilon}_{i0}(\theta, \Delta x, \phi) = \int_{x_i}^{x_{i+1}} \left\{ U^{\epsilon}_{\theta,\Delta x}(x, 0) - U_0(x) \right\} \phi(x, 0) dx$$

$$\equiv \int_{x_i}^{x_{i+1}} [U^{\epsilon}_{i0}] \phi| dx. \tag{6.16}$$

(We take definitions (6.14)–(6.16) as applying also at $\epsilon = 0$, $U^0_{\Delta x} = U_{\Delta x}$.) It follows directly from (6.15) and (6.16) that

$$\left| D^{\epsilon}_{ij}(\theta, \Delta x, \phi) \right| \leq |\mathrm{Supp}(\phi)| \; ||\phi||_{\infty} \Delta x \mathrm{Var}_u U^{\epsilon}_{ij}. \tag{6.17}$$

Now let $O(1)$ denote a constant that is independent of $\epsilon, \hat{\epsilon}, \delta$, and Δx.

Claim. *The following estimate holds:*

$$\left| \int_{\Theta} D^{\epsilon}_{ij} D^{\epsilon}_{kl} d\theta \right| \equiv |\langle D^{\epsilon}_{ij}, D^{\epsilon}_{kl} \rangle| \leq O(1) \epsilon \Delta x^2 \cdot \mathrm{Var}_u U^{\epsilon}_{ij} \cdot \mathrm{Var}_u U^{\epsilon}_{kl}. \tag{6.18}$$

Proof of claim. First, neglecting higher order terms in Δx, we can assume without loss of generality that ϕ is constant on mesh rectangles, $\phi = \phi_{ij} = \mathrm{const}$ on $\mathcal{R}_{ij}$. Following the argument in [6], we first note that if $j < l$, then U_{ij} is independent of a_{kl}, and so we can pass da_k through the integral to the factor $\int_0^1 \int_{x_k}^{x_{k+1}} D_{kl} dx da_k$, which is equal to zero as in Glimm's original argument. Thus,

$$\int_{\Theta} D^{\epsilon}_{ij} D_{kl} d\theta = 0. \tag{6.19}$$

Therefore,

$$\begin{aligned} \left| \langle D^{\epsilon}_{ij}, D^{\epsilon}_{kl} \rangle \right| &= \left| \langle D^{\epsilon}_{ij}, D^{\epsilon}_{kl} \rangle - \langle D^{\epsilon}_{ij}, D_{kl} \rangle \right| \\ &= \left| \langle D^{\epsilon}_{ij}, D^{\epsilon}_{kl} - D_{kl} \rangle \right| \leq ||D^{\epsilon}_{ij}||_{\infty} ||D^{\epsilon}_{kl} - D_{kl}||_{\infty} \\ &\leq O(1) \mathrm{Var}_u U^{\epsilon}_{ij} ||D^{\epsilon}_{kl} - D_{kl}||_{\infty} \Delta x. \end{aligned} \tag{6.20}$$

But

$$||D^{\epsilon}_{kl} - D_{kl}||_{\infty} \leq \int_{x_k}^{x_{k+1}} |[U^{\epsilon}_{kl}] - [U_{kl}]| \, |\phi| dx \leq O(1) \epsilon \Delta x \mathrm{Var}_u U_{kl}, \tag{6.21}$$

and using this in (6.20) gives

$$\left| \langle D^{\epsilon}_{ij}, D^{\epsilon}_{kl} \rangle \right| \leq O(1) \epsilon \Delta x^2 \cdot \mathrm{Var}_u U_{ij} \cdot \mathrm{Var}_u U_{ij} \tag{6.22}$$

as claimed.

Thus we can estimate

$$\int_\Theta \left(R_\phi^\epsilon\right)^2 d\theta = \int_\Theta \left(\sum_{ij} D_{ij}^\epsilon\right)^2 d\theta$$

$$= \sum_i \int_\Theta \left(D_{ij}^\epsilon\right)^2 d\theta + \sum_{ij,kl} \int_\Theta D_{ij}^\epsilon D_{kl}^\epsilon d\theta \tag{6.23}$$

$$= I + II, \tag{6.24}$$

where

$$|I| \le \sum_{ij} \left(\int_{x_i}^{x_{i+1}} [U_{ij}^\epsilon]\phi dx\right)^2 \le O(1) \sum_{ij} \Delta x^2 \left(\mathrm{Var}_u U_{ij}^\epsilon\right)^2$$

$$\le O(1) \left\{C_0\hat\epsilon + \hat K(\hat\epsilon)\right\} \Delta x \tag{6.25}$$

and

$$|II| \le \sum_{ij,kl} O(1)\epsilon\Delta x^2 \cdot \mathrm{Var}_u U_{ij} \cdot \mathrm{Var}_u U_{kl}$$

$$\le \sum_{ij,kl} O(1)\epsilon\Delta x^2 \cdot (C_0\hat\epsilon + \hat K(\hat\epsilon))^2. \tag{6.26}$$

Thus

$$|I| + |II| \le O(1) \left\{\hat\epsilon + \hat K(\hat\epsilon)\Delta x + \epsilon(C_0\hat\epsilon + \hat K(\hat\epsilon))^2\right\}, \tag{6.27}$$

which verifies (6.13) of Lemma 6.4. ◻

Now that we have an estimate for R_ϕ^ϵ in Lemma 6.4; we obtain an estimate for $R_\phi \equiv R_{\Delta x}(a_{\Delta x}, u_{\Delta x}, \varphi)$ by estimating the difference $|R_\phi^\epsilon - R_\phi|$,

$$|R_\phi| \le \left|R_\phi^\epsilon - R_\phi\right| + \left|R_\phi^\epsilon\right|. \tag{6.28}$$

Lemma 6.5. *The following estimate holds:*

$$\left|R_\phi^\epsilon - R_\phi\right| \le O(1) \left\{\epsilon + o(\Delta x)K(\epsilon) + \frac{\Delta x}{\delta} + \int\int_E |(g\cdot U^\epsilon)_\delta - g(U^\epsilon)|\, dx\, dt + \int\int_E |(g\cdot U^\epsilon)_\delta - g(U)|\, dx\, dt\right\}, \tag{6.29}$$

where E denotes the support of ϕ and $O(1)$ denotes a constant independent of $\epsilon, \hat\epsilon, \delta$, and Δx.

Proof of Lemma 6.5. Starting with (6.5) and (6.12), we obtain

$$\begin{aligned}\left|R_\phi^\epsilon - R_\phi\right| &\le \int\int_{t\ge 0} |U_{\Delta x}^\epsilon - U_{\Delta x}|\,|\phi_t|\, dx\, dt \\ &\quad + \int\int_{t\ge 0} |f(U_{\Delta x}^\epsilon) - f(U_{\Delta x})|\,|\phi_x|\, dx\, dt \\ &\quad + \left|\int\int_{t\ge 0} \{a_\epsilon' g(U_{\Delta x}^\epsilon) - a' g(U_{\Delta x})\}\, \phi\, dx\, dt\right| \\ &= I_1 + I_2 + I_3.\end{aligned} \tag{6.30}$$

It follows from (6.9) that

$$|I_1| \le O(1)\epsilon, \tag{6.31}$$

$$|I_2| \le O(1)\epsilon, \tag{6.32}$$

and it remains to estimate I_3. But

$$\begin{aligned}|I_3| \le \int\int_{t\ge 0} &|a_\epsilon'|\,|g(U^\epsilon_{\Delta x}) - (g\cdot U^\epsilon_{\Delta x})_\delta|\,|\phi|\,dx\,dt \\ &+ \int\int_{t\ge 0} |a_\epsilon'|\,|(g\cdot U^\epsilon_{\Delta x})_\delta - g(U_{\Delta x})|\,|\phi|\,dx\,dt \\ &\qquad + \left|\int\int_{t\ge 0} \frac{d}{dx}(a_\epsilon - a)\,(g\cdot U^\epsilon_{\Delta x})_\delta\,\phi\,dx\,dt\right| \\ = I_{3a} + I_{3b} + I_{3c},&\end{aligned}$$

and using Lemma 6.3 we obtain

$$|I_{3a}| \le O(1)\left\{\frac{1}{\epsilon}\int\int_{t\ge 0} |(g\cdot U^\epsilon)_\delta - g(U^\epsilon)|\;dx\,dt + o(\Delta x)K(\epsilon)\right\} \tag{6.33}$$

and

$$|I_{3b}| \le O(1)\left\{\int\int_{t\ge 0} |(g\cdot U^\epsilon)_\delta - g(U)|\;dx\,dt + o(\Delta x)K(\epsilon)\right\}. \tag{6.34}$$

Finally, integrating I_{3c} by parts and using (6.9) we obtain

$$|I_{3c}| \le O(1)\int\int_{t\ge 0} |a_\epsilon - a|\left|\frac{d}{dx}(g\cdot U^\epsilon)_\delta\right|\,dx\,dt \le O(1)\frac{\Delta x}{\delta}. \tag{6.35}$$

Putting (6.31)–(6.35) into (6.30) yields (6.29) of Lemma 6.5. ∎

We can now give the following proof.

Proof of Theorem 6.1. To establish (6.6) for $R_{\Delta x}(a_{\Delta x}, u_{\Delta x}, \varphi) \equiv R_\phi$, we show that

$$\lim_{\Delta x\to 0}\int_\Theta R_\phi^2\,d\theta = 0. \tag{6.36}$$

To this end, using (6.28) we can write

$$\begin{aligned}\int_\Theta R_\phi^2\,d\theta &\le 2\int_\Theta \left(R_\phi^\epsilon\right)^2\,d\theta + 2\int_\Theta \left|R_\phi^\epsilon - R_\phi\right|^2\,d\theta \\ &\le O(1)\left\{\hat\epsilon + \left[\hat K(\hat\epsilon)\Delta x\right]_1 + \left[\epsilon(\hat\epsilon + \hat K(\hat\epsilon)\right]_2\right\} \\ &\quad + O(1)\left\{\left[\epsilon + \int\int_E |(g\cdot U^\epsilon)_\delta - g(U^\epsilon)|\;dx\,dt\right.\right. \\ &\qquad\qquad \left. + \int\int_E |(g\cdot U^\epsilon)_\delta - g(U)|\;dx\,dt\right]_3 \\ &\qquad\qquad\qquad \left. + \left[o(\Delta x)K(\epsilon) + \frac{\Delta x}{\delta}\right]_4\right\}^2,\end{aligned} \tag{6.37}$$

where we have applied (6.13) and (6.29). Now let τ be any small positive number. Then, to make $\int_\Theta R_\phi^2\, d\theta < \tau$, choose $\hat\epsilon, \epsilon, \delta$ and $\Delta x \ll 1$ in order as follows (the brackets and $O(1)$ refer to quantities defined in (6.37)). First choose $\hat\epsilon \ll 1$ so that

$$O(1)\hat\epsilon < \frac{\tau}{4}; \tag{6.38}$$

choose $\epsilon < \epsilon_0 \ll 1$ so that

$$O(1)\,[\cdot]_2 < \frac{\tau}{4}; \tag{6.39}$$

choose $\epsilon \ll \epsilon_0$ and $\delta \ll 1$ so that

$$[\cdot]_3 < \frac{1}{2}\sqrt{\left\{\frac{\tau/4}{O(1)}\right\}}; \tag{6.40}$$

finally, choose $\Delta x \ll 1$ so that

$$[\cdot]_1 < \frac{\tau}{4} \quad \text{and} \quad [\cdot]_4 < \frac{1}{2}\sqrt{\left\{\frac{\tau/4}{O(1)}\right\}}. \tag{6.41}$$

Putting (6.38)–(6.41) into (6.37), we obtain

$$\int_\Theta R_\phi^2\, d\theta < \frac{\tau}{4} + \frac{\tau}{4} + \frac{\tau}{4} + \frac{1}{2}\left(\frac{\tau}{4} + \frac{\tau}{4}\right) < \tau. \tag{6.42}$$

From (6.42) we conclude (6.36), from which we conclude that $R_\phi \to 0$ off a set of measure zero in Θ. Theorem 6.1 now follows by taking a countable dense set of test functions, extracting a set of measure for each one, and taking $\theta \in \Theta/\mathcal{N}$, where $\mathcal{N}$ is the union of the measure zero sets for each of the countable list of test functions; cf. [4]. This completes the proof of Theorem 6.1. $\square$

7. Appendix. In this appendix, we verify Propositions 4.3 and 5.2.

Proof of Proposition 4.3. Let $\bar\gamma_1\bar\gamma_0\bar\gamma_2$ be any connected sequence of incoming waves that take $U_L \to U_R$, and let $[U_L, U_R] = \gamma_1\gamma_0\gamma_2$. To verify the proposition, we can list the sixteen possibilities for $\bar\gamma_1\bar\gamma_0\bar\gamma_2$ according to whether γ_i are shock waves or rarefaction waves ($i = 1, 2$, four cases), whether γ_0 lies to the left or right of $\mathcal{T}$, and whether a increases or decreases across γ_0. (Since the issue involves only the location of the standing wave curves, it is not important whether $g_u > 0$ or $g_u < 0$.) In each case it is easy to verify that the rarefaction waves in the solution of the Riemann problem lie within the standing wave curves that bound the rarefaction waves among the incoming waves $\bar\gamma_1\bar\gamma_0\bar\gamma_2$. It follows that $Traj(\gamma_r^L) \subseteq Traj(\bar\gamma_r^L)$ and $Traj(\gamma_r^R) \subseteq Traj(\bar\gamma_r^R)$ in each case. The details are omitted. $\square$

Proof of Proposition 5.2. We show that $F(\gamma_1\gamma_0\gamma_2) \le F(\bar\gamma_1\bar\gamma_0\bar\gamma_2)$ for any connected sequence of incoming waves $\bar\gamma_1\bar\gamma_0\bar\gamma_2$ that take U_L to U_R, where the outgoing waves $\gamma_1\gamma_0\gamma_2 = [U_L, U_R]_P$. (Recall that $[U_L, U_R]_P$ is obtained from $[U_L, U_R]$ by replacing every triple composite wave by its projection. Note that no wave can precede or follow, a triple composite wave in $[U_L, U_R]$ when $g_u < 0$ or $g_u > 0$, respectively, so that $[U_L, U_R]_P$ always consists of three waves $\gamma_1\gamma_0\gamma_2$.) We verify $F(\gamma_1\gamma_0\gamma_2) \le F(\bar\gamma_1\bar\gamma_0\bar\gamma_2)$ in four salient cases diagrammed in Figure 31. All other cases follow by a concatenation of these cases.

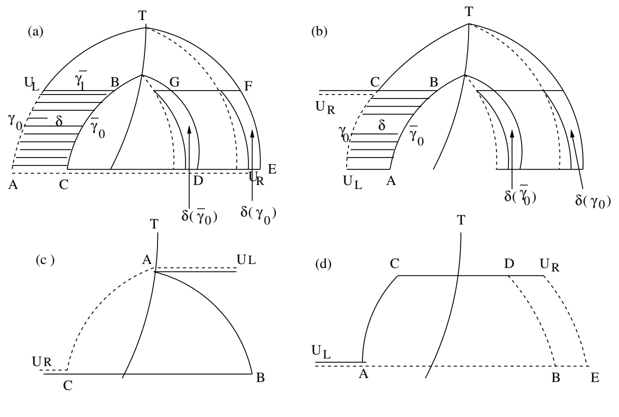

FIG. 31.

Cases (a) and (b) of Figure 31 deal with regular interactions in which the standing waves lie on the same side of $\mathcal{T}$ before and after interaction (the case $\gamma_0, \bar{\gamma}_0 < \mathcal{T}$ is considered). The point here is that when $\bar{\gamma}_0$ interacts with a shock wave (Case (a)) F decreases because L_w^* decreases on standing wave–shock wave interactions, and this decrease dominates the change in the corrective terms $\delta(\bar{\gamma}_0)$ and $\delta(\gamma_0)$ which were added to make L_w continuous. Cases (c) and (d) deal with the case when the standing waves $\bar{\gamma}_0$ and γ_0 lie on opposite sides of the transition curve, and the crossing occurs by rarefaction wave and shock wave, respectively. We now discuss the cases (a)–(d) of Figure 31 in detail.

Case (a). In this case, $F([U_L, U_R]_P) - F(\bar{\gamma}_1\bar{\gamma}_0\bar{\gamma}_2) = L_w(U_L \to A \to C) - L_w(U_L \to B \to C)$. When $g_u < 0$, $\delta(\bar{\gamma}_0) = 0 = \delta(\gamma_0)$ because these correction terms are added to standing waves on the right of $\mathcal{T}$ in this case. So when $g_u < 0$, $L_w(U_L \to A \to C) - L_w(U_L \to B \to C) = L_w^*(U_L \to A \to C) - L_w^*(U_L \to B \to C) < 0$ by Lemma 2.7. On the other hand, when $g_u > 0$, we have $L_w(U_L \to A \to C) - L_w(U_L \to B \to C) = L_w^*(U_L \to A \to C) - L_w^*(U_L \to B \to C) + \delta(\gamma_0) - \delta(\bar{\gamma}_0) - \delta < 0$, because $\delta + \delta(\gamma_0) - \delta(\bar{\gamma}_0) < 0$ by Lemma 2.7. (That is, the decrease $-\delta$ in L_w^* due to interaction with the shock wave $A \to C$ dominates the change $\delta(\gamma_0) - \delta(\bar{\gamma}_0)$.)

Case (b). In this case, $F([U_L, U_R]_P) - F(\bar{\gamma}_1\bar{\gamma}_0\bar{\gamma}_2) = L_w(U_L \to C) - L_w(U_L \to A \to B \to C)$. (Here, $d_r(\bar{\gamma}_0, \bar{\gamma}_2) = 0$ because the waves $\bar{\gamma}_0$ and $\bar{\gamma}_2$ do not approach.) Now if $g_u < 0$, then $\delta(\gamma_0) = 0 = \delta(\bar{\gamma}_0)$ (because these correction terms are added to the waves on the right of $\mathcal{T}$ when $g_u < 0$), so we have $L_w(U_L \to C) - L_w(U_L \to A \to B \to C) = L_w^*(U_L \to C) - L_w^*(U_L \to A \to B \to C) \leq 0$ by Lemma 2.7. On the other hand, when $g_u > 0$ we have $L_w(U_L \to C) = L_w^*(U_L \to C) + \delta(\gamma_0)$ and $L_w(U_L \to A \to B \to C) = L_w^*(U_L \to A \to B \to C) + \delta(\bar{\gamma}_0)$. But $L_w^*(\gamma_0) + L_w^*(C \to B) - L_w^*(U_L \to A \to B) = -\delta < 0$, and so $L_w^*(\gamma_0) - L_w^*(U_L \to A \to B \to C) \leq -\delta - L_w^*(C \to B)$. Thus, $L_w(U_L \to C) - L_w(U_L \to A \to B \to C) = -\delta - L_w^*(C \to B) + \delta(\gamma_0) - \delta(\bar{\gamma}_0) \leq 0$ as claimed.

Case (c). In this case, $F([U_L, U_R]_P) - F(\bar{\gamma}_1\bar{\gamma}_0\bar{\gamma}_2) = L_w(A \to C) - L_w(A \to B \to U_R) \leq L_w^*(A \to B \to C) - L_w^*(A \to B \to U_R) = -L_w^*(C \to U_R) < 0$ by Lemma 2.7. (Here, $d_r(\bar{\gamma}_0, \bar{\gamma}_2) = 0$ because $B \to C$ lies below the standing wave curve through $\bar{\gamma}_0$, and $C \to U_R$ does not approach $\bar{\gamma}_0$.)

Case (d). In this case, choose D between C and U_R and B between A and E such that A, C, B, D lie on the same standing wave curve and $L_w(A \to C \to D) = L_w(A \to B \to D)$. Then $F([U_L, U_R]_P) - F(\bar{\gamma}_1\bar{\gamma}_0\bar{\gamma}_2) = L_w(A \to E \to U_R) - L_w(A \to$

$C \to U_R) = L_w(A \to E \to U_R) - L_w(A \to B \to D \to U_R) = L_w(B \to E \to U_R) - L_w(B \to D \to U_R) \le 0$ by the analysis of Case (a) (that is, we reduced the problem to the case of regular interaction on the right of $\mathcal{T}$).

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