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SHOCK WAVE COSMOLOGY INSIDE A BLACK HOLE: THE CASE OF NON-CRITICAL EXPANSION

JOEL SMOLLER*
Department of Mathematics, University of Michigan
Ann Arbor, MI 48109, USA

BLAKE TEMPLE†
Department of Mathematics, University of California, Davis
Davis CA 95616, USA

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Abstract. We derive and analyze the equations that extend the results in [20, 21] to the case of non-critical expansion $k \neq 0$. By an asymptotic argument we show that the equation of state $p = \frac{c^2}{3}\rho$ plays the same distinguished role in the analysis when $k \neq 0$ as it does when $k = 0$: only for this equation of state does the shock emerge from the Big Bang at a finite nonzero speed — the speed of light. We also obtain a simple closed system that extends the case $\sigma = \frac{p}{\rho} =$ const. considered in [20, 21] to the case of a general positive, increasing, convex equation of state $p = p(\rho)$.

Keywords: Cosmology; general relativity; shock waves; black hole; white hole.

1. Introduction

We derive the equations that extend the work in [20, 21] to the case of non-critical expansion $k \neq 0$. In [20, 21] we derived a system of equations that describe a shock wave cosmology in which a wave at the leading edge of the expansion of the galaxies is modeled by a shock wave in a critically expanding Friedmann-Robertson-Walker (FRW) universe. In the shock wave model, the Big Bang is a bounded explosion of finite total mass that generates a shock wave, and for the case in [20, 21], the shock wave lies beyond one Hubble length from the FRW center. The main point is that when the shock wave lies beyond one Hubble length, an arbitrarily large region of

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uniform expansion can be created behind the wave at the instant of the Big Bang, but this also implies that the total mass inside the shock wave satisfies $\frac{2M}{r} > 1$, and in this sense the solution evolves *inside a Black Hole*. One of the real surprises in the analysis was that the equation of state $p = \sigma\rho$, $\sigma = \frac{c^2}{3}$, correct for the earliest stage of Big Bang physics, plays a distinguished role in the analysis — only for $\sigma = \frac{c^2}{3}$ does the shock wave emerge from the Big Bang at a finite nonzero speed, and this speed calculates out to be exactly the speed of light (the shock wave then decelerates to subluminous speed for all times after the Big Bang). This naturally raises the question as to whether this result is special to the case of critical expansion $k = 0$. In this note we show that this is *not* the case. That is, we derive the corresponding equations for the case $k \neq 0$, and by asymptotic analysis we show that again the shock wave emerges at the Big Bang at finite nonzero speed only for the case $\sigma = \frac{c^2}{3}$, and again this nonzero speed is the speed of light.

In either of the cases $k = 0$ or $k \neq 0$, there are two ODE's that determine the evolution of the FRW metric inside the shock wave, and three ODE's that determine the metric beyond the shock (which we called the TOV metric *inside the Black Hole*); since one also needs an equation for the shock position, it might appear at the start that in total six equations are necessary. We showed in [20,21] that when $k = 0$, certain transformations of the variables uncouple the equations, and thus reduce the six equations to the solution of a non-autonomous scalar equation which can be completely analyzed in the phase plane. However, when $k \neq 0$, the corresponding changes of variables lead only to a reduction to a system of three coupled, non-autonomous equations, and this is more complicated to analyze. For this reason we content ourselves with an asymptotic analysis at the rest point that corresponds to the Big Bang. In [20,21] we only considered the case $\sigma =$ const., $k = 0$, but an unexpected corollary of the analysis here leads to the derivation of a simplified set of equations for the general case when σ is *not* constant, and the FRW equation of state $p = p(\rho)$ is any given function of the density ρ satisfying $p'(\rho) > 0$, $p''(\rho) > 0$. Furthermore, when $\sigma =$ const. but $k \neq 0$, the transformation of the variables leads to a reduction to a system of two coupled non-autonomous equations.

In Sec. 2 we derive the shock equations for the case $k \neq 0$, and in Sec. 3 we do the asymptotics that give the shock speed as a function of σ at the rest point that corresponds to the Big Bang. In the final section we discuss the case when $\sigma = \frac{p}{\rho}$ is not constant. For references to shock waves and general relativity see [1,3,6,8,9,22,23] as well as the authors' prior work in [9–21].

2. Derivation of the Equations

In [20,21] we derived a set of ODE's that describe the matching of a $k = 0$, FRW metric to what we termed the TOV metric *inside the Black Hole*, across a shock wave interface. The initial conditions were then determined by an entropy condition (the condition that the shock be compressive, cf. [4,7]) and this condition, which we showed held globally along the orbit, broke the time symmetry of the solution and picked the explosion over the implosion. In this section, we derive the corresponding equations when $k \neq 0$.

To start, recall that the line element for the FRW metric in standard coordinates takes the form

$$ds^2 = -dt^2 + R(t)^2\left(\frac{dr^2}{1-kr^2} + r^2\,d\Omega^2\right), \tag{2.1}$$

and the line element for the TOV metric *inside the Black Hole* takes the form

$$ds^2 = -B\,d\bar{t}^{\,2} + A^{-1}\,d\bar{r}^2 + \bar{r}^2\,d\Omega^2, \tag{2.2}$$

with $A = A(\bar{r})$ and $B = B(\bar{r})$. Both metrics assume a stress tensor for a perfect fluid co-moving with the metric g,

$$T^{ij} = (\rho c^2 + p)w^i w^j + p g^{ij}, \tag{2.3}$$

where p is the pressure, ρ is the density, $\mathbf{w}$ is the velocity vector, $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$ denotes the line element on the unit 2-sphere, and we assume the speed of light $c = 1$ when convenient. For the TOV metric we assume

$$A = 1 - \frac{2M}{\bar{r}} < 0, \tag{2.4}$$

which is equivalent to the condition $\frac{2M}{\bar{r}} > 1$. This implies that $\bar{t}$ is the spacelike variable and $\bar{r}$ is the timelike variable in the TOV metric (2.2), and it is in this sense that we say the solution lies *inside the Black Hole.*

In the case of the FRW metric, $R(t)$ is the cosmological scale factor, sign(k) determines the curvature of the 3-surfaces at constant t, and by rescaling the radial coordinate r, we can assume k takes one of the values $\{-1, 0, 1\}$. The Einstein equations $G = \kappa T$ for the FRW metric reduce to the two equations

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{\kappa}{3}\rho - \frac{k}{R^2}, \tag{2.5}$$

$$\dot{\rho} = -3(p + \rho)H, \tag{2.6}$$

cf. [23]. Here H denotes the Hubble constant

$$H \equiv H(t) = \frac{\dot{R}(t)}{R(t)}, \tag{2.7}$$

and the variable $\bar{r}$,

$$\bar{r} = R(t)r, \tag{2.8}$$

measures arc length in the radial direction at each fixed time of an FRW spacetime.

In the case of the TOV metric *inside the Black Hole*, the Einstein equations, derived in [20, 21], take the form[a]

$$\bar{p}' = \frac{\bar{p}+\bar{\rho}}{2}\frac{N'}{N-1}, \tag{2.9}$$

$$N' = -\left\{\frac{N}{\bar{r}} + \kappa\bar{p}\bar{r}\right\}, \tag{2.10}$$

$$\frac{B'}{B} = -\frac{1}{N-1}\left\{\frac{N}{\bar{r}} + \kappa\bar{\rho}\right\}, \tag{2.11}$$

where *prime* denotes $\frac{d}{d\bar{r}}$, and the equations only apply when $N = 1 - A > 1$.

To match the FRW to the TOV metric *inside the Black Hole* across a shock interface, we transform the FRW metric over to standard Schwarzschild coordinates where the line element takes the form (2.2) with $A \equiv A_{\mathrm{FRW}}(\bar{r},\bar{t})$ and $B \equiv B_{\mathrm{FRW}}(\bar{r},\bar{t})$, and equate the corresponding $A(\bar{r})$, $B(\bar{r})$ of the TOV metric *inside the Black Hole*. Formulas for $A_{\mathrm{FRW}}(\bar{r},\bar{t})$ and $B_{\mathrm{FRW}}(\bar{r},\bar{t})$, were derived in [20, 21], and are given by

$$A_{\mathrm{FRW}} = 1 - kr^2 - (\bar{r}H)^2, \tag{2.12}$$

$$B_{\mathrm{FRW}} = \frac{1}{\psi^2}\left(\frac{1-kr^2}{1-kr^2-(\bar{r}H)^2}\right). \tag{2.13}$$

Thus, matching the $d\bar{r}^2$, $d\bar{t}^2$ and $d\Omega^2$ components of the metric lead to the following identities that hold *at the shock surface*:

$$N = 1 - A = kr^2 + \bar{r}^2H^2, \tag{2.14}$$

$$AB = A_{\mathrm{FRW}}B_{\mathrm{FRW}} = \frac{1-kr^2}{\psi^2}, \tag{2.15}$$

$$\bar{r} = R(t)r. \tag{2.16}$$

Here, we let A, B unsubscripted denote the TOV metric components that depend only on $\bar{r}$, and ψ is an integrating factor that arises from the transformation of the FRW metric over to standard Schwarzschild coordinates, cf. [20, 21]. Putting (2.5) into (2.14) gives the condition which holds at the shock surface,

$$M = \frac{\kappa}{6}\rho\bar{r}^3 \Leftrightarrow N = \frac{\kappa}{3}\rho\bar{r}^2, \tag{2.17}$$

where $M = M(\bar{r})$ comes from the TOV side of the shock and $\rho = \rho(t)$ comes from the FRW side. Equation (2.17) implicitly determines the shock surface $\bar{r} = \bar{r}(t)$. Equations (2.12)–(2.17) determine a Lipschitz continuous matching of a general FRW metric to a general TOV metric *inside the Black Hole*. It remains to derive a conservation condition that rules out any delta function sources that can lie on

[a] Recall that the essential reason why the equations for the TOV metric *inside the Black Hole* take a different form than the standard TOV equations "outside the Black Hole", is that the assumption that the fluid is *co-moving* with the metric puts the nonzero component of the velocity vector **w** on the timelike coordinate $\bar{r}$ when $A < 0$, and on the timelike coordinate $\bar{t}$ when $A > 0$, cf. [20, 21].

the shock surface due to the fact that the Einstein equations are second order in the metric components. We showed in [20, 21] that, for this purpose, it suffices to determine a condition that implies $\det[T^{\mu\nu}_{\rm FRW} - T^{\mu\nu}_{\rm TOV}] = 0$ in standard (barred) Schwarzschild coordinates, on the shock surface. (Note that the necessity of this condition follows from the Rankine–Hugoniot jump conditions because $[T] \cdot \mathbf{n} = 0$ implies $\det[T] = 0$.) To obtain such a condition, a calculation gives that

$$T^{\mu\nu}_{\rm TOV} = -(\bar\rho + \bar p)\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \bar p A \begin{bmatrix} 1 & 0 \\ 0 & -\dfrac{\psi^2}{1-kr^2} \end{bmatrix}, \tag{2.18}$$

$$T^{\mu\nu}_{\rm FRW} = (\rho + p)\begin{bmatrix} H^2\bar r^2 & \psi H \bar r \\ \psi H \bar r & \psi^2 \end{bmatrix} + pA \begin{bmatrix} 1 & 0 \\ 0 & -\dfrac{\psi^2}{1-kr^2} \end{bmatrix}. \tag{2.19}$$

The latter is obtained by mapping the components of the FRW stress tensor over from unbarrred to barred coordinates using the Jacobian

$$\frac{\partial \bar x^\mu}{\partial x^\alpha} = \bar p A \begin{bmatrix} 1 - kr^2 - A & R \\ \psi & \dfrac{\psi R}{1-kr^2}(1 - kr^2 - A) \end{bmatrix},$$

where $x^\alpha = (t, r)$, $\alpha = (0, 1)$, but $\bar x^\mu = (\bar r, \bar t)$, $\mu = (0, 1)$ because $A < 0$ implies that $\bar r$ is timelike *inside the Black Hole*. Subtracting $T^{\mu\nu}_{\rm TOV}$ from the second matrix in (2.19) gives

$$A \begin{bmatrix} p + \bar\rho & 0 \\ 0 & -\psi^2 \dfrac{p - \bar p}{1 - kr^2} \end{bmatrix}. \tag{2.20}$$

Using this gives that $T^{\mu\nu}_{\rm FRW} - T^{\mu\nu}_{\rm TOV}$ is equal to

$$(\rho + p)\begin{bmatrix} (1 - kr^2) - \dfrac{\rho - \bar\rho}{\rho + p}A & \psi\sqrt{1 - kr^2 - A} \\ \psi\sqrt{1 - kr^2 - A} & \dfrac{\psi^2}{1 - kr^2}\left[(1 - kr^2) - \dfrac{p - \bar p}{\rho + p}A\right] \end{bmatrix}. \tag{2.21}$$

A calculation using (2.21) shows that

$$\frac{1}{\psi^2}\det[T] = (\rho - \bar\rho)(p - \bar p)\frac{A^2}{1 - kr^2} + (\rho + p)(\bar\rho + \bar p)A,$$

so $\det[T] = 0$ holds if and only if

$$\begin{aligned} 0 &= \rho(p - \bar p)A \\ &\quad - \bar\rho[(p - \bar p)A - \bar\rho(\rho + p)(1 - kr^2) + \bar p(\rho + p)(1 - kr^2)]. \end{aligned} \tag{2.22}$$

Solving (2.22) for $\bar\rho$ gives

$$\bar\rho = \frac{\rho + \frac{\rho + p}{p - \bar p}\frac{1 - kr^2}{A}\bar p}{1 - \frac{\rho + p}{p - \bar p}\frac{1 - kr^2}{A}}.$$

which, using the identities $N = 1 - A$, $\sigma = \frac{\underline{p}}{\rho}$, $u = \frac{\bar{p}}{\rho}$, and $v = \frac{\bar{\rho}}{\rho}$, is equivalent to

$$v = \frac{1 + \frac{1+\sigma}{\sigma-u}\frac{1-kr^2}{1-N}u}{1 - \frac{1+\sigma}{\sigma-u}\frac{1-kr^2}{1-N}}. \tag{2.23}$$

We call condition (2.23) the *conservation constraint* because it is equivalent to $\det[T] = 0$, and so (2.23) implies that there are no delta functions sources at the shock, cf. [20, 21]. We can now derive the equations that give the simultaneous evolution of the TOV metric *inside the Black Hole*, together with the shock position.

Using the shock identity $N = \frac{\kappa}{3}\rho\bar{r}^2$ equivalent to (2.17), Eq. (2.10) is equivalent to

$$\frac{d\bar{r}}{dN} = -\frac{\bar{r}}{N}\left(\frac{1}{1+3u}\right). \tag{2.24}$$

Using this, we obtain that (2.9) is equivalent to

$$\frac{d\bar{p}}{dN} = \frac{\bar{p}'}{N'} = \frac{\bar{\rho}+\bar{p}}{2}\frac{1}{N-1}. \tag{2.25}$$

This leads to

$$\frac{du}{dN} = \frac{d(\frac{\bar{p}}{\rho})}{dN} = \frac{1}{\rho}\left[\frac{d\bar{p}}{dN} - u\frac{d\rho}{dN}\right]. \tag{2.26}$$

But using $N = \frac{\kappa}{3}\rho\bar{r}^2$ again we get

$$\frac{1}{\rho}\frac{d\rho}{dN} = \frac{3}{\kappa\rho}\frac{d}{dN}\left(\frac{N}{\bar{r}^2}\right) = \frac{3}{N}\frac{1+u}{1+3u}, \tag{2.27}$$

where we have used (2.24). Therefore, (2.26) can be written in the form

$$\frac{du}{dN} = \frac{u+v}{2}\frac{1}{N-1} - 3u\frac{1+u}{1+3u}\frac{1}{N}. \tag{2.28}$$

To obtain our final form for Eq. (2.28), we eliminate v by means of the conservation constraint (2.23). A calculation shows that

$$u + v = \frac{1+u}{1 - \frac{1+\sigma}{\sigma-u}\frac{1-kr^2}{1-N}},$$

and thus

$$\frac{du}{dN} = \frac{1}{2}\frac{(1+u)(\sigma-u)}{[(N-1)(\sigma-u)+(1+\sigma)(1-kr^2)]} - \frac{3u(1+u)}{1+3u}\frac{1}{N},$$

which simplifies to the final form

$$\frac{du}{dN} = \frac{-(1+u)}{2(1+3u)N}\cdot\left\{\frac{(\sigma-u)(3u-1)N + 6u(u+1-(1+\sigma)kr^2)}{(\sigma-u)(N-1)+(1-kr^2)(1+\sigma)}\right\}. \tag{2.29}$$

We conclude that the Eq. (2.29) together with (2.24) gives the equations that describe the evolution of the shock position $\bar{r}$ and the TOV density u when N is taken as the independent variable. The difference between the case $k = 0$ and

$k \neq 0$ is that Eq. (2.29) contains the variable r which represents the shock position in FRW variables, and this couples the equations to the FRW metric when $k \neq 0$, even when σ is constant. Thus, to close the system, we need an additional equation for $\frac{dr}{dN}$, or some equivalent FRW variable. We state the result in the following theorem:

Theorem 2.1. *Assume that a given* FRW *metric is fixed, and let* $z = \ln(r^2)$. *Then the system of equations*

$$\frac{du}{dN} = \frac{-(1+u)}{2(1+3u)N} \cdot \left\{ \frac{(\sigma - u)(3u-1)N + 6u(1+u-(1+\sigma)ke^z)}{(\sigma - u)(N-1) + (1-ke^z)(1+\sigma)} \right\}, \tag{2.30}$$

$$\frac{dz}{dN} = \frac{2}{N} \frac{(u-\sigma)}{(1+3u)(1+\sigma)}, \tag{2.31}$$

$$\frac{d\bar{r}}{dN} = -\frac{\bar{r}}{N}\left(\frac{1}{1+3u}\right), \tag{2.32}$$

determines the pressure $u = \frac{\bar{p}}{\rho}$ *together with the shock positions* $\bar{r}$ *and* $r = \sqrt{e^z}$ *as a function of the variable* $N > 1$. *In particular, the system closes when the* FRW *equation of state satisfies* $\sigma \equiv \frac{p}{\rho} = const.$

Note that the first two equations alone close when $\sigma =$ const. Note also that by (2.14), the variable $\sqrt{N - kr^2}$ is exactly equal to the number of Hubble lengths from the shock wave to the FRW center at fixed FRW time, reducing to exactly $\sqrt{N}$ when $k = 0$, cf. [20, 21]. Of course, Eqs. (2.30)–(2.32) only apply for $N > 1$, that is, they apply *inside the Black Hole*, because the TOV system (2.9)–(2.11) is only valid for $N > 1$.

Proof. The Eqs. (2.30) and (2.32) follow directly from (2.29) and (2.24) upon eliminating r in favor of z. Thus it remains only to derive (2.31). For this, a calculation starting from (2.17) gives

$$\dot{N} = N'\dot{\bar{r}} = \frac{\kappa}{3}[\dot{\rho}\bar{r}^2 + 2\rho\bar{r}\dot{\bar{r}}] = N\left[\frac{\dot{\rho}}{\rho} + \frac{2\dot{\bar{r}}}{\bar{r}}\right], \tag{2.33}$$

so that

$$\frac{\dot{N}}{N} = \frac{\dot{\rho}}{\rho} + 2\frac{\dot{\bar{r}}}{\bar{r}}. \tag{2.34}$$

On the other hand, by (2.32),

$$\frac{N'}{N}\dot{\bar{r}} = -\frac{1+3u}{\bar{r}}\dot{\bar{r}}, \tag{2.35}$$

so using this in (2.33) leads to

$$\frac{\dot{\bar{r}}}{\bar{r}} = -\frac{1}{3(1+u)}\frac{\dot{\rho}}{\rho}. \tag{2.36}$$

Using (2.36) to eliminate $\frac{\dot{\bar{r}}}{\bar{r}}$ from (2.35), we obtain

$$\frac{\dot{N}}{N} = \frac{1+3u}{3(1+u)}\frac{\dot{\rho}}{\rho},$$

and using (2.6) to eliminate $\frac{\dot{\rho}}{\rho}$ from this last equation gives

$$\frac{\dot{N}}{N} = -\frac{(1+3u)(1+\sigma)}{(1+u)}H. \tag{2.37}$$

We use (2.37) to obtain an expression for $\frac{dR}{dN}$ as follows:

$$R\frac{dN}{dR} = \frac{R}{\dot{R}}\frac{dN}{dR}\dot{R} = \frac{1}{H}\dot{N} = -\frac{(1+3u)(1+\sigma)}{1+u}N,$$

so that

$$\frac{dR}{dN} = -\frac{1+u}{(1+3u)(1+\sigma)}\frac{R}{N}. \tag{2.38}$$

Note that at this stage, Eq. (2.38) closes the system (2.30), (2.32) by means of the identity $z = \ln(r^2) = 2\ln\left(\frac{\bar{r}}{R}\right)$.

Finally, to obtain the Eq. (2.31) for z directly, write

$$\frac{dz}{dN} = 2\frac{R}{\bar{r}}\frac{d}{dN}\left(\frac{\bar{r}}{R}\right) = 2\frac{R}{\bar{r}}\left[\frac{1}{R}\frac{d\bar{r}}{dN} - \frac{\bar{r}}{R^2}\frac{dR}{dN}\right], \tag{2.39}$$

which, upon using (2.32) and (2.38), gives

$$\frac{dz}{dN} = \frac{2}{N}\frac{(u-\sigma)}{(1+3u)(1+\sigma)},$$

as desired. □

3. The Shock Speed at the Big Bang

In this section we assume $\sigma = \frac{p}{\rho} = \text{const.}$, $0 < \sigma < c^2 = 1$, and study solutions of system (2.30)–(2.32) asymptotically in the limit of the Big Bang[b] $N \to \infty$. Our goal here is to show that the shock emerges at the instant of the Big Bang at finite nonzero speed only at the value of $\sigma = \frac{c^2}{3}$, (correct for the earliest stage of Big Bang physics [23]), in which case the shock emerges at the speed of light. This extends the corresponding result in [20, 21] to the case $k \neq 0$.

Following the lead in [20, 21], we first transform system (2.30)–(2.32) over to the independent variable $S \equiv \frac{1}{N}$, (so that the domain $\infty > N > 1$ goes over to $0 < S < 1$, and $S \to 0$ corresponds to the Big Bang) and then we study the rest

[b]By (2.32) the limit $N \to \infty$ is the limit $\bar{r} \to 0$, (assuming u tends to a finite positive value). By (2.17), this is the limit $\rho \to \infty$, which is the limit of the Big Bang on the FRW side of the shock.

points of the resulting system on $S = 0$. Making the substitution $S = 1/N$, system (2.30)–(2.32) is equivalent to the following system which we now study:

$$\frac{du}{dS} = \frac{1}{S}\frac{(1+u)}{2(1+3u)} \cdot \left\{ \frac{(\sigma - u)(3u-1) + 6u[1+u-(1+\sigma)ke^z]S}{(\sigma - u) + [1+u-(1+\sigma)ke^z]S} \right\}, \tag{3.1}$$

$$\frac{d\bar{r}}{dS} = \frac{\bar{r}}{S}\left(\frac{1}{1+3u}\right), \tag{3.2}$$

$$\frac{dz}{dS} = -\frac{2}{S}\frac{(u-\sigma)}{(1+3u)(1+\sigma)}. \tag{3.3}$$

Following the analysis in [20,21] for the case $k = 0$, we write Eq. (3.1) as a first-order autonomous system

$$u' = (1+u)\left\{(\sigma - u)(3u-1) + 6u[1+u-(1+\sigma)ke^z]S\right\}, \tag{3.4}$$

$$S' = 2S(1+3u)\left\{(\sigma - u) + [1+u-(1+\sigma)ke^z]S\right\}. \tag{3.5}$$

Note that when $k = 0$, these equations decouple from the z equation, and there are two rest points on $S = 0$, namely, $P_1 = (0, \sigma)$ and $P_2 = (0, 1/3)$. In [20, 21], we proved that there is a unique solution $u_\sigma(S)$ of (3.4)–(3.5) satisfying the correct entropy condition, and $u_\sigma(S)$ increases in backwards "time" $S \to 0$ into the rest point P_1 when $0 < \sigma < 1/3$, and into the rest point P_2 when $1/3 \le \sigma < 1$ (that is, the orbit tends to the minimum of $\{\sigma, 1/3\}$ as $S \to 0$). Moreover, $u_\sigma(S) < \sigma$ for all $S > 0$ when $k = 0$, and thus by Eq. (3.3), $z = \ln(r^2)$ decreases to its minimum value $z_* = \ln(r_*^2)$ in backwards time as $S \to 0$. In the case $k \neq 0$, the system (3.4)–(3.5) has the same two rest points P_1 and P_2 at $S = 0$. We now show asymptotically that near $S = 0$ there are solutions in the cases $k \neq 0$ that are qualitatively the same as the $u_\sigma(S)$ of the case $k = 0$. That is, we assume the existence of solutions that tend to the same rest points, and are qualitatively the same as $u_\sigma(S)$ asymptotically near $S = 0$, and then we verify that the asymptotics area consistent.

Thus, in the case $\sigma < 1/3$, we assume a solution of (3.1)–(3.3) that tends to rest point P_1 with asymptotic form

$$u_\sigma(S) = \sigma - m_* S - n_* S^2 + \text{h.o.t.}, \quad \text{as } S \to 0. \tag{3.6}$$

In the case $\sigma = 1/3$, we assume a solution of (3.1)–(3.3) that tends to rest point $P_1 = P_2$ with asymptotic form

$$u_\sigma(S) = \sigma - m_* \sqrt{S} + \text{h.o.t.}, \quad \text{as } S \to 0. \tag{3.7}$$

In the case $\sigma > 1/3$ we assume only that the orbit asymptotically satisfies

$$u_\sigma(S) \to \frac{1}{3}, \quad \text{as } S \to 0. \tag{3.8}$$

Note that in all three cases, $u < \sigma$ near $S = 0$, and therefore (3.3) implies that z is monotone decreasing in backwards time, so that z decreases to its minimum value

$z_* > 0$ as $S \to 0$. Thus we can assume

$$ke^z \to kr_*^2 \quad \text{as } S \to 0. \tag{3.9}$$

From this we can conclude that $u_\sigma(S)$ asymptotically satisfies the single equation obtained from (3.1) by replacing e^z with the constant value $r_*^2 > 0$, namely,

$$\frac{du}{dS} = \frac{1}{S}\frac{(1+u)}{2(1+3u)} \cdot \left\{ \frac{(\sigma-u)(3u-1)+6u[1+u-(1+\sigma)kr_*^2]S}{(\sigma-u)+[1+u-(1+\sigma)kr_*^2]S} \right\}. \tag{3.10}$$

It remains then to determine the values of m_* and n_* such that $u_\sigma(S)$ satisfies (3.10) asymptotically as $S \to 0$. The value of m_* determines the shock speed at the Big Bang $S = 0$.

Consider first the case $0 < \sigma < 1/3$. Substituting (3.6) into (3.10) and retaining terms to order S we obtain

$$S\frac{du}{dS} = \frac{1+\sigma}{2(1+3\sigma)} \left\{ \frac{A}{B} \right\}, \tag{3.11}$$

where

$$\begin{aligned} A = &\left\{-(1-3\sigma)m_* + 6\sigma(1+\sigma)(1-kr_*^2)\right\} \\ &+\{-(1-3\sigma)n_* + 3m_*(-m_* - 2\sigma - 2(1+\sigma)(1-kr_*^2))\}S, \end{aligned} \tag{3.12}$$

$$B = m_* + (1+\sigma)(1-kr_*^2) + O(S). \tag{3.13}$$

Now noticing that the LHS of (3.11) is $O(S)$ as $S \to 0$, it follows that, for the ansatz (3.6) to be consistent, we must choose m_* so that the leading order term in A vanishes. This gives

$$m_* = \frac{6\sigma(1+\sigma)(1-kr_*^2)}{1-3\sigma}. \tag{3.14}$$

(In particular, this tells us that, as in the case $k = 0$, the solution orbit comes into the rest point P_1 tangent to the u-isocline when $\sigma < 1/3$, [20, 21].) Assuming this value of m_*, the consistency of the ansatz (3.6) when $\sigma < 1/3$ follows from the fact that for this choice, both sides of (3.11) are $O(S)$, and since the coefficient of n_* in (3.13) is equal to $(3\sigma - 1) \neq 0$, we can solve for a unique value of $n*$ to satisfy $S\frac{du}{dS} = -m_*S$ to leading order in (3.11). Note that $1 - kr_*^2 > 0$ in an FRW metric even when $k > 0$. We conclude from (3.14) that $m_* > 0$ as required by the assumption $u < \sigma$ in (3.3). This verifies the asymptotics of $u_\sigma(S)$ when $k \neq 0$ and $0 < \sigma < 1/3$.

Next we consider the case $\sigma = 1/3$, the case when $P_1 = P_2$ is a non-regular singular point. Putting the ansatz (3.7) into (3.11) and keeping only the leading $O(\sqrt{S})$ terms gives

$$S\frac{du}{dS} = \frac{1}{3}\frac{-3m_*^2 + \frac{8}{3}(1-kr_*^2)}{m_*}\sqrt{S}. \tag{3.15}$$

Using the fact that the ansatz (3.7) implies that

$$\frac{du}{dS} = -\frac{m_*}{2\sqrt{S}},$$

we can set the RHS of (3.15) equal to $-\frac{m_*}{2}\sqrt{S}$ and solve for m_* to obtain

$$m_* = \frac{4}{3}\sqrt{1-kr_*^2} > 0. \tag{3.16}$$

We can now prove the following theorem for the case $k \neq 0$.

Theorem 3.1. *The solution trajectories $u_\sigma(S)$ given asymptotically by* (3.6)–(3.8) *determine the shock speed $s_\sigma(S)$ at the limit of the Big Bang $S \to 0$ as follows:*

If $\sigma < 1/3$,

$$\lim_{S\to 0} s_\sigma(S) = 0, \tag{3.17}$$

if $\sigma = 1/3$,

$$\lim_{S\to 0} s_\sigma(S) = 1, \tag{3.18}$$

and if $\sigma > 1/3$,

$$\lim_{S\to 0} s_\sigma(S) = \infty. \tag{3.19}$$

For the proof we need the following lemma:

Lemma 3.2. *The speed of the shock wave relative to an observer fixed with the fluid co-moving with the* FRW *metric, as determined by* (3.1)–(3.3), *is given by*

$$s_\sigma(S) = \sqrt{\frac{1-kr^2S}{1-kr^2}}\left(\frac{\sigma-u}{1+u}\right)\frac{1}{\sqrt{S}}. \tag{3.20}$$

Assuming the lemma we can give the following proof:

Proof of Theorem 3.1. Substituting the ansatz (3.6), (3.8) into (3.20) and taking the limit $S \to 0$ immediately gives (3.17), (3.19) respectively.

Consider now the most interesting case $\sigma = 1/3$. In this case, substituting (3.7) into (3.20) yields an expression that is $O(1)$ as $S \to 0$ because, special to the case $\sigma = 1/3$, $\sigma - u$ is $O(\sqrt{S})$ as $S \to 0$, and this cancels the $O(\frac{1}{\sqrt{S}})$ in (3.20). Thus, when $\sigma = 1/3$, the value of $s_\sigma(S)$ in the limit $S \to 0$ is determined by the value of m_* in (3.16). Putting (3.7) and (3.16) into (3.20) and taking the limit $S \to 0$ gives

$$\begin{aligned}\lim_{S\to 0} s_{\frac{1}{3}}(S) &= \lim_{S\to 0}\sqrt{\frac{1-kr^2S}{1-kr^2}}\left(\frac{\left(\frac{4}{3}\sqrt{1-kr^2}\right)}{1+u}\right)\\ &= \sqrt{\frac{1}{1-kr_*^2}}\left(\frac{\left(\frac{4}{3}\sqrt{1-kr_*^2}\right)}{1+1/3}\right) = 1.\end{aligned} \tag{3.21}$$

This completes the proof of Theorem 3.1. □

Proof of Lemma 3.2. We first show that the speed of the shock wave relative to a locally inertial coordinate frame co-moving with the FRW fluid at any given point in a solution of system (3.1)–(3.3), is given by

$$s_\sigma = \dot{r}\frac{R}{\sqrt{1-kr^2}}. \tag{3.22}$$

To see this, recall that the (t,r)-portion of the FRW line element is given by

$$ds^2 = -dt^2 + R^2\frac{dr^2}{1-kr^2}. \tag{3.23}$$

Since the FRW fluid is fixed with respect to coordinate r, to obtain the invariant speed of a shock with coordinate speed $\dot{r}$ relative to the FRW fluid at a given point, we need only transform the r coordinate over to $r = \phi(\tilde{r})$ so that the FRW line element in $(t,\tilde{r})$-coördinates is Minkowskian at the point, and then

$$s_\sigma = \frac{d\tilde{r}}{dt}. \tag{3.24}$$

Changing (3.23) over to $(t,\tilde{r})$ coordinates gives

$$ds^2 = -dt^2 + R^2\frac{\phi'(\tilde{r})^2\,d\tilde{r}^2}{1-kr^2},$$

and thus the resulting metric is Minkowskian at a point where

$$\phi'(\tilde{r}) = \frac{\sqrt{1-kr^2}}{R}. \tag{3.25}$$

We conclude from (3.24) that the shock speed at a given point in a solution of system (3.1)–(3.3) is given by

$$s_\sigma = \frac{d\tilde{r}}{dt} = \frac{\dot{r}}{\phi'(\tilde{r})} = \dot{r}\frac{R}{\sqrt{1-kr^2}}, \tag{3.26}$$

as claimed in (3.22). Thus the proof of Lemma 3.2 and the verification of (3.20) is complete once we show that the shock speed $\dot{r}$ at a point in a solution of (3.1)–(3.3) satisfies

$$\dot{r} = \frac{\sqrt{N-kr^2}}{R}\left(\frac{\sigma-u}{1+u}\right). \tag{3.27}$$

That is, (3.20) follows from (3.26) upon making the substitution (3.27), and recalling that $N = 1/S$.

To finish, we now verify (3.27). Differentiating (2.17) with respect to t we obtain

$$\begin{aligned} \dot{N} &= \frac{\kappa}{3}\dot{\rho}\bar{r}^2 + \frac{2\kappa}{3}\rho\bar{r}\dot{\bar{r}} \\ &= \frac{\kappa}{3}\rho\bar{r}^2\left(-3(1+\sigma)H\right) + \frac{\kappa}{3}\rho\bar{r}^2\frac{2\dot{\bar{r}}}{\bar{r}} \\ &= -3N(1+\sigma)H + \frac{2N\dot{\bar{r}}}{\bar{r}}, \end{aligned} \tag{3.28}$$

where we have used (2.6). On the other hand, using (2.10), we can write

$$\dot{N} = N'\dot{\bar{r}} = -\frac{1}{\bar{r}}\left\{N + \kappa\rho\bar{r}^2 u\right\}\dot{\bar{r}}$$
$$= \frac{N}{\bar{r}}(1+3u)\dot{\bar{r}}, \tag{3.29}$$

where we have used (2.17). Equating (3.28) and (3.29) and solving for $\dot{\bar{r}}$ we obtain

$$\dot{\bar{r}} = \frac{1+\sigma}{1+u}H\bar{r}. \tag{3.30}$$

But,

$$\dot{r} = \frac{d}{dt}\left(\frac{\bar{r}}{R}\right) = \frac{\bar{r}}{R}\left(\dot{\bar{r}} - H\bar{r}\right)$$
$$= \frac{H\bar{r}}{R}\frac{\sigma - u}{1+u}, \tag{3.31}$$

where we have used (3.30). Using Eq. (2.5) to eliminate ρ in the equation for N in (2.17), we obtain

$$N = H^2\bar{r}^2 + kr^2,$$

which gives

$$H^2\bar{r}^2 = N - kr^2. \tag{3.32}$$

Substituting (3.32) into (3.31) gives (3.27) as claimed. This completes the proof of Lemma 3.2. □

4. Closing the Equations for General $\sigma = \frac{p}{\rho}$

In this section we obtain an equation for $\frac{d\rho}{dN}$ that closes the system (2.30)–(2.31) when $\sigma = \frac{p}{\rho}$ is not constant, and the FRW equation of state $p = p(\rho)$ is given. To this end, starting with (2.17), write

$$\frac{d\rho}{dN} = \frac{d}{dN}\left(\frac{3}{\kappa}\frac{N}{\bar{r}^2}\right) = \frac{3}{\kappa}\left\{\frac{1}{\bar{r}^2} - \frac{N}{\bar{r}^3}\frac{d\bar{r}}{dN}\right\}, \tag{4.1}$$

and so using (2.32) we obtain the equation for $\frac{d\rho}{dN}$ that closes system (2.30)–(2.31), namely,

$$\frac{d\rho}{dN} = \frac{3}{\kappa\bar{r}^2}\left(\frac{2+3u}{1+3u}\right) = \frac{\rho}{N}\left(\frac{2+3u}{1+3u}\right). \tag{4.2}$$

As a corollary, we prove the following theorem:

Theorem 4.1. *Assume that the* FRW *equation of state* $p = p(\rho) \geq 0$ *is given and satisfies* $p'(\rho) > 0$, $p''(\rho) \geq 0$. *Then* σ *is a non-decreasing function of* N *along solutions of system* (2.30)–(2.32), (4.2), *so long as* $u \geq 0$ *and* $N > 1$.

Proof. Using $\sigma = \frac{p(\rho)}{\rho}$ and (4.2), we can write

$$\frac{d\sigma}{dN} = \frac{d\sigma}{d\rho}\frac{d\rho}{dN} = \left(p'(\rho) - \frac{p(\rho)}{\rho}\right)\frac{3}{\kappa\rho\bar{r}^2}\left\{\frac{2+3u}{1+3u}\right\},$$

and so

$$\frac{d\sigma}{dN} = \frac{1}{N}\left(p'(\rho) - \frac{p(\rho)}{\rho}\right)\left\{\frac{2+3u}{1+3u}\right\}. \tag{4.3}$$

The result follows from (4.3) upon noting that

$$\left(p'(\rho) - \frac{p(\rho)}{\rho}\right) \geq 0,$$

because p is a positive, increasing, convex function of ρ. □

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