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Shock Waves and Cosmology

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1. Introduction

The standard model of cosmology is based on a Friedmann-Robertson-Walker metric

$$ds^2 = dt^2 + R(t)^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2\right], \tag{1}$$

which satisfies Einstein's equations of general relativity with a perfect fluid source. In this model, which accounts for things on the largest length scale, the universe is approximated by a space of uniform density and pressure at each fixed time, and the expansion rate is determined by the cosmological scale factor $R(t)$ that evolves according to the Einstein equations. Astronomical observations show that the galaxies are uniform on a scale of about one billion lightyears, and the expansion is critical – that is, $k = 0$ in (**1**) – and so, according to (**1**), on the largest scale, the universe is infinite flat Euclidean space R^3 at each fixed time. Here $R(t)$ determines the Hubble "constant", $H(t) = \dot{R}(t)/R(t)$ and $c/H(t)$ is called the Hubble distance at time $t > 0$, (c is the speed of light).

In the standard model, the Hubble constant, which measures the recessional velocity of the galaxies, applies to the entire universe of infinite mass and extent, at each time $t > 0$. That is, the entire universe burst out from the Big-Bang at $t = 0$, and the universe is of infinite mass and extent at each time $t > 0$. In order to avoid this seemingly unphysical situation, we have been exploring the possibility of a new cosmological model, in which the expansion of the galaxies is a bounded expansion of finite total mass. In our model, the FRW universe emerges from the Big-Bang with a shock wave at the leading edge of the expansion, analogous to a classical shock wave explosion. In order to close the resulting Einstein equations, we take as equation of state

$$p = \sigma^2\rho, \tag{2}$$

where σ is (constant) the sound speed satisfying $0 < \sigma < c, \rho$ is the density and p is the pressure.

In this paper, we summarize our work in [**4, 5**], whereby we describe a two parameter family of exact solutions of the Einstein equations that refine the FRW metric by a spherical shock wave cut-off. In these exact solutions the expanding FRW metric is reduced to a region of finite extent and finite total mass at each fixed time, and this FRW region is bounded by an entropy satisfying shock wave that emerges form the origin, (the center of the explosion), at the instant of the Big Bang $t = 0$. The shock wave, which marks the leading edge of the FRW expansion, propagates outward into a larger ambient spacetime from time $t = 0$ onward. Thus, in this refinement of the FRW metric, the Big Bang that set the galaxies in motion is an explosion of finite mass that looks more like a classical shock wave explosion than does the Big Bang of the Standard Model. The family of solutions is then determined by two parameters, $0 < \sigma \leq 1$ and $r_* \geq 0$. The second parameter r_* is the FRW radial coordinate r of the shock in the limit $t \to 0$, the instant of the Big Bang. The FRW radial coordinate r is singular with respect to radial arclength $\bar{r} = rR$ at the Big Bang $R = 0$, so setting $r_* > 0$ does not place the shock wave away from the origin at time $t = 0$. The distance from the FRW center to the shock wave tends to zero in the limit $t \to 0$ even when $r_* > 0$. In the limit $r_* \to \infty$ we recover from the family of solutions, the usual (infinite) FRW metric with equation of state $p = \sigma\rho$. That is we recover the standard FRW metric in the limit that the shock wave is infinitely far out. In this sense our family of exact solutions of the Einstein equations represents a two parameter refinement of the standard Friedmann-Robertson-Walker metric.

The exact solutions for the case $r_* = 0$ were first constructed in [**3**], and are qualitatively different from the solutions when $r_* > 0$, which were constructed later in [**5**]. The difference is that when $r_* = 0$, the shock wave lies closer than one Hubble length from the center of the FRW spacetime through its motion, but when $r_* > 0$, the shock wave emerges at the Big Bang at a distance beyond one Hubble length. (The Hubble length depends on time, and tends to zero as $t \to 0$.) We show in [**5**] that one Hubble length, equal to $\frac{c}{H}$ where $H = \frac{\dot{R}}{R}$, is a critical length scale in a $k = 0$ FRW metric because the total mass inside one Hubble length has a Schwarzschild radius equal to exactly one Hubble length. That is one Hubble length marks precisely the distance at which the Schwarzschild radius $\bar{r}_s \equiv 2M$ of the mass M inside a radial shock wave at distance $\bar{r}$ from the FRW center, crosses from inside ($\bar{r}_s < \bar{r}$) to outside ($\bar{r}_s > \bar{r}$) the shock wave. If the shock wave is at a distance closer than one Hubble length from the FRW center, then $2M < \bar{r}$ and we say that the solutions lies *outside the Black Hole*, but if the shocks wave is at a distance greater than one Hubble length, then $2M > \bar{r}$ at the shock, and we say the solution lies *inside the Black Hole*. Since M increases like $\bar{r}^3$, it follows that $2M < \bar{r}$ for $\bar{r}$ sufficiently small, and $2M > \bar{r}$ for $\bar{r}$ sufficiently large, so there must be a critical radius at which $2M = \bar{r}$, and in Section 2, (taken from [**5**]), we show that when $k = 0$, this critical radius is exactly the Hubble length. When the parameter $r_* = 0$, the family of solutions for $0 < \sigma \leq 1$ starts at the Big Bang, and evolves thereafter *outside* the Black Hole, satisfying $\frac{2M}{\bar{r}} < 1$ everywhere from $t = 0$ onward. But when $r_* > 0$, the shock wave is further out than one Hubble length at the instant of the Big Bang, and the solution begins with $\frac{2M}{\bar{r}} > 1$ at the shock wave. From this time onward, the spacetime expands until eventually the Hubble length catches up to the shock wave at $\frac{2M}{\bar{r}} = 1$ and then passes the shock wave, making $\frac{2M}{\bar{r}} < 1$ thereafter. Thus when $r_* > 0$, the whole spacetime begins *inside the Black Hole*, (with $\frac{2M}{\bar{r}} > 1$ for sufficiently large $\bar{r}$), but eventually evolves to a solution *outside the Black Hole*. The time when $\bar{r} = 2M$ actually marks the event horizon of a *White Hole*, (the time reversal of a Black Hole), in the ambient spacetime beyond the shock wave. We show that when $r_* > 0$, the time when the Hubble length catches up to the shock wave comes after the time when the shock wave comes into view at the FRW center, and when $2M = \bar{r}$, (assuming t is so large that we can neglect the pressure from this time onward), the whole solution emerges from the White Hole as a finite ball of mass expanding into empty space, satisfying $\frac{2M}{\bar{r}} < 1$ everywhere thereafter. In fact, when $r_* > 0$, the zero pressure Oppenheimer-Snyder solution *outside the Black Hole* gives the large time asymptotics of the solution, (c.f. [**1, 2, 5**]. The main difference between the cases $r_* > 0$ and $r_* = 0$ is that when $r_* > 0$, a large region of uniform expansion is created behind the shock wave at the instant of the Big Bang. Thus, when $r_* > 0$, lightlike information about the shock wave propagates *inward from the wave*, rather than *outward from the center*, as is the case when $r_* = 0$ and the shock lies inside one Hubble length. It follows that when $r_* > 0$, an observer positioned in the FRW spacetime *inside* the shock wave, will see exactly what the standard model of cosmology predicts, up until the time when the shock wave comes into view in the far field. In this sense, the case $r_* > 0$ gives a Black Hole cosmology that refines the standard FRW model of cosmology to the case of finite mass. One of the surprising differences between the case $r_* = 0$ and the case $r_* > 0$ is that, when $r_* > 0$, the important equation of state $p = \frac{1}{3}\rho$ comes out of the analysis as special at the Big Bang. When $r_* > 0$ the shock wave emerges at the instant of the Big Bang at a finite non-zero speed, (the speed of light), *only* for the special value $\sigma = 1/3$. In this case, the equation of state on both sides of the shock wave tends to the correct relation $p = \frac{1}{3}\rho$ as $t \to 0$, and the shock wave decelerates to subliminal speed for all positive times thereafter, (see [**5**] and Theorem 4 below).

In all cases $0 < \sigma \le 1$, the spacetime metric that lies beyond the shock wave is taken to be a metric of Tolmann-Oppenheimer-Volkoff (TOV) form,

$$ds^2 = -B(\bar{r})d\bar{t}^2 + A^{-1}(\bar{r})d\bar{r}^2 + \bar{r}^2\left[d\theta^2 + \sin^2\theta d\phi^2\right]. \tag{3}$$

The metric (3) is in standard Schwarzschild coordinates, and the metric components depend only on the radial coordinate $\bar{r}$. Barred coordinates are used to distinguish TOV coordinates from unbarred FRW coordinates for shock matching. The mass function $M(\bar{r})$ enters as a metric component through the relation,

$$A = 1 - \frac{2M(\bar{r})}{\bar{r}}.$$

The TOV metric (3) has a very different character depending on whether $A > 0$ or $A < 0$; that is, depending on whether the solution lies *outside the Black Hole* or *inside the Black Hole.* In the case $A > 0$, $\bar{r}$ is spacelike coordinate, and the TOV metric describes a static fluid sphere in general relativity. When $A < 0$, $\bar{r}$ is the timelike coordinate, and (3) is a dynamical metric that evolves in time. The exact shock wave solutions are obtained by taking $\bar{r} = R(t)r$ to match the spheres of symmetry, and then matching the metrics (1) and (3) at an interface $\bar{r} = \bar{r}(t)$ across which the metrics are Lipschitz continuous. This can be done in general. In order for the interface to be a physically meaningful shock surface, we use the result (see [**1**]), that a single additional conservation constraint is sufficient to rule out delta function sources at the shock, (the Einstein equations $G = \kappa T$ are second order in the metric, and so delta function sources will in general be present at a Lipschitz continuous matching of metrics), and guarantee that the matched metric solves the Einstein equations in the weak sense. The Lipschitz matching of the metrics, together with the conservation constraint, leads to a system of ordinary differential equations (ODE's) that determine the shock position, together with the TOV density and pressure at the shock. Since the TOV metric depends only on $\bar{r}$, the equations thus determine the TOV spacetime beyond the shock wave. To obtain a physically meaningful outgoing shock wave, we impose the constraint $\bar{p} \leq \bar{\rho}$ to ensure that the equation of state on the TOV side of the shock is qualitatively reasonable, and as the entropy condition we impose the condition that the shock be compressive. For an outgoing shock wave, this is the condition $\rho > \bar{\rho}, p > \bar{p}$, that the pressure and density be larger on the side of the shock that receives the mass flux – the FRW side when the shock wave is propagating away from the FRW center. This condition breaks the time reversal symmetry of the equations, and is sufficient to rule out rarefaction shocks in classical gas dynamics. The ODE's, together with the equation of state bound and the conservation and entropy constraints, determine a *unique* solution of the ODE's for every $0 < \sigma \leq 1$ and $\bar{r}_* \geq 0$, and this provides the two parameter family of solutions discussed here, [**4, 5**]. The Lipschitz matching of the metrics implies that the total mass M is continuous across the interface, and so when $r_* > 0$, the total mass of the entire solution, inside and outside the shock wave, is finite at each time $t > 0$, and both the FRW and TOV spacetimes emerge at the Big Bang. The total mass M on the FRW side of the shock has the meaning of total mass inside radius $\bar{r}$ at fixed time, but on the TOV side of the shock, M does not evolve according to equations that give it the interpretation as a total mass because the metric is *inside the Black Hole.* Nevertheless, after the spacetime emerges from the Black Hole, the total mass takes on its usual meaning outside the Black Hole, and time asymptotically the Big Bang ends with an expansion of finite total mass in the usual sense. Thus, when $r_* > 0$, out shock wave refinement of the FRW metric leads to a Big Bang of *finite total mass.*

We now consider these remarks in somewhat more detail.

According to Einstein's Theory of General Relativity, all properties of the gravitational field are determined by a Lorentzian spacetime metric tensor, g, whose line element in a given coordinate system $x = (x^0, ..., x^3)$ is given by

$$ds^2 = g_{ij}dx^i dx^j. \tag{4}$$

(We use the Einstein summation convention whereby repeated up-down indices are assumed summed from 0 to 3.) The components g_{ij} of the gravitational metric g satisfy the Einstein equations,

$$G^{ij} = \kappa T^{ij}, \quad T^{ij} = (\rho c^2 + p)(w^i w^j + p g^{ij}), \tag{5}$$

where we assume the stress-energy tensor T of a perfect fluid. Here G is the Einstein curvature tensor,

$$\kappa = \frac{8\pi\mathcal{G}}{c^4}$$

is the coupling constant, $\mathcal{G}$ is Newton's gravitational constant, c is the speed of light ρc^2 is the energy density, p is the pressure, and $\mathbf{w} = (w^0, ..., w^3)$ are the components of the 4-velocity of the fluid, and we use the convention that $c = 1$ and $\mathcal{G} = 1$ when convenient.

Putting the metric (1) into the Einstein equations (5) gives the equations for the FRW metric,

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{\kappa}{3}\,\rho - \frac{k}{R^2}, \tag{6}$$

and

$$\dot{\rho} = -3(p+\rho)H. \tag{7}$$

The unknowns R, ρ and p are assumed to be functions of the FRW coordinate time t alone, and "dot" denotes differentiation when respect to t.

To verify that the Hubble length $\bar{r}_{crit} = 1/H$ is the limit for FRW-TOV shock matching outside a Black Hole, write the FRW metric (1) in standard Schwarzschild coordinates $\bar{\mathbf{x}} = (\bar{r}, \bar{t})$ where the metric takes the form

$$ds^2 = -B(\bar{r})d\bar{t}^2 + A(\bar{r},\bar{t})^{-1}d\bar{r}^2 + \bar{r}^2 d\Omega^2,$$

and the mass function $M(\bar{r},\bar{t})$ is defined through the relation

$$A = 1 - \frac{2M}{\bar{r}}.$$

Substituting $\bar{r} = Rr$ into (1) and diagonalizing the resulting metric we obtain (see [**5**] for details),

$$ds^2 = -\frac{1}{\psi^2}\left\{\frac{1-kr^2}{1-kr^2-H^2\bar{r}^2}\right\}d\bar{t}^2 + \left\{\frac{1}{1-kr^2-H^2\bar{r}^2}\right\}d\bar{r}^2 + \bar{r}^2 d\Omega^2,$$

where ψ is an integrating factor that solves the equation

$$\frac{\partial}{\partial \bar{r}}\left(\psi\,\frac{1-kr^2-H^2\bar{r}^2}{1-kr^2}\right) = \frac{\partial}{\partial t}\left(\psi\,\frac{H\bar{r}}{1-kr^2}\right) = 0,$$

and the time coordinate $\bar{t} = \bar{t}(t,\bar{r})$ is defined by the exact differential

$$d\bar{t} = \left(\psi\,\frac{1-kr^2-H^2\bar{r}^2}{1-kr^2}\right)dt + \left(\psi\,\frac{H\bar{r}}{1-kr^2}\right)d\bar{r}. \tag{8}$$

Now from (6), it follows that

$$M(t,\bar{r}) = \frac{\kappa}{2}\int_0^{\bar{r}} \rho(t)s^2 ds = \frac{1}{3}\,\frac{\kappa}{2}\,\rho\bar{r}^3. \tag{9}$$

Since in the FRW metric $\bar{r} = Rr$ measures arclength along radial geodesics at fixed time, we see from (9) that $M(t,\bar{r})$ has the physical interpretation as the total mass inside radius $\bar{r}$ at time t in the FRW metric. Restricting to the case of critical expansion $k=0$, we see from (6) and (9) that $\bar{r} = H^{-1}$ is equivalent to $\frac{2M}{\bar{r}} = 1$, and so at fixed time t, we have

$$\bar{r} = H^{-1} \;\text{ iff }\; \frac{2M}{\bar{r}} = 1 \;\text{ iff }\; A = 0. \tag{10}$$

Thus $\bar{r} = H^{-1}$ is the critical length scale for the FRW metric at fixed time t in the sense that $A = 1 - \frac{2M}{\bar{r}}$ changes sign at $\bar{r} = H^{-1}$. Thus the universe lies *inside a Black Hole* beyond $\bar{r} = H^{-1}$, as claimed above. Now we showed in [**2**] that the standard TOV metric outside the Black Hole cannot be continued into $A = 0$. So if we want to do shock matching beyond one Hubble length we require a metric of a different type. In [**5**] we introduce the TOV metric *inside the Black Hole* – a metric of TOV form, with $A < 0$, whose fluid is co-moving with the timelike radial coordinate $\bar{r}$.

It is interesting that the Hubble length $\bar{r}_{crit} = \frac{c}{H}$ is also the critical distance at which the outward expansion of the FRW metric exactly cancels the inward advance of a radial light ray impinging on an observer positioned at the origin of a $k = 0$ FRW metric. Indeed, by (1), a light ray traveling radially inward toward the center of an FRW coordinate system satisfies,

$$c^2 dt^2 = R^2 dr^2,$$

so that

$$\frac{d\bar{r}}{dt} = \dot{R}r + R\dot{r} = H\bar{r} - c = H\Big(\bar{r} - \frac{c}{H}\Big) > 0,$$

if and only if

$$\bar{r} > \frac{c}{H}.$$

Thus the arclength distance from the origin to an inward light ray at fixed time t in a $k = 0$ FRW metric will actually *increase* as long as the light ray lies beyond the Hubble length. An inward moving light ray will, however, eventually cross the Hubble length and reach the origin in finite proper time, due to the increase in the Hubble length with time.

The next theorem gives closed form solutions of the FRW equations (6), (7), in the case when $\sigma = const.$

THEOREM 1. *Assume $k = 0$ and the equation of state*

$$p = \sigma\rho,$$

where σ is a constant,

$$0 \leq \sigma \leq 1.$$

Then, (assuming an expanding universe $\dot{R} > 0$), the solution of system (6), (7) satisfying $R = 0$ at $t = 0$ and $R = 1$ at $t = t_0$ is given by,

$$\rho = \frac{4}{3\kappa(1+\sigma)^2}\,\frac{1}{t^2}, \tag{11}$$

$$R = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\sigma)}}, \tag{12}$$

$$\frac{H}{H_0} = \frac{t_0}{t}. \tag{13}$$

Moreover, the age of the universe t_0 given in terms of the Hubble length, is

$$t_0 = \frac{2}{3(1+\sigma)}\,\frac{1}{H_0}\,.$$

When the shock wave is beyond one Hubble length from the FRW center, we obtain a family of shock wave solutions for each $0 < \sigma \leq 1$ and $r_* > 0$ by shock matching the FRW metric (1) to a TOV metric under the assumption that

$$A(\bar{r}) = 1 - \frac{2M(\bar{r})}{\bar{r}} \equiv 1 - N(\bar{r}) < 0.$$

In this case, $\bar{r}$ is the timelike variable. Assuming the stress tensor T is taken to be that of a perfect fluid co-moving with the TOV metric, the Einstein equations $G = \kappa T$, inside the Black Hole, take the form, [**5**],

$$\bar{p}' = \frac{\bar{p} + \bar{\rho}}{2} \frac{N'}{N-1}, \tag{14}$$

$$N' = -\left\{ \frac{N}{\bar{r}} + \kappa \bar{p} \bar{r} \right\}, \tag{15}$$

$$\frac{B'}{B} = -\frac{1}{N-1} \left\{ \frac{N}{\bar{r}} + \kappa \bar{\rho} \right\}. \tag{16}$$

The system (14–16) defines the simplest class of gravitational metrics that contain matter, evolve *inside the Black Hole*, and such that the mass function $M(\bar{r}) < \infty$ at each fixed time $\bar{r}$. System (14–16) for $A < 0$ differs substantially from the TOV equations for $A > 0$ because, for example, the energy density T^{00} is equated with the timelike component G^{rr} when $A < 0$, but with G^{tt} when $A > 0$. In particular, this implies that, *inside the Black Hole*, the mass function $M(\bar{r})$ does not have the interpretation as a total mass inside radius $\bar{r}$ as it does *outside the Black Hole*.

Now we assume the FRW solution (11–13) and we derive the ODE's that describe the TOV metrics that match this FRW metric Lipschitz continuously across a shock surface. Afterward we impose the conservation (weak form of the Rankine-Hugoniot conditions), entropy, and equation of state constraints. Matching a given $k = 0$ FRW metric to a TOV metric *inside the Black Hole* across a shock interface, leads to the system of ODE's, (see [**5**] for details),

$$\frac{du}{dN} = -\left\{ \frac{(1+u)}{2(1+3u)N} \right\} \left\{ \frac{(3u-1)(\sigma - u)N + 6u(1+u)}{(\sigma - u)N + (1+u)} \right\}, \tag{17}$$

$$\frac{d\bar{r}}{dN} = -\frac{1}{1+3u} \frac{\bar{r}}{N}, \tag{18}$$

with conservation constraint

$$v = \frac{-\sigma(1+u) + (\sigma - u)N}{(1+u) + (\sigma - u)N}, \tag{19}$$

where

$$u = \frac{\bar{p}}{\rho}, \; v = \frac{\bar{\rho}}{\rho} \;\; \sigma = \frac{p}{\rho}. \tag{20}$$

Here ρ and p are the (given) FRW density and pressure, and all the variables are evaluated at the shock. Solutions (17–19) determine the (unknown) TOV metric Lipschitz continuously across a shock surface and the conservation of energy and momentum hold across the shock, and there are no delta function sources at the shock, [**1**]. Notice that the dependence of (17)–(19) on the FRW metric is only through the variable σ so that the entire solution is determined by the scalar equation (17) when σ is a constant. For the entropy condition we require

$$0 < \bar{p} < p, \quad 0 < \bar{\rho} < \rho, \tag{21}$$

and in order to obtain a physically reasonable solution we require the following equation of state requirement on the TOV side of the shock:

$$0 < \bar{p} < \bar{\rho}. \tag{22}$$

The condition (21) assures that outgoing shock waves are compressive. We prove that (21) and (22) hold provided that (see [**5**])

$$\frac{1}{N} < \frac{(1-u)(\sigma-u)}{(1+u)(\sigma+u)}. \tag{23}$$

Since σ is constant equation (17) decouples from (18), so that solutions of (17–19) are determined by the single equation (17). Letting $S = \frac{1}{N}$ (transforming the Big Bang $N \to \infty$ to $S \to 0$) we find

$$\frac{du}{dS} = \left\{\frac{(1+u)}{2(1+3u)S}\right\}\left\{\frac{(3u-1)(\sigma-u)+6u(1+u)S}{(\sigma-u)+(1+u)S}\right\}. \tag{24}$$

Note that the conditions $N > 1$ and $0 < \bar{p} < p$ restrict the domain of (24) to the region $0 < u < \sigma < 1,\ 0 < S < 1$. The next theorem gives the existence of solutions for $0 < \sigma \le 1, r_* > 0$, *inside the Black Hole*, c.f. [**5**]:

THEOREM 2. *For every $\sigma, 0 < \sigma < 1$ there exists a unique solution $u_\sigma(S)$ of (24) such that (23) holds on the solution for all $S, 0 < S < 1$, and on this solution, $0 < u_\sigma(S) < \bar{u}$, where*

$$\bar{u} = Min\{1/3, \sigma\}, \tag{25}$$

and

$$\lim_{S\to 1} \bar{p} = 0 = \lim_{S\to 1} \bar{\rho}. \tag{26}$$

For each of those solutions $u_\sigma(S)$, the shock position is determined by the solution of (24), which in turn is determined by an initial condition which can be taken to be the FRW radial position of the shock wave at the instant of the Big Bang,

$$r_* = \lim_{S\to 0} r(s) > 0. \tag{27}$$

(Shock mating outside the black hole implies $r_* = 0$.)

As for the shock speed we have

THEOREM 3. *Let $0 < \sigma < 1$; then the shock speed is everywhere subliminal (the shock speed $s_\sigma(S) \equiv s(u_0(S)) < 1$ for all $S,\ 0 < s \le 1$ if and only if $\sigma \le 1/3$.*

For the shock speed at the Big Bang $S = 0$, we have

THEOREM 4. *The shock speed at the Big Bang $S = 0$ is given by:*

$$\lim_{S\to 0} s_\sigma(S) = 0, \quad \sigma < 1/3, \tag{28}$$

$$\lim_{S\to 0} s_\sigma(S) = \infty, \quad \sigma > 1/3, \tag{29}$$

$$\lim_{S\to 0} s_\sigma(S) = 1, \quad \sigma = 1/3. \tag{30}$$

Thus Theorem 4 shows that the equation of state $p = \frac{1}{3}\,\rho$ plays a special role in the analysis when $r_* > 0$, and only for this equation of state does the shock wave emerge at the Big Bang at a finite non-zero speed, the speed of light. Moreover, (26) implies that the solution continues to a $k = 0$ Oppenheimer-Snyder solution outside of the black hole for $S > 1$. We prove that the shock wave will first become visible at the FRW center $\bar{r} = 0$ at the moment $t = t_0$, $(R(t_0) = 1)$, when the Hubble length $H_0^{-1} = H_{t_0}^{-1}$ satisfies

$$\frac{1}{H_0} = \frac{1 + 3\sigma}{2}\, r_*, \tag{31}$$

(recall that r_* is the FRW position of the shock wave at the Big Bang). Furthermore, the time $t_{crit} > t_0$ at which the shock wave will emerge from the White Hole given that t_0 is the first instant at which the shock becomes visible at the FRW center, can be estimated by

$$\frac{2}{1 + 3\sigma}\, e^{\frac{1}{4}\,\sigma} \leq \frac{t_{crit}}{t_0} \leq \frac{2}{1 + 3\sigma}\, e^{\frac{2\sqrt{3\sigma}}{1+\sigma}}. \tag{32}$$

We believe that the existence of a wave at the leading edge of the expansion of the galaxies is the most likely possibility. The alternatives are that either the universe of expanding galaxies goes on out to infinity, or else the universe is not simply connected. Although the first possibility has been believed for most of the history of cosmology based on the Friedmann universe, we find this implausible and arbitrary in light of the shock wave refinements of the FRW metric discussed here. The second possibility, that the universe is not simply connected, has received considerable attention recently. However, since we have not seen, and cannot create, any non-simply connected 3-spaces on any other length scale, and since there is no observational evidence to support this, we view this as less likely than the existence of wave at the leading edge of the expansion of the galaxies, left over from the Big Bang.

A final comment is in order regarding our overall philosophy. The family of exact shock wave solutions described here are rough models in the sense that the equation of state on the FRW side satisfies $\sigma = const.$, and the equation of state on the TOV side is determined by the equations, and therefore cannot be imposed. Nevertheless, the bounds on the equations of state imply that the equations of state are qualitatively reasonable, and we expect that this family of solutions will capture the gross dynamics of solutions when more general equations of state are imposed. For more general equations of state, other waves, such as rarefaction waves and entropy waves, would need to be present to meet the conservation constraint, and thereby mediate the transition across the shock wave. Such transitional waves would be pretty much impossible to model in an exact solution. But the fact that we can find global solutions that meet our physical bounds, and that are qualitatively the same for all values of $\sigma \in (0, 1]$ and all initial shock positions, strongly suggests that such a shock wave would be the dominant wave in a large class of problems.

References

[1] J. Smoller and B. Temple, "Shock-wave solutions of the Einstein equations: the Oppenheimer-Snyder model of gravitational collapse extended to the case of non-zero pressure", *Arch. Rat. Mech. Anal.*, **128**, (1994), pp. 249–297.

[2] J. Smoller and B. Temple, "Astrophysical shock-wave solutions of the Einstein equations", *Phys. Rev. D*, **142**, No. 6 (1998), pp. 177–191.

[3] J. Smoller and B. Temple, "Cosmology with a shock-wave", *Commun. Math. Phys.*, **210**, (2000), 275–308.

[4] J. Smoller and B. Temple, "Shock wave cosmology inside a White Hole, *PNAS*, **100**, no. 20, (2003), 11216–11218.

[5] Smoller, J., and Temple, B., Cosmology, black holes, and shock waves beyond the Hubble length, *Methods and Appl. of Anal.*, **11**, (2004), 77–132.

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