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Points of general relativistic shock wave interaction are 'regularity singularities' where space–time is not locally flat

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We show that the regularity of the gravitational metric tensor in spherically symmetric space–times cannot be lifted from $C^{0,1}$ to $C^{1,1}$ within the class of $C^{1,1}$ coordinate transformations in a neighbourhood of a point of shock wave interaction in General Relativity, without forcing the determinant of the metric tensor to vanish at the point of interaction. This is in contrast to Israel's theorem, which states that such coordinate transformations always exist in a neighbourhood of a point on a smooth *single* shock surface. The results thus imply that points of shock wave interaction represent a new kind of *regularity singularity* for perfect fluids evolving in space–time, singularities that make perfectly good sense physically, that can form from the evolution of smooth initial data, but at which the space–time is not locally Minkowskian under any coordinate transformation. In particular, at regularity singularities, delta function sources in the second derivatives of the metric exist in all coordinate systems of the $C^{1,1}$-atlas, but due to cancellation, the full Riemann curvature tensor remains *supnorm bounded*.

Keywords: shock wave interactions; general relativity; regularity singularity

1. Introduction

The guiding principle in Albert Einstein's pursuit of General Relativity (GR) was the principle that space–time should be *locally inertial* (we say also *locally Lorentzian*, *locally Minkowskian*). That is, an observer in freefall through a gravitational field should observe all of the physics of special relativity, except for the second-order acceleration effects due to space–time curvature (gravity). But assuming space–time is locally inertial is equivalent to assuming the gravitational metric tensor g is smooth enough to pursue the construction of Riemann normal coordinates at a point p, coordinates in which g is exactly the Minkowski metric at p, all first-order derivatives of g vanish at p, and such that all second-order derivatives of g bounded in a neighbourhood of p. However, the Einstein equations are a system of partial differential equations (PDEs) for the metric tensor g coupled to the sources, and the PDEs by themselves determine the smoothness of the gravitational metric tensor by the evolution they impose. Thus, the condition on space–time that it be locally inertial at every point cannot be assumed at the start, but must be determined by regularity theorems for the Einstein equations.

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The presence of shock waves makes this issue all the more interesting for the Einstein–Euler equations, the equations that describe the time evolution of a perfect fluid. In this case, the Einstein equations $G = \kappa T$ imply the GR compressible Euler equations Div $T = 0$ through the Bianchi identities, and the compressible Euler equations create shock waves whenever the flow is sufficiently compressive. At a shock wave, the fluid density, pressure, velocity and hence T are discontinuous; so the Einstein equations imply that the curvature G must also become discontinuous at shocks. But discontinuous curvature by itself is not inconsistent with the assumption that space–time be locally inertial. For example, if the gravitational metric tensor were $C^{1,1}$ (differentiable with Lipschitz continuous first derivatives, Smoller & Temple 1994), then second derivatives of the metric are at worst discontinuous, and the metric has enough smoothness for there to exist coordinate transformations which transform g to the Minkowski metric at p, with zero derivatives at p, and bounded second derivatives as well (Smoller & Temple 1994). Furthermore, Israel's theorem asserts that a space–time metric of regularity $C^{0,1}$, i.e. Lipschitz continuous, across a smooth *single* shock surface, is lifted to $C^{1,1}$ by the $C^{1,1}$ coordinate map to Gaussian normal coordinates, and this is smooth enough to ensure the existence of locally inertial coordinate frames at each point. In fact, when discontinuities in the fluid are present, $C^{1,1}$ coordinate transformations constitute the atlas of transformations capable of lifting the regularity of the metric one order, while still preserving the weak formulation of the Einstein equations (Smoller & Temple 1994). It is common in GR to assume the gravitational metric tensor is at least $C^{1,1}$, and, for example, this assumption is taken at the start in singularity theorems of Hawking & Ellis (1973). Groah & Temple (2004) set out a framework to rigorously address these issues, by providing the first general existence theory for spherically symmetric shock wave solutions of the Einstein–Euler equations, allowing for arbitrary numbers of interacting shock waves of arbitrary strength. In coordinates where their analysis is feasible, standard Schwarzschild coordinates (SSCs; a general spherically symmetric metric can generically be transformed to an SSC, cf. Weinberg 1972), the gravitational metric is only $C^{0,1}$ at shock waves, and it has remained an open problem as to whether general weak solutions constructed by Groah & Temple could be smoothed to $C^{1,1}$ by coordinate transformation, like the single shock surfaces addressed by Israel.

In this paper, we resolve the open problem of Groah & Temple by proving that there do not exist $C^{1,1}$ coordinate transformations that can lift the regularity of a gravitational metric tensor from $C^{0,1}$ to $C^{1,1}$ at a point of shock wave interaction in a spherically symmetric space–time, without forcing the determinant of the metric tensor to vanish at the point of interaction. Consequently, in contrast to Israel's theorem for single shock surfaces, shock wave solutions cannot in general be continued as $C^{1,1}$ strong solutions of the Einstein equations beyond the first point of shock wave interaction. We emphasize that perfect fluid solutions are *supnorm-bounded* and *free of delta function sources* at points of shock wave interaction.

To state the main result precisely, let $g_{\mu\nu}$ denote a spherically symmetric space–time metric in SSC—that is, the metric takes the form

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -A(t,r)\,\mathrm{d}t^2 + B(t,r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2. \tag{1.1}$$

Here either t or r can be taken to be timelike, and $d\Omega^2 = d\vartheta^2 + \sin^2(\vartheta)\, d\varphi^2$ is the line element on the unit 2-sphere (cf. Groah & Temple 2004). In §2, we make precise the definition of a point of regular shock wave interaction in SSC. Essentially, this is a point in (t, r)-space where two distinct shock waves enter or leave a point p at distinct speeds, such that the metric is Lipschitz continuous across the shocks and smooth away from the shocks, the Rankine–Hugoniot (RH) jump conditions hold across each shock curve continuously up to the point of interaction p, derivatives are continuous up to the shock boundaries and the SSC Einstein equations hold weakly in a neighbourhood of p and strongly away from the shocks (Smoller 1983).

The main result of the paper is the following theorem (cf. definition 3.1 and theorem 7.1):

Theorem 1.1. *Assume that p is a point of regular shock wave interaction in SSCs. Then there does not exist a $C^{1,1}$ regular coordinate transformation, defined in a neighbourhood of p, such that the metric components are C^1 functions of the new coordinates and such that the metric has a non-zero determinant at p.*

The proof of theorem 1.1 is constructive in the sense that we characterize the Jacobians of (t, r) coordinate transformations that smooth the components of the gravitational metric in a deleted neighbourhood of a point p of a regular shock wave interaction, and then prove that any such Jacobian must have a vanishing determinant at p itself. We refer to Reintjes (2011) for a proof that extends theorem 1.1 to $C^{1,1}$ transformations, allowing for changes of angular variables.

Our assumptions in theorem 1.1 apply to the upper half $(t \geq 0)$ and the lower half $(t \leq 0)$ of a shock wave interaction (at $t = 0$) separately, suitable for the initial value problem, and also general enough to include the case of two timelike interacting shock waves of opposite families that cross at the point p, but also general enough to include the cases of two outgoing shock waves created by the focusing of compressive rarefaction waves, or two incoming shock waves of the same family that interact at p to create an outgoing shock wave of the same family and an outgoing rarefaction wave of the opposite family (cf. Smoller 1983). In particular, our framework and our theorems are general enough to incorporate and apply to the shock wave interaction that was numerically simulated in Vogler & Temple (2012).

We emphasize that although points of shock wave interaction are straightforward to construct for the relativistic compressible Euler equations in flat space–time, we know of no construction of a point of regular shock wave interaction in GR with complete mathematical rigour. However, the assumptions are straightforward, the existence theory of Groah & Temple (2004) establish shock wave solutions of the Einstein–Euler equations for interacting shock waves within the assumed $C^{0,1}$ regularity class and simulations of points of shock wave interaction numerically verify the regular assumptions in our definition (Vogler & Temple 2012). We conclude that all evidence indicates that regular shock wave interactions exist in SSCs, and in fact cannot be avoided in solutions consisting of, say, an outgoing spherical shock wave (the blast wave of an explosion) evolving inside an incoming spherical shock wave (the leading edge of an implosion). Taken on whole, we interpret this as definitive physical proof that points of shock wave interaction create a new kind of *regularity singularity* where the gravitational metric tensor cannot be smoothed from $C^{0,1}$ to $C^{1,1}$ by any $C^{1,1}$ coordinate transformation.

It is instructive at this point to clarify the difference between the essential $C^{0,1}$ singularities in the metric at points of shock wave interaction, and the essential $C^{0,1}$ singularities at surface layers like the 'thin shells' introduced in Israel's (1966) illuminating paper. (See also Geroch & Traschen 1987.) To start, recall that because the Einstein equation $G = \kappa T$ implies the compressible Euler equations Div $T = 0$, it can be shown that at singularities where the gravitational metric is only $C^{0,1}$, both $G = \kappa T$ and Div $T = 0$ hold in the weak sense of the theory of distributions (cf. Israel 1966; Smoller & Temple 1994). But on surface layers, Div $T \neq 0$ weakly for the *pointwise* T defined a.e. on each side of the surface. (For spherically symmetric solutions, this can happen only when the RH jump relations fail at the surface, cf. Smoller & Temple (1994).) Thus pointwise values of T must be augmented by distributional sources on the surface that come from second derivatives of G through the Einstein equation $G = \kappa T$ (true delta function sources because $g \in C^{0,1}$), in order to meet the Bianchi identity Div $T = 0$ weakly. Conclude that on surface layers, the pointwise T must be augmented by distributional sources to describe the true sources in T, and these delta function sources are the *cause* of the essential $C^{0,1}$ singularity in the metric g because second derivatives of g must have distributional sources, and consequently g cannot be $C^{1,1}$ in any regular coordinate system.

For shock wave solutions of $G = \kappa T$, the issue is more delicate because the pointwise T solves Div $T = 0$ weakly at shock waves without any delta function sources. Thus the constraint that G have delta function sources is removed, and there is in principle no clear obstacle to the existence of coordinate systems that smooth the metric to $C^{1,1}$. Israel's theorem confirms that there is no obstacle to $C^{1,1}$ smoothness in the special case of single shock surfaces, but the methods in Groah & Temple (2004) are only sufficient to prove the existence of solutions in $C^{0,1}$, and the question as to whether there is an obstacle for more complicated solutions with interactions has remained unresolved until now.

The argument in the present paper resolves this issue by proving that at points of shock wave interaction, the Einstein–Euler equations in SSC generate an *essential* $C^{0,1}$ singularity in the metric that *cannot* be smoothed to $C^{1,1}$ by coordinate transformation, even though Div $T = 0$ is solved exactly, and there are no delta function sources present in T or G to explain the $C^{0,1}$ singularity in the metric g.

We conclude that points of shock wave interaction are a new kind of *regularity singularity* in the gravitational field that are not generated by delta function sources in T, that can form from the evolution of smooth initial data and that correctly reflect the physics of the equations even though the space–time is not locally Minkowskian under any local $C^{1,1}$ coordinate transformation, and where the metric tensor does not have sufficient regularity to satisfy strongly the Einstein–Euler equations in any coordinate system of the $C^{1,1}$ atlas. At such singularities, delta function sources in the second derivatives of the gravitational metric tensor exist in all coordinate systems of the $C^{1,1}$ atlas, but due to cancellation, the Einstein tensor remains uniformly bounded. (In fact, even the Riemann tensor is uniformly bounded, as will be addressed in a follow-up paper by Reintjes.)

In §3, we set out the framework of shock waves in GR, and define what we call a point of *regular shock wave interaction* in SSC. In §4, we introduce a canonical form for functions $C^{0,1}$ across a hypersurface. In §5, we derive a canonical form for Jacobians of general $C^{1,1}$ coordinate transformations necessary and sufficient to lift the regularity of a metric tensor from $C^{0,1}$ to C^1 at points on a shock surface. In §6, we give a new constructive proof of Israel's theorem for spherically symmetric space–times, combining the results from §§4 and 5. In particular, we show that the freedom to add an arbitrary C^1-function to our canonical form suffices for the Jacobians to be integrable to actual coordinate transformations if and only if the RH jump conditions hold.

The main step towards theorem 1.1 is achieved in §7, where we prove that at a point of regular shock wave interaction in SSC, there exists no coordinate transformation of the (t,r)-plane that lifts the metric regularity to C^1. The essential point is that the C^1 gauge freedom in our canonical forms cannot satisfy the integrability condition on the Jacobians, without forcing the determinant of the Jacobian to vanish at the point of interaction.

2. Preliminaries

Let g denote a Lorentzian metric g of signature $(-1,1,1,1)$ on a four-dimensional space–time manifold M. We call M a C^k-manifold if it is endowed with a C^k-atlas, a collection of four-dimensional local diffeomorphisms from M to $\mathbb{R}^4$, such that any composition of two local diffeomorphisms x and y of the form $x \circ y^{-1}$ is C^k regular. ($x \circ y^{-1}$ is referred to as a coordinate transformation.) In this paper, we consider $C^{1,1}$-manifolds.

We use standard index notation for tensors whereby Greek versus Roman indices distinguish coordinate systems, and repeated up–down indices are assumed summed from 0 to 3. Under coordinate transformation, tensors transform by contraction with the Jacobian $J^\mu_j = \partial x^\mu/\partial x^j$, J^j_ν denotes the inverse Jacobian, and indices are raised and lowered with the metric and its inverse g^{ij}, which transform as bilinear forms, $g_{\mu\nu} = J^i_\mu J^j_\nu g_{ij}$ (cf. Weinberg 1972). We use the fact that a matrix of functions J^μ_j is the Jacobian of a regular local coordinate transformation if and only if the curls vanish, i.e.

$$J^\mu_{i,j} = J^\mu_{j,i} \quad \text{and} \quad \mathrm{Det}(J^\mu_j) \neq 0, \tag{2.1}$$

where $f_{,j} = \partial f/\partial x^j$ denotes partial differentiation with respect to the coordinate x^j and $\mathrm{Det}(J^\mu_j)$ denotes the determinant of the Jacobian.

In this paper, we consider the *Einstein–Euler equations*

$$G^{ij} = \kappa T^{ij}, \tag{2.2}$$

which couples the metric tensor g_{ij} to the undifferentiated perfect fluid sources

$$T^{ij} = (p+\rho)u^i u^j + pg^{ij}, \tag{2.3}$$

through the second-order Einstein curvature tensor $G^{ij} \equiv R^{ij} - \frac{1}{2}Rg^{ij}$, and

$$\mathrm{Div}\ T = 0 \tag{2.4}$$

follows from Div $G=0$. Here κ is the coupling constant, ρ is the energy density, u_i the 4-velocity and p the pressure (cf. Weinberg 1972). Equation (2.4) reduces to the relativistic compressible Euler equations when g_{ij} is the Minkowski metric, and the Euler equations close when an equation of state (e.g. $p=p(\rho)$) is imposed. Shock waves form from smooth solutions of the relativistic compressible Euler equations when the initial data is sufficiently compressive (Smoller 1983).

Across a smooth shock surface Σ, the RH jump conditions hold,

$$[T^{\mu\nu}]n_\nu = 0, \tag{2.5}$$

where $[f]=f_{\mathrm{L}}-f_{\mathrm{R}}$ denotes the jump in f from right to left across Σ, and n_ν is the surface normal. The RH condition (2.5) is equivalent to the weak formulation of (2.4) across Σ (cf. Smoller 1983).

In this paper, we restrict to time-dependent spherically symmetric metrics in SSC (1.1), where the metric takes the form Weinberg (1972), where $B(t,r)=(1-2M(t,r)/r)$ defines the mass function M. The Einstein equations for a metric in SSC are given by

$$B_r + B\frac{B-1}{r} = \kappa AB^2 rT^{00}, \tag{2.6}$$

$$B_t = -\kappa AB^2 rT^{01}, \tag{2.7}$$

$$A_r - A\frac{1+B}{r} = \kappa AB^2 rT^{11} \tag{2.8}$$

and

$$B_{tt} - A_{rr} + \Phi = -2\kappa ABr^2 T^{22}, \tag{2.9}$$

with

$$\Phi = -\frac{BA_tB_t}{2AB} - \frac{B_t^2}{2B} - \frac{A_r}{r} + \frac{AB_r}{rB} + \frac{A_r^2}{2A} + \frac{A_rB_r}{2B}.$$

Note that the first three Einstein equations in SSC imply that the metric cannot be any smoother than Lipschitz continuous if the source T is discontinuous, for example, $T^{ij}\in L^\infty$, and in this paper we make the assumption throughout that A and B are Lipschitz continuous, i.e. $C^{0,1}$ functions, of t and r. We now make precise the notion of a point of *regular shock wave interaction*.

3. A point of regular shock wave interaction in Standard Schwarzschild coordinates

In this paper, we restrict attention to radial shock waves, by which we mean hypersurfaces Σ locally parametrized by

$$\Sigma(t,\vartheta,\varphi) = (t, x(t), \vartheta, \varphi), \tag{3.1}$$

across which A and B are $C^{0,1}$ and T in (2.3) satisfies (2.5). Then, for each t, Σ is a 2-sphere with radius $x(t)$ and centre $r=0$. Treating ϕ and θ as constant, we introduce γ, the restriction of a shock surface Σ to the (t,r)-plane,

$$\gamma(t) = (t, x(t)), \tag{3.2}$$

with normal 1-form

$$n_\sigma = (\dot{x}, -1). \tag{3.3}$$

For radial shock surfaces (3.1) in SSC, the RH jump conditions (2.5) take the simplified form

$$[T^{00}]\dot{x} = [T^{01}] \tag{3.4}$$

and

$$[T^{10}]\dot{x} = [T^{11}]. \tag{3.5}$$

Now suppose two timelike shock surfaces Σ_i are parametrized in SSC by

$$\Sigma_i(t,\theta,\phi) = (t, x_i(t), \theta, \phi), \quad i = 1, 2. \tag{3.6}$$

Let $\gamma_i(t)$ denote their corresponding restrictions to the (t,r)-plane,

$$\gamma_i(t) = (t, x_i(t)), \tag{3.7}$$

with normal 1-forms

$$(n_i)_\sigma = (\dot{x}_i, -1), \tag{3.8}$$

and use the notation that $[f]_i(t)$ denotes the jump in the quantity f across the surface $\gamma_i(t)$.

For our theorem, it suffices to restrict attention to the lower or upper part of a shock wave interaction that occurs at $t = 0$. That is, in either the lower or upper half plane

$$\mathbb{R}^2_- = \{(t,r) : t < 0\} \quad \text{or} \quad \mathbb{R}^2_+ = \{(t,r) : t > 0\},$$

respectively, whichever half plane contains two shock waves that intersect at p with distinct speeds. (We denote with $\overline{\mathbb{R}^2_\pm}$ the closure of $\mathbb{R}^2_\pm$.) Thus, without loss of generality, let $\gamma_i(t) = (t, x_i(t))$, $(i = 1, 2)$, be two shock curves in the lower (t,r)-plane that intersect at a point $(0, r_0)$, $r_0 > 0$, i.e.

$$x_1(0) = r_0 = x_2(0). \tag{3.9}$$

We now introduce the precise definition of a point of *regular shock wave interaction* in SSC. Without loss of generality, we assume a lower shock wave interaction in $\mathbb{R}^2_-$. Our assumptions are what one would expect of a shock wave solution of the Einstein–Euler equations, namely, a $C^{0,1}$ metric smooth enough to solve the Einstein equations strongly away from the shocks and smooth enough to ensure the RH conditions to hold across each shock.

Definition 3.1. Let $r_0 > 0$, and let $g_{\mu\nu}$ be an SSC metric in $C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$, where $\mathcal{N} \subset \mathbb{R}^2$ is a neighbourhood of a point $p = (0, r_0)$ of intersection of two timelike shock curves $\gamma_i(t) = (t, x_i(t)) \in \mathbb{R}^2_-$, $t \in (-\epsilon, 0)$. Assume the shock speeds $\dot{x}_i(0) = \lim_{t\to 0} \dot{x}_i(t)$ exist and are distinct, and let $\hat{\mathcal{N}}$ denote the neighbourhood consisting of all points in $\mathcal{N} \cap \mathbb{R}^2_-$ not in the closure of the two intersecting curves $\gamma_i(t)$. Then we say that p is a point of regular shock wave interaction in SSC if:

— The pair (g, T) is a strong solution of the SSC Einstein equations (2.6)–(2.9) in $\hat{\mathcal{N}}$, with $T^{\mu\nu} \in C^0(\hat{\mathcal{N}})$ and $g_{\mu\nu} \in C^2(\hat{\mathcal{N}})$.

— The limits of T and of metric derivatives $g_{\mu\nu,\sigma}$ exist on both sides of each shock curve $\gamma_i(t)$ for all $-\epsilon < t < 0$.
— The jumps in the metric derivatives $[g_{\mu\nu,\sigma}]_i(t)$ are C^1 function with respect to t for $i = 1, 2$ and for $t \in (-\epsilon, 0)$.
— The limits

$$\lim_{t\to 0}[g_{\mu\nu,\sigma}]_i(t) = [g_{\mu\nu,\sigma}]_i(0)$$

exist for $i = 1, 2$.
— The metric g is continuous across each shock curve $\gamma_i(t)$ separately, but no better than Lipschitz continuous in the sense that, for each i there exists μ, ν such that

$$[g_{\mu\nu,\sigma}]_i(n_i)^\sigma \neq 0$$

at each point on γ_i, $t \in (-\epsilon, 0)$ and

$$\lim_{t\to 0}[g_{\mu\nu,\sigma}]_i(n_i)^\sigma \neq 0.$$

— The stress tensor T is bounded on $\mathcal{N} \cap \overline{\mathbb{R}^2_-}$ and satisfies the RH jump conditions

$$[T^{\nu\sigma}]_i(n_i)_\sigma = 0$$

at each point on $\gamma_i(t)$, $i = 1, 2$, $t \in (-\epsilon, 0)$, and the limits of these jumps exist up to p as $t \to 0$.

4. Functions $C^{0,1}$ across a hypersurface

In this section, we give a precise definition of a function $C^{0,1}$ across a hypersurface, and use this to derive a canonical form for such functions.

Definition 4.1. Let Σ be a smooth (timelike) hypersurface in some open set $\mathcal{N} \subset \mathbb{R}^d$. We call a function f 'Lipschitz continuous across Σ' (or $C^{0,1}$ acros Σ), if $f \in C^{0,1}(\mathcal{N})$, f is smooth ($f \in C^2(\mathcal{N} \setminus \Sigma)$ suffices) in $\mathcal{N} \setminus \Sigma$, and limits of derivatives of f exist and are smooth functions on each side of Σ separately. We call a metric $g_{\mu\nu}$ Lipschitz continuous across Σ in coordinates x^μ if all metric components are $C^{0,1}$ across Σ.

The main point of the earlier-mentioned definition is that we assume smoothness of f (or $g_{\mu\nu}$), away and tangential to the hypersurface Σ. Note that the continuity of f across Σ implies the continuity of all derivatives of f tangent to Σ, i.e.

$$[f_{,\sigma}]v^\sigma = 0, \tag{4.1}$$

for all v^σ tangent to Σ. Moreover, definition 4.1 allows for the normal derivative of f to be discontinuous, that is,

$$[f_{,\sigma}]n^\sigma \neq 0, \tag{4.2}$$

where n^σ is normal to Σ with respect to some (Lorentz) metric $g_{\mu\nu}$ defined on $\mathcal{N}$.

We can now clarify the connections between the Einstein equations and the RH jump conditions (3.4), (3.5) for SSC metrics only $C^{0,1}$ across a hypersurface. To this end, consider a spherically symmetric space–time metric (1.1) given in

SSC, assume that the first three Einstein equations (2.6)–(2.8) hold, and assume that the stress tensor T is discontinuous across a smooth radial shock surface described in the (t, r)-plane by $\gamma(t)$ as in (3.1)–(3.3). Condition (4.1) across γ applied to each metric component $g_{\mu\nu}$ in SSC (1.1) then reads

$$[B_t] = -\dot{x}[B_r] \tag{4.3}$$

and

$$[A_t] = -\dot{x}[A_r]. \tag{4.4}$$

On the other hand, the first three Einstein equations in SSC (2.6)–(2.8) imply

$$[B_r] = \kappa A B^2 r [T^{00}], \tag{4.5}$$

$$[B_t] = -\kappa A B^2 r [T^{01}] \tag{4.6}$$

and

$$[A_r] = \kappa A B^2 r [T^{11}]. \tag{4.7}$$

Now, using the jumps in Einstein equations (4.5)–(4.7), we find that (4.3) is equivalent to the first RH jump condition (3.4) (cf. lemma 9, page 286, of Smoller & Temple 1994), while the second condition (4.4) is independent of equations (4.5)–(4.7), because A_t does not appear in the first-order SSC equations (2.6)–(2.8). The result, then, is that in addition to the assumption that the metric be $C^{0,1}$ across the shock surface in SSC, the RH conditions (3.4) and (3.5), together with the Einstein equations (4.5)–(4.7), yield only one additional condition over and above (4.3) and (4.4), namely

$$[A_r] = -\dot{x}[B_t]. \tag{4.8}$$

The RH jump conditions together with the Einstein equations will enter our method in §§5–7 only through the three equations (4.8), (4.3) and (4.4).

The following lemma provides a canonical form for any function f that is Lipschitz continuous across a *single* shock curve γ in the (t, r)-plane, under the assumption that the vector n^μ, normal to γ, is obtained by raising the index in (3.3) with respect to a Lorentzian metric g that is $C^{0,1}$ across γ. (Note that by definition 4.1, n^μ varies C^1 in directions tangent to γ, and we suppress the angular coordinates.)

Lemma 4.2. *Suppose f is $C^{0,1}$ across a smooth curve $\gamma(t) = (t, x(t))$ in the sense of definition 4.1, $t \in (-\epsilon, \epsilon)$, in an open subset $\mathcal{N}$ of $\mathbb{R}^2$. Then there exists a function $\Phi \in C^1(\mathcal{N})$ such that*

$$f(t, r) = \tfrac{1}{2}\varphi(t)|x(t) - r| + \Phi(t, r), \tag{4.9}$$

where

$$\varphi(t) = \frac{[f_{,\mu}]n^\mu}{n^\sigma n_\sigma} \in C^1(-\epsilon, \epsilon), \tag{4.10}$$

and $n_\mu(t) = (\dot{x}(t), -1)$ is a 1-form normal to the tangent vector $v^\mu(t) = \dot{\gamma}^\mu(t)$. In particular, it suffices that indices are raised and lowered by a Lorentzian metric $g_{\mu\nu}$ that is $C^{0,1}$ across γ.

In words, the canonical form (4.9) separates off the $C^{0,1}$ kink of f across γ from its more regular C^1 behaviour away from γ: the kink is incorporated into $|x(t)-r|$, φ gives the smoothly varying strength of the jump and Φ encodes the remaining C^1 behaviour of f.

In §7, we need a canonical form analogous to (4.9) for two shock curves, but such that it allows for the Jacobian to be in the weaker regularity class $C^{0,1}$ away from the shock curves. To this end, suppose two timelike shock surfaces described in the (t,r)-plane by, $\gamma_i(t)$, such that (3.6)–(3.9) applies. To cover the generic case of shock wave interaction, we assume each $\gamma_i(t)$ is smooth away from $t=0$ with continuous tangent vectors up to $t=0$, and it suffices to restrict to lower shock wave interactions in $\mathbb{R}^2_-$.

Corollary 4.3. *Let $\gamma_i(t)=(t,x_i(t))$ be two smooth curves defined on $I=(-\epsilon,0)$, some $\epsilon>0$, such that the limits $\lim_{t\to 0-}\gamma_i(t)=(0,r_0)$ and $\dot{x}_i(0)=\lim_{t\to 0-}\dot{x}_i(t)$ both exist for $i=1,2$. Let f be a function in $C^{0,1}(\mathcal{N}\cap\mathbb{R}^2_-)$ for $\mathcal{N}$ a neighbourhood of $(0,r_0)$ in $\mathbb{R}^2$, so that f meets condition (4.1) on each γ_i. Then there exists a $C^{0,1}$ function Φ defined on $\mathcal{N}\cap\mathbb{R}^2_-$, such that*

$$[\Phi_t]_i \equiv 0 \equiv [\Phi_r]_i, \quad i=1,2 \tag{4.11}$$

and

$$f(t,r)=\frac{1}{2}\sum_{i=1,2}\varphi_i(t)|x_i(t)-r|+\Phi(t,r), \tag{4.12}$$

for all (t,r) in $\mathcal{N}\cap\mathbb{R}^2_-$, where

$$\varphi_i(t)=\frac{[f_{,\mu}]_i n_i^{\mu}}{n_i^{\mu}n_{i\mu}} \quad \in C^{0,1}(I). \tag{4.13}$$

In particular, φ_i has discontinuous derivatives wherever $f\circ\gamma_i$ does, and again it suffices that indices are raised and lowered by a Lorentzian metric $g_{\mu\nu}$ that is $C^{0,1}$ across each γ_i.

5. A necessary and sufficient condition for smoothing metrics

In this section, we derive a necessary and sufficient pointwise condition on the Jacobians of a coordinate transformation that it lifts the regularity of a $C^{0,1}$ metric tensor to $C^{1,1}$ in a neighbourhood of a point on a single shock surface Σ. This is the starting point for §§6 and 7.

We begin with the transformation law

$$g_{\alpha\beta}=J^{\mu}_{\alpha}g_{\mu\nu}J^{\nu}_{\beta}, \tag{5.1}$$

for the metric components at a point on a hypersurface Σ for a general $C^{1,1}$ coordinate transformation $x^{\mu}\to x^{\alpha}$, where, as customary, the indices indicate the coordinate system. J^{μ}_{α} denotes the Jacobian of the transformation, that is, $J^{\mu}_{\alpha}=\partial x^{\mu}/\partial x^{\alpha}$. Assume now that the metric components $g_{\mu\nu}$ are only Lipschitz continuous with respect to x^μ across Σ. Then differentiating (5.1) in the direction $w = w^\sigma(\partial/\partial x^\sigma)$ we obtain

$$[g_{\alpha\beta,\gamma}]w^\gamma = J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\sigma}]w^\sigma + g_{\mu\nu} J^\mu_\alpha [J^\nu_{\beta,\sigma}]w^\sigma + g_{\mu\nu} J^\nu_\beta [J^\mu_{\alpha,\sigma}]w^\sigma, \tag{5.2}$$

where $[f]$ denotes the jump in the quantity f across the shock surface Σ. Thus, since both g and J^μ_α are in general Lipschitz continuous across Σ, the jumps appear only on the derivatives. Equation (5.2) gives a necessary and sufficient condition for the metric g to be $C^{1,1}$ in x^α coordinates. Namely, taking $w = \partial/\partial x^\sigma$, (5.2) implies that $[g_{\alpha\beta,\gamma}] = 0$ for every $\alpha, \beta, \gamma = 0, \ldots, 3$ if and only if

$$[J^\mu_{\alpha,\gamma}]J^\nu_\beta g_{\mu\nu} + [J^\nu_{\beta,\gamma}]J^\mu_\alpha g_{\mu\nu} + J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\gamma}] = 0. \tag{5.3}$$

Note that if the coordinate transformation is C^2, so that J^μ_α is C^1, then the jumps in J vanish, and (5.2) reduces to

$$[g_{\alpha\beta,\gamma}]w^\gamma = J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\sigma}]w^\sigma,$$

which is tensorial because the non-tensorial terms cancel out in the jump $[g_{\alpha\beta,\gamma}]$. It is precisely the lack of covariance in (5.2) for $C^{1,1}$ transformations that provides the necessary degrees of freedom in the jumps $[J^\mu_{\alpha,\gamma}]$ to lift the smoothness of a Lipshitz metric one order at single shock surface.

We now exploit linearity in (5.3) to solve for the $[J^\mu_{\alpha,\gamma}]$ associated with a given $C^{1,1}$ coordinate transformation. To this end, suppose we are given a single radial shock surface Σ in SSC locally parametrized by

$$\Sigma(t, \theta, \phi) = (t, x(t), \theta, \phi). \tag{5.4}$$

For such a hypersurface in SSC, the angular variables play a passive role, and the essential issue regarding smoothing the metric components by $C^{1,1}$ coordinate transformation lies in the atlas of (t, r)-coordinate transformations. Thus, we restrict to the atlas of (t, r)-coordinate transformations for a general $C^{0,1}$ metric in SSC (cf. (1.1)). The following lemma gives the unique solution $[J^\mu_{\alpha,\gamma}]$ of (5.3) for (t, r)-transformations of $C^{0,1}$ metrics g in SSC.

Lemma 5.1. *Let*

$$g_{\mu\nu} = -A(t, r)\,\mathrm{d}t^2 + B(t, r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

be a given metric expressed in SSC, let Σ denote a single radial shock surface (5.4) across which g is only Lipschitz continuous. Then the unique solution $[J^\mu_{\alpha,\gamma}]$ of (5.3) that satisfies the integrability condition (cf. (2.1))

$$[J^\mu_{\alpha,\beta}] = [J^\mu_{\beta,\alpha}], \tag{5.5}$$

is given by

$$\left.\begin{aligned}
&[J^t_{0,t}] = -\frac{1}{2}\left(\frac{[A_t]}{A}J^t_0 + \frac{[A_r]}{A}J^r_0\right); \quad [J^t_{0,r}] = -\frac{1}{2}\left(\frac{[A_r]}{A}J^t_0 + \frac{[B_t]}{A}J^r_0\right)\\
&[J^t_{1,t}] = -\frac{1}{2}\left(\frac{[A_t]}{A}J^t_1 + \frac{[A_r]}{A}J^r_1\right); \quad [J^t_{1,r}] = -\frac{1}{2}\left(\frac{[A_r]}{A}J^t_1 + \frac{[B_t]}{A}J^r_1\right)\\
&[J^r_{0,t}] = -\frac{1}{2}\left(\frac{[A_r]}{B}J^t_0 + \frac{[B_t]}{B}J^r_0\right); \quad [J^r_{0,r}] = -\frac{1}{2}\left(\frac{[B_t]}{B}J^t_0 + \frac{[B_r]}{B}J^r_0\right)\\
\text{and}\quad &[J^r_{1,t}] = -\frac{1}{2}\left(\frac{[A_r]}{B}J^t_1 + \frac{[B_t]}{B}J^r_1\right); \quad [J^r_{1,r}] = -\frac{1}{2}\left(\frac{[B_t]}{B}J^t_1 + \frac{[B_r]}{B}J^r_1\right).
\end{aligned}\right\} \tag{5.6}$$

(*We use the notation* $\mu, \nu \in \{t, r\}$ *and* $\alpha, \beta \in \{0, 1\}$, *so that* t, r *are used to denote indices whenever they appear on the Jacobian* J.)

Proof. Equation (5.3) as an inhomogeneous 6×6 linear system in eight unknowns $[J^\mu_{\alpha,\gamma}]$. Imposing the integrability condition in the form of (5.5) gives two additional equations that complete (5.3) to an 8×8 system that is uniquely solvable for $[J^\mu_{\alpha,\gamma}]$. The result is a purely algebraic system, whose unique solution (5.6) is obtained by a lengthy calculation aided by MAPLE (cf. Reintjes 2011 for details.) ∎

Condition (5.3) is a necessary and sufficient condition for $[g_{\alpha\beta,\gamma}] = 0$ at a point on a smooth single shock surface. Because lemma 5.1 tells us that we can uniquely solve (5.3) for the Jacobian derivatives, it follows that a necessary and sufficient condition for $[g_{\alpha\beta,\gamma}] = 0$ is also that the jumps in the Jacobian derivatives be exactly the functions of the jumps in the original SSC metric components recorded in (5.6). In the light of this, lemma 5.1 immediately implies the following corollary:

Corollary 5.2. *Let* p *be a point on a single smooth shock curve* γ, *and let* $g_{\mu\nu}$ *be a metric tensor in SSC, which is* $C^{0,1}$ *across* γ *in the sense of definition 4.1. Suppose* J^μ_α *is the Jacobian of an actual coordinate transformation defined on a neighbourhood* $\mathcal{N}$ *of* p. *Then the metric in the new coordinates* $g_{\alpha\beta}$ *is in* $C^{1,1}(\mathcal{N})$ *if and only if* J^μ_α *satisfies (5.6).*

We have shown in corollary 5.2 that (5.6) is a necessary and sufficient condition on a Jacobian derivative J^μ_α for lifting the SSC metric regularity to $C^{1,1}$ in a neighbourhood of a shock curve.

6. Metric smoothing on single shock surfaces and a constructive proof of Israel's theorem

In this section, we give an alternative constructive proof of Israel's theorem for spherically symmetric space–times. For this, we address the issue of how to obtain Jacobians of actual coordinate transformations defined on a whole neighbourhood of a shock surface that satisfy (5.6). That is, we need a set of functions J^μ_α that satisfies (5.6), and also satisfies the integrability condition (2.1) in a whole neighbourhood.

Theorem 6.1 (Israel's theorem). *Suppose $g_{\mu\nu}$ is an SSC metric that is $C^{0,1}$ across a radial shock surface γ in the sense of definition 4.1, such that it solves the Einstein equations (2.6)–(2.9) strongly away from γ, and assume $T^{\mu\nu}$ is everywhere bounded and in C^0 away from γ. Then around each point p on γ, there exists a $C^{1,1}$ coordinate transformation of the (t, r)-plane, defined in a neighbourhood $\mathcal{N}$ of p, such that the transformed metric components $g_{\alpha\beta}$ are $C^{1,1}$ functions of the new coordinates, if and only if the RH jump conditions (3.4), (3.5) hold on γ in a neighbourhood of p.*

The main step is to construct Jacobians acting on the (t, r)-plane that satisfy the smoothing condition (5.6) on the shock curve, the condition that guarantees $[g_{\alpha\beta,\gamma}] = 0$. The following lemma gives an explicit formula for functions J^μ_α satisfying (5.6). The main point is that, in the case of single shock curves, both the RH jump conditions and the Einstein equations are necessary and sufficient for such functions J^μ_α to exist.

Lemma 6.2. *Let p be a point on a single shock curve γ across which the SSC metric $g_{\mu\nu}$ is Lipschitz continuous in the sense of definition 4.1 in a neighbourhood $\mathcal{N}$ of p. Then there exists a set of functions $J^\mu_\alpha \in C^{0,1}(\mathcal{N})$ satisfying the smoothing condition (5.6) on $\gamma \cap \mathcal{N}$ if and only if (4.8) holds on $\gamma \cap \mathcal{N}$. Furthermore, all J^μ_α that are in $C^{0,1}(\mathcal{N})$ and satisfy (5.6) on $\gamma \cap \mathcal{N}$ are given by*

$$\left.\begin{aligned} J^t_0(t,r) &= \frac{[A_r]\phi(t) + [B_t]\omega(t)}{4A\circ\gamma(t)}|x(t) - r| + \Phi(t,r) \\ J^t_1(t,r) &= \frac{[A_r]\nu(t) + [B_t]\zeta(t)}{4A\circ\gamma(t)}|x(t) - r| + N(t,r) \\ J^r_0(t,r) &= \frac{[B_t]\phi(t) + [B_r]\omega(t)}{4B\circ\gamma(t)}|x(t) - r| + \Omega(t,r) \\ \text{and} \qquad J^r_1(t,r) &= \frac{[B_t]\nu(t) + [B_r]\zeta(t)}{4B\circ\gamma(t)}|x(t) - r| + Z(t,r), \end{aligned}\right\} \tag{6.1}$$

for arbitrary functions $\Phi, \Omega, Z, N \in C^{0,1}(\mathcal{N})$, where

$$\phi = \Phi\circ\gamma, \quad \omega = \Omega\circ\gamma, \quad \nu = N\circ\gamma, \quad \zeta = Z\circ\gamma. \tag{6.2}$$

Moreover, each arbitrary function $U = \Phi, \Omega, Z$ or N satisfies

$$[U_r] = 0 = [U_t]. \tag{6.3}$$

Proof. Suppose there exists a set of functions $J^\mu_\alpha \in C^{0,1}(\mathcal{N})$ satisfying (5.6), then their continuity implies that tangential derivatives along γ match across γ, that is

$$[J^\mu_{\alpha,t}] = -\dot{x}[J^\mu_{\alpha,r}] \tag{6.4}$$

for all $\mu \in \{t, r\}$ and $\alpha \in \{0, 1\}$. Imposing (6.4) in (5.6) and using (4.3)–(4.4) yields (4.8).

To prove the opposite direction, it suffices to show that all t and r derivatives of J^μ_α, defined in (6.1) satisfy (5.6) for all $\mu \in \{t, r\}$ and $\alpha \in \{0, 1\}$. This follows directly from (4.3), (4.4) and (4.8), upon noting that (6.2) implies the identities

$$\phi = J^t_0 \circ \gamma, \quad \nu = J^t_1 \circ \gamma, \quad \omega = J^r_0 \circ \gamma, \quad \zeta = J^r_1 \circ \gamma. \tag{6.5}$$

This proves the existence of functions J^μ_α satisfying (5.6). Applying (the one shock version of) corollary 4.3 (which allows Φ to have the lower regularity $\Phi \in C^{1,1}$ but imposes the jumps (7.4) along γ) confirms that all such functions can be written in the canonical form (6.1). ■

To complete the proof of Israel's theorem, we must prove the existence of coordinate transformations $x^\mu \to x^\alpha$ that lift the $C^{0,1}$ regularity of $g_{\mu\nu}$ to $C^{1,1}$. It remains, then, to show that the functions J^μ_α defined earlier in ansatz (6.1) can be integrated to coordinate functions, i.e. that they satisfy the integrability condition (2.1) in a whole neighbourhood. This is accomplished in the following two lemmas.

Lemma 6.3. *The functions J^μ_α defined in (6.1) satisfy the integrability condition (2.1) if an only if the free functions Φ, Ω, N and Z satisfy the following system of two PDEs*:

$$\begin{aligned}(\dot\alpha|X| + \Phi_t)(\beta|X| + N) + \Phi_r(\epsilon|X| + Z) - (\alpha|X| + \Phi)(\dot\beta|X| + N_t)\\ - N_r(\delta|X| + \Omega) + fH(X) = 0\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}(\dot\delta|X| + \Omega_t)(\beta|X| + N) + \Omega_r(\epsilon|X| + Z) - (\dot\epsilon|X| + Z_t)(\alpha|X| + \Phi)\\ - Z_r(\delta|X| + \Omega) + hH(X) = 0,\end{aligned} \tag{6.7}$$

where $X(t,r) = x(t) - r$, $H(\cdot)$ denotes the Heaviside step function,

$$\left.\begin{aligned}\alpha = \frac{[A_r]\phi(t) + [B_t]\omega(t)}{4A\circ\gamma(t)}; \quad \beta = \frac{[A_r]\nu(t) + [B_t]\zeta(t)}{4A\circ\gamma(t)};\\ \delta = \frac{[B_t]\phi(t) + [B_r]\omega(t)}{4B\circ\gamma(t)}; \quad \epsilon = \frac{[B_t]\nu(t) + [B_r]\zeta(t)}{4B\circ\gamma(t)};\end{aligned}\right\} \tag{6.8}$$

and

$$\left.\begin{aligned}f &= (\beta\delta - \alpha\epsilon)|X| + \alpha\dot x N - \beta\dot x\Phi + \beta\Omega - \alpha Z,\\ h &= (\beta\delta - \alpha\epsilon)\dot x|X| + \delta\dot x N - \epsilon\dot x\Phi + \epsilon\Omega - \delta Z,\end{aligned}\right\} \tag{6.9}$$

where, α, β, δ and ϵ are C^1 functions of t and f and h are in $C^{0,1}$.

The proof of lemma 6.3 follows by substituting ansatz (6.1) into the integrability condition (2.1) and identifying the terms in the resulting first-order differential equations for J^μ_α. (For details, see Reintjes 2011.)

The proof of Israel's theorem is complete, once we prove the existence of solutions Φ, Ω, N and Z of (6.6), (6.7) that are $C^{0,1}$, such that they satisfy (6.3). For this, it suffices to choose N and Z arbitrarily, so that (6.6), (6.7) reduces to a system of two linear first-order PDE's for the unknown functions Φ and Ω. The condition (6.3) essentially imposes that Φ, Ω, N and Z be C^1 across the shock γ. Since (6.6), (6.7) are linear equations for Φ and Ω, they can be solved along characteristics, and so the only obstacle to solutions Φ and Ω with the requisite smoothness to satisfy the condition (6.3) is the presence of the Heaviside function $H(X)$ on the right-hand side of (6.6), (6.7). Lemma 3.3 thus isolates the discontinuous behaviour of equations (6.6), (6.7) in the functions f and h, the coefficients of H. Israel's theorem is now a consequence of the following lemma which states that these coefficients of $H(X)$ vanish precisely when the RH jump conditions hold on γ. (See Reintjes 2011 for details.)

Lemma 6.4. *Assume the SSC metric $g_{\mu\nu}$ is $C^{0,1}$ across γ and solves the first three Einstein equations strongly away from γ. Then the coefficients f and g of $H(X)$ in (6.6), (6.7) vanish on γ if and only if the RH jump conditions (2.5) hold on γ.*

We can now complete the proof of Israel's theorem. Assuming that the Einstein equations hold strongly away from the shock curve (in fact, it suffices to assume that only the first three equations hold), we have that there exist functions J^{μ}_{α} satisfying the smoothing condition (5.6) if and only if the RH jump conditions hold (cf. lemma 6.2). Furthermore, by lemmas 6.3 and 6.4, a solution to the integrability condition with the required regularity holds if and only if the RH jump conditions hold (in the sense of (4.8)). Thus, under the assumption that the Einstein equations hold strongly away from γ, we can integrate the Jacobians J^{μ}_{α} to coordinate functions that smooth the metric g to $C^{1,1}$ if and only if the RH jump conditions hold. This completes the proof of theorem 6.

7. Shock wave interactions as regularity singularities in GR: transformations in the (t, r)-plane

The main step in the proof of theorem 1.1 is to prove that there do not exist $C^{1,1}$ coordinate transformations of the (t, r)-plane in a neighbourhood of a point p of regular shock wave interaction in SSC that lifts the regularity of the metric g from $C^{0,1}$ to $C^{1,1}$ in a neighbourhood of p. We formulate the main step precisely for lower shock wave interactions in $\mathbb{R}^2_-$ in the following theorem, which is the topic of this section. A corresponding result applies to upper shock wave interactions in R^2_+, as well as two wave interactions in a whole neighbourhood of p.

Theorem 7.1. *Suppose that p is a point of regular shock wave interaction in SSC, in the sense of definition 3.1, for the SSC metric $g_{\mu\nu}$. Then there does not exist a $C^{1,1}$ coordinate transformation $x^{\alpha} \circ (x^{\mu})^{-1}$ of the (t, r)-plane, defined on $\mathcal{N} \cap \mathbb{R}^2_-$ for a neighbourhood $\mathcal{N}$ of p in $\mathbb{R}^2$, such that the metric components $g_{\alpha\beta}$ are C^1 functions of the coordinates x^{α} in $\mathcal{N} \cap \mathbb{R}^2_-$ and such that the metric has a non-vanishing determinant at p (that is, such that $\lim_{q \to p} \mathrm{Det}(g_{\alpha\beta}(q)) \neq 0$).*

In the remainder of this section, we outline the proof of theorem 7.1, which mirrors the constructive proof of Israel's theorem 6.1 in that it uses the extension (7.1) of ansatz (6.1) to construct all $C^{1,1}$ coordinate transformations that can smooth the gravitational metric to $C^{1,1}$ in a neighbourhood of a point p of regular shock wave interaction. The negative conclusion is then reached by proving that any such coordinate transformation must have a vanishing Jacobian determinant at p. But now, to prove non-existence, we must show the ansatz (7.1) is general enough to include all $C^{0,1}$ Jacobians that could possibly lift the regularity of the metric. For this, we use condition (5.6) to construct a canonical form for the Jacobians in a neighbourhood of p, that generalizes (6.1) to the case of two shock curves, with the weaker assumption of $C^{0,1}$ regularity on the functions Φ, Ω, Z, N. We conclude the proof by showing that this canonical form is inconsistent with the assumption that $\mathrm{Det}(g_{\alpha\beta}) \neq 0$ at p, by using the continuity of the Jacobians up to p.

To implement these ideas, the main step is to show that the canonical form (4.12) of corollary 4.3 can be applied to the Jacobians J^μ_α in the presence of a shock wave interaction. The result is recorded in the following lemma.

Lemma 7.2. *Let p be a point of regular shock wave interaction in SSC in the sense of definition 3.1, corresponding to the SSC metric $g_{\mu\nu}$ defined on $\mathcal{N} \cap \overline{\mathbb{R}^2_-}$. Then there exists a set of functions $J^\mu_\alpha \in C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ satisfying the smoothing condition (5.6) on $\gamma_i \cap \mathcal{N}$, $i = 1, 2$, if and only if (4.8) holds on each shock curve $\gamma_i \cap \mathcal{N}$. In this case, all J^μ_α in $C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ assume the canonical form*

$$\left.\begin{aligned} J^t_0(t,r) &= \sum_i \alpha_i(t)|x_i(t) - r| + \Phi(t,r),\\ J^t_1(t,r) &= \sum_i \beta_i(t)|x_i(t) - r| + N(t,r),\\ J^r_0(t,r) &= \sum_i \delta_i(t)|x_i(t) - r| + \Omega(t,r)\\ \text{and}\quad J^r_1(t,r) &= \sum_i \epsilon_i(t)|x_i(t) - r| + Z(t,r), \end{aligned}\right\} \tag{7.1}$$

where

$$\left.\begin{aligned} \alpha_i(t) &= \frac{[A_r]_i\,\phi_i(t) + [B_t]_i\,\omega_i(t)}{4A\circ\gamma_i(t)},\\ \beta_i(t) &= \frac{[A_r]_i\,\nu_i(t) + [B_t]_i\,\zeta_i(t)}{4A\circ\gamma_i(t)},\\ \delta_i(t) &= \frac{[B_t]_i\,\phi_i(t) + [B_r]_i\,\omega_i(t)}{4B\circ\gamma_i(t)}\\ \text{and}\quad \epsilon_i(t) &= \frac{[B_t]_i\,\nu_i(t) + [B_r]_i\,\zeta_i(t)}{4B\circ\gamma_i(t)}, \end{aligned}\right\} \tag{7.2}$$

with

$$\phi_i = \Phi\circ\gamma_i,\quad \omega_i = \Omega\circ\gamma_i,\quad \zeta_i = Z\circ\gamma_i \quad\text{and}\quad \nu_i = N\circ\gamma_i, \tag{7.3}$$

and where $\Phi, \Omega, Z, N \in C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ have matching derivatives on each shock curve $\gamma_i(t)$,

$$[U_r]_i = 0 = [U_t]_i, \tag{7.4}$$

for $U = \Phi, \Omega, Z, N$, $t \in (-\epsilon, 0)$.

Equation (7.1) gives a canonical form for all functions J^μ_α that meet the necessary and sufficient condition (5.6) for $[g_{\alpha\beta,\gamma}] = 0$. The essence of this canonical form is that the jumps in derivatives across the shock waves have been taken out of the functions Φ, Ω, Z, N in (7.4). However, for J^μ_α to be proper Jacobians that can be integrated to a coordinate system, we must use the free functions Φ, Ω, Z, N to meet the integrability condition (2.1). To finish the proof of theorem 7.1, we show that, as a consequence of (7.4) (that is, the free functions are C^1 regular at the shocks), the Jacobian determinant $\mathrm{Det} J^\mu_\alpha$ must vanish at the point of shock interaction, which then implies $\mathrm{Det}(g_{\alpha\beta}) = 0$.

Thus, using the canonical form (7.1) restricted to the shock curve and taking the determinant of the resulting J^μ_α leads directly to

$$\mathrm{Det}(J^\mu_\alpha \circ \gamma_i(t)) = (J^t_0 J^r_1 - J^t_1 J^r_0)|_{\gamma_i(t)} = \phi_i(t)\zeta_i(t) - \nu_i(t)\omega_i(t). \tag{7.5}$$

Since J^μ_α is continuous, we obtain the same limit $t \to 0$ for $i = 1, 2$,

$$\lim_{t\to 0^+} \mathrm{Det}(J^\mu_\alpha \circ \gamma_i(t)) = \phi_i(0)\zeta_i(0) - \nu_i(0)\omega_i(0) = \phi_0\zeta_0 - \nu_0\omega_0. \tag{7.6}$$

Therefore, the final step in the proof of theorem 7.1 is the following lemma.

Lemma 7.3. *Let $p \in \mathcal{N}$ be a point of regular shock wave interaction in SSC in the sense of definition 3.1. Then if the integrability condition*

$$J^\mu_{\alpha,\beta} = J^\mu_{\beta,\alpha} \tag{7.7}$$

holds in $\mathcal{N} \cap \mathbb{R}^2_-$ for the functions J^μ_α defined in (7.1) (so that Φ, Ω, N and Z satisfy (7.4)), then

$$\frac{1}{4B}\left(\frac{\dot{x}_1\dot{x}_2}{A} + \frac{1}{B}\right)[B_r]_1[B_r]_2(\dot{x}_1 - \dot{x}_2)(\phi_0\zeta_0 - \nu_0\omega_0) = 0. \tag{7.8}$$

Proof. Substituting the J^μ_α in (7.1) into (7.7) gives equations (6.6), (6.7) except that we now sum over two shock curves instead of one. The difference is the appearance of additional mixed terms in the coefficients f and h of the discontinuous terms multiplying the Heaviside function $H(X)$. The proof is accomplished by showing that, unlike f and g in (6.6), (6.7), these mixed terms do not vanish by the jump conditions for the Einstein equations alone. Finally, a lengthy calculation to evaluate the limit $t \to 0$ demonstrates that imposing the condition that these additional mixed terms should vanish, which is necessary for (7.4) to hold, implies the final equation (7.8). (See Reintjes 2011 for details.)

To finish the proof of theorem 7.1, observe that the first three terms in (7.8) are non-zero by our assumption that shock curves are non-null, and have distinct speeds at $t = 0$. Thus, (7.8) implies

$$\mathrm{Det}\, J^\mu_\alpha(p) = (\phi_0\zeta_0 - \nu_0\omega_0) = 0, \tag{7.9}$$

as claimed. ■

In summary, we remark that the derivatives of J^μ_α are uniquely solvable in condition (5.6), and in principle there are enough free functions in Φ, Ω, Z, N in the canonical form of lemma 7.1, to arrange for the discontinuous term in the integrability condition to vanish, as in lemma 6.1 of Israel's theorem. But taking the limit to the point p of shock wave interaction, the condition (7.4), expressing that $[g_{\alpha\beta,\gamma}]$ vanishes at shocks, has the effect of freezing out all the freedom in

Φ, Ω, Z, N, thereby forcing condition (7.9), implying that the determinant of the Jacobian must vanish at p. The answer was not apparent until the very last step, and thus we find the result quite remarkable and surprising.

8. The loss of locally inertial frames

Definition 8.1. We call x^j locally inertial at p if the metric g_{ij} in coordinates x^j satisfies

— $g_{ij}(p) = \eta_{ij}$, $\eta_{ij} = \text{diag}(-1, 1, 1, 1)$,
— $g_{ij,l}(p) = 0$ for all $i, j, l \in \{0, \ldots, 3\}$,
— $g_{ij,kl}$ are bounded in every compact neighbourhood of p.

This condition ensures that the physical equations in curved space–time differ from flat by only *gravitational effects*, i.e. effects that are second order in the metric derivatives. In most of the literature on GR, the gravitational metric is assumed to be at least $C^{1,1}$ (cf. Hawking & Ellis 1973), which then directly implies condition (3) of definition 8.1.

By theorem 1.1, there exist second-order derivatives of the metric which are unbounded in every neighbourhood of p. Therefore, the following corollary is a straightforward consequence of theorem 1.1.

Corollary 8.2. *Let p be a point of regular shock wave interaction in SSC in the sense of definition 3.1. Then there does not exist a $C^{1,1}$ coordinate transformation such that the resulting metric g_{ij} is locally Minkowskian around p.*

9. Conclusion

We conclude that the essential $C^{0,1}$ singularities in the gravitational metric at points of shock wave interaction, where the pointwise a.e. T is supnorm-bounded and satisfies $Div\ T = 0$ weakly, are a new kind of singularity in GR created without the presence of delta function singularities in the sources. We name them *regularity singularities*, and maintain that such singularities in perfect fluids are fundamental to the mathematical theory of GR shock waves, and the partial differential equations that describe them.

Because the gravitational metric tensor is not locally inertial at points of shock wave interaction, it begs the question as to whether there are general relativistic gravitational effects at points of shock wave interaction that cannot be predicted from the compressible Euler equations in special relativity alone. At a regularity singularity, the unbounded second derivatives in g cancel out in the curvature tensor G, but the curvature is not the only measurable effect of the gravitational field; so one would expect there to exist measurable general relativistic effects at points of shock wave interaction that are physical. Indeed, even if there are dissipativity terms, like those of the Navier–Stokes equations, which regularize the gravitational metric at points of shock wave interaction, our results assert that the steep gradients in the second derivatives of the metric tensor at small viscosity cannot be removed uniformly while keeping the metric determinant uniformly bounded away from zero. We thus wonder whether shock wave interactions might provide a physical regime where new general relativistic effects might be observed. Said differently, a regularity singularity is not hidden behind an event horizon; so it is a sort of counterexample to the cosmic censorship conjecture in the sense that it gives rise to unbounded second-order metric derivatives, which by themselves might yield physically measurable effects that resemble some effects of unbounded curvature.

The ideas and methods presented here are the creation of Moritz Reintjes. The detailed proofs are due to him and can be found in his doctoral thesis (Reintjes 2011), which was supervised by Blake Temple. Both authors were partially supported by NSF Grant, where the problem was first proposed.

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