Subluminality and damping of plane waves in causal dissipative relativistic fluid dynamics

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Abstract

In this paper and its companion [2], the authors introduce a causal hyperbolic modification of the Eckart-Landau-Weinberg dissipation tensor, for the dynamics of collision-dominated radiation. The resulting system of partial differential equations for relativistic fluid dynamics is shown to exclusively admit subluminal, decaying modes for any finite wavelength. This validates the new relativistic Navier-Stokes-Fourier type theory as both *causal* and *dissipative*.

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1 Introduction

It has been a longstanding open problem as to whether there exists a causal relativistic theory of dissipation that parallels the classical non-relativistic Navier-Stokes equations of fluid dynamics. Current theories of dissipation in relativity suffer from one of two deficits: either they are not causal, or they fail to provide shock profiles. In this paper and its companion [2], the authors introduce a modification of the relativistic Eckart-Landau-Weinberg (ELW) dissipation tensor that ultimately provides a *causal* relativistic theory of viscosity and heat conduction, and a corresponding relativistic version of Navier-Stokes with shock profiles, for the fluid dynamics of pure radiation.

In relativity, the local state of a barotropic[1] perfect fluid is completely characterized by its energy density ρ and its 4-velocity U, or equivalently, by a 4×4 tensor T composed from ρ and U. The spatio-temporal evolution of these quantities is given by the vanishing of the 4-divergence of this *perfect fluid stress-energy tensor* T,

$$\nabla \cdot T = 0. \tag{1.1}$$

This is a hyperbolic system of four first-order nonlinear partial differential equations giving the conservation laws for energy and momentum. In the case of *imperfect* fluids, the conventional ansatz[2] for the dynamics is obtained by modifying (1.1) as

$$\nabla \cdot (T + \Delta T) = 0, \qquad \Delta T = L(\partial U, \partial \theta), \tag{1.2}$$

where, in analogy with non-relativistic Navier-Stokes-Fourier theory, the *dissipation stress tensor* ΔT is a linear function L of velocity and temperature gradients. Eckart found that a certain well-motivated choice

$$\Delta T = \Delta T_{ELW} = L_{ELW}(\partial U, \partial \theta) \tag{1.3}$$

for the dissipation stress tensor leads to consistency with thermodynamics in the sense of everywhere positive entropy production [1]. The Eckart-Landau-Weinberg (ELW) tensor ΔT_{ELW} (cf. (1.11) below) has three free parameters χ, η, ζ that represent heat conductivity, shear viscosity and bulk viscosity, respectively. Unfortunately, ELW theory is not causal (cf. [7] and references therein). This follows essentially from the parabolic character of the second-order system of partial differential equations (1.2) that result when (1.3) is assumed, a characteristic feature of parabolicity being unboundedness of propagation speeds. Following Eckart's work, other more elaborate theories of dissipative relativistic fluid dynamics were proposed in which additional variables are introduced and the conservation laws (1.1) are supplemented with first order partial differential equations, sometimes hyperbolic, as in the work of Geroch and Lindblom [5], sometimes of unknown mathematical type, but particularly firmly anchored in kinetic theory, as most notably in the comprehensive framework set out

[1] The case when pressure is a function of the energy density alone.

[2] This ansatz was first successfully used by Eckart [1] and later adopted by Weinberg [13]. A different, ultimately equivalent ansatz was introduced by Landau [9].

by Israel and Stewart [8]. As with their non-relativistic counterparts [11], only weak shock waves typically have smooth profiles in these extended theories [12].

Since hyperbolicity is the most natural criterion for guaranteeing finite speed of propagation and well-posedness for systems of partial differential equations, our idea here is to accept the original Navier-Stokes type ansatz (1.2) and attempt to derive ΔT as a linear function in the temperature and velocity gradients in such a way that *the resulting second-order system of partial differential equations is hyperbolic.* In [2], we have shown that this is possible. More precisely, it can be done in such a way that (i) the principal part of the differential operator is causal, and (ii) profiles exist for shock waves of arbitrary strength. Assertions (i) and (ii) are Theorems 1 and 2 in [2]. As it is the fundamental hyperbolicity notion of Hughes, Kato and Marsden that we use, we also obtain there nonlinear wellposedness of our version of (1.2), as a direct corollary of [6].

Our derived dissipation stress tensor

$$\Delta T = \tilde{\Delta} T = \tilde{L}(\partial U, \partial \theta) \tag{1.4}$$

(cf. (1.12) below) is determined by the same three free parameters χ, η, ζ for heat conductivity, shear viscosity and bulk viscosity as ELW, but it is composed differently. Based on the results in this paper and in [2], we propose the second order system of four equations

$$\nabla \cdot (T + \tilde{\Delta} T) = 0, \tag{1.5}$$

as the proper, causal, relativistic counterpart of the classical Navier-Stokes-Fourier theory.

The purpose of the present paper consists in providing an independent justification of the new equations by demonstrating causality and dissipativity at the level of modes. As in [2], we consider the fluid model of collision-dominated radiation, in which energy density, pressure, and temperature are linked by the Stefan-Boltzmann law

$$\rho = 3p = a\theta^4, \tag{1.6}$$

with a the Stefan-Boltzmann constant, [13]. Differently from [2], we now focus on linear plane waves, the Fourier-Laplace modes associated with full linearizations of our proposed new PDE system (1.5). Here, *full* means that we do not restrict attention to the leading second order part of the equations, but include the first order *acoustic* part as well. We prove the subluminality and decay of all modes for this mixed-order combination. This is by no means an automatic consequence of the results in [2]. Through a careful analysis of the system's dispersion relation, we show:

Theorem 1. *Assume* (1.6)*, and assume* $\chi > 0, \eta > 0, \zeta \geq -\eta/3$. *Then each finite-wavelength Fourier-Laplace mode for the linearization of* (1.5) *about any constant state, travels at a speed strictly slower than the speed of light, and decays with time at a non-vanishing rate.*

Theorem 1, proven in Section 2, completes our justification of $\tilde{\Delta}T$ by establishing that the full system (1.5) is *causal* and *dissipative* in the proper technical sense of these words. In particular, this demonstrates our evolution equations have an irreversible arrow of time.

Written in the fluid's rest frame at any given spacetime point, our equations (1.5) reduce at that point to their simplest form

$$\begin{pmatrix} \chi(\frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2} - \Delta\theta) \\ \sigma\frac{1}{c^2}\frac{\partial^2 \mathbf{v}}{\partial t^2} - (\eta\nabla\cdot(\mathcal{S}\mathbf{v}) + \zeta\nabla(\nabla\cdot\mathbf{v})) \end{pmatrix} = R, \tag{1.7}$$

where the right hand side R contains no higher than first derivatives. On the left, we have written c instead of 1 for the speed of light c, to show that in the limit $c \to \infty$ one recovers the classical Navier-Stokes-Fourier dissipation.

Paper [2] and this article started from the observation that a covariant dissipation stress tensor that is linear in velocity and temperature gradients and treats the temporal derivatives at the same level as the spatial ones could, in principle, be of a very general form involving no less than seven parameters $\chi, \eta, \zeta, \mu, \kappa, \sigma, \omega$, (c.f. (1.12) below). Leaving χ, η, ζ free and adapting the *new* coefficients $\kappa, \mu, \sigma, \omega$ as

$$\kappa = \omega = -\chi, \qquad \sigma = -\mu = \frac{4}{3}\eta + \zeta, \tag{1.8}$$

yields (1.5) and induces both the above Theorem 1 and the results of [2]. It was indeed by looking for choices of the seven parameters $\chi, \eta, \zeta, \mu, \kappa, \sigma, \omega$ sufficient to guarantee subluminality and decay of Fourier modes that we first discovered the proposed dissipation tensor. In other words, the considerations reported in this paper preceded, temporally and logically, the progress made in [2].

The methods used in this paper and in [2] can be used for other fluids (cf. [4]). We have addressed the fluid model of pure radiation first because of its canonical nature, its simplicity and its importance. In particular, the equations of pure radiation apply to the radiation phase of the Standard Model of Cosmology, which lasts from very shortly after the Big Bang up until the time when radiation does not dominate matter any more, some thirty thousand years later, [29]. During the radiation phase, the frames of isotropy of the energy evolve like the particle paths of a perfect fluid with constant sound speed $s = c/\sqrt{3}$. One important motivation for carefully considering the mechanisms for viscosity and heat conduction in this fluid dynamical model is the crucial role that these mechanisms are regarded to have played in the later isotropy of the universe [10, 14]. A related, but different motivation consists in the fact that the compressible Euler equations associated with pure radiation are a highly nonlinear system of conservation laws, so that inevitably, shock waves form. The details of the dissipative mechanisms must be known in order to correctly determine the internal structure of these shock waves. Thus the fact that our new causal relativistic dissipation tensor for a fluid with equation of state (1.6) incorporates viscosity and heat conduction in a naturally unifying, genuinely covariant manner, with the property that all shock waves possess a unique smooth internal structure has both an indicative meaning for general fluids and a concrete physical interpretation in a central model of astrophysics.

We conclude the introduction by explicitly giving T and our proposed $\Delta T = \tilde{\Delta} T$ by their representation in the fluid's restframe. For the inviscid part this is

$$T|_0 = \begin{pmatrix} \rho & \frac{4}{3}\rho v^i \\ \frac{4}{3}\rho v^j & \frac{1}{3}\rho \delta^{ij} \end{pmatrix},$$

where $\mathbf{v} = (v^1, v^2, v^3)$ is the 3-velocity and $i, j = 1, 2, 3$. The general form of a covariant dissipation stress tensor that is linear in velocity and temperature gradients and treats the temporal derivatives at the same level as the spatial ones is

$$-\Delta T|_0 = \begin{pmatrix} \kappa\dot{\theta} + \sigma\nabla\cdot\mathbf{v} & \chi\frac{\partial\theta}{\partial x_j} + \mu\dot{v}^j \\ \chi\frac{\partial\theta}{\partial x_i} + \mu\dot{v}^i & \eta(\mathcal{S}\mathbf{v})^{ij} + (\zeta\nabla\cdot\mathbf{v} + \omega\dot{\theta})\delta^{ij} \end{pmatrix} \tag{1.9}$$

where

$$\mathcal{S}\mathbf{v} = D\mathbf{v} + (D\mathbf{v})^T - \frac{2}{3}\nabla\cdot\mathbf{v}\, I \quad \text{with } (D\mathbf{v})^{ij} = \frac{\partial v^i}{\partial x_j} \tag{1.10}$$

is twice the symmetric tracefree part of the velocity gradient $D\mathbf{v}$.
From (1.9), one recovers ELW theory by specifying $\kappa = \sigma = \omega = 0,\ \mu = \chi$:

$$-\Delta T_{ELW}|_0 = \begin{pmatrix} 0 & \chi(\frac{\partial\theta}{\partial x_j} + \theta\dot{v}^j) \\ \chi(\frac{\partial\theta}{\partial x_i} + \theta\dot{v}^i) & \eta(\mathcal{S}\mathbf{v})^{ij} + \zeta(\nabla\cdot\mathbf{v})\delta^{ij} \end{pmatrix}. \tag{1.11}$$

Our own proposal (1.8) yields

$$-\tilde{\Delta}T|_0 = \begin{pmatrix} -\chi\dot{\theta} + \sigma\nabla\cdot\mathbf{v} & \chi\frac{\partial\theta}{\partial x_j} - \sigma\dot{v}^j \\ \chi\frac{\partial\theta}{\partial x_i} - \sigma\dot{v}^i & \eta(\mathcal{S}\mathbf{v})^{ij} + (\zeta\nabla\cdot\mathbf{v} - \chi\dot{\theta})\delta^{ij} \end{pmatrix}, \quad \sigma = \frac{4}{3}\eta + \zeta. \tag{1.12}$$

The reader checks easily that the "separation"on the left hand side of (1.7) into one hyperbolic operator acting only on θ and another hyperbolic operator acting only on $\mathbf{v}$ is due to cancellations of mixed derivatives involving the terms accompanied by σ in the first row of (1.12) and the terms accompanied by χ in the second row. These terms thus do not influence the leading order part of the equations. But since χ and σ depend on the temperature, these terms do give rise to quadratic gradient terms in the nonlinear problem. and therefore would indeed, if our theory is correct, correspond to new physical effects.

2 Subluminality and damping

This section serves to prove Theorem 1. Written out, the linearized equations read

$$\begin{aligned} \left\{3\frac{\partial\theta}{\partial t} + \bar{\theta}\nabla\cdot\mathbf{v}\right\} + \left\{\hat{\chi}\left(\frac{\partial^2\theta}{\partial t^2} - \Delta\theta\right)\right\} = 0, \\ \left\{\frac{1}{\bar{\theta}}\nabla\theta + \frac{\partial\mathbf{v}}{\partial t}\right\} + \left\{\hat{\sigma}\frac{\partial^2\mathbf{v}}{\partial t^2} - (\hat{\eta}\,\nabla\cdot\mathcal{S}\mathbf{v} + \hat{\zeta}\,\nabla(\nabla\cdot\mathbf{v}))\right\} = 0 \end{aligned} \tag{2.1}$$

where $\bar{\theta}$ denotes the temperature at the constant state at which the linearization is taken, and $\hat{\sigma} = (4\hat{\eta}/3) + \hat{\zeta}$.

A Fourier-Laplace mode

$$\begin{pmatrix} \hat{\rho} \\ \hat{\mathbf{v}} \end{pmatrix} e^{\lambda t + i\boldsymbol{\xi}\cdot\boldsymbol{x}} = \begin{pmatrix} \hat{\rho} \\ \hat{\mathbf{v}} \end{pmatrix} e^{Re\{\lambda\}t} e^{i(Im\{\lambda\}t + \boldsymbol{\xi}\cdot\boldsymbol{x})}, \quad (\lambda, \boldsymbol{\xi}) \in \mathbb{C} \times \mathbb{R}^3 \tag{2.2}$$

solving (2.1) is called *dissipative* if $Re\{\lambda\} \leq 0$; it is called *strictly dissipative* if $Re\{\lambda\} < 0$ and *dissipation-free* or *neutral* if $Re\{\lambda\} = 0$. The speed of a mode is given by

$$s = -\frac{Im\{\lambda\}}{|\boldsymbol{\xi}|};$$

it is called *subluminal* if $s^2 < c^2$; it is called *luminal* if $s^2 = c^2$.

The existence of a Fourier-Laplace mode (2.2) is equivalent to the dispersion relation

$$\det M(\lambda, |\boldsymbol{\xi}|) = 0 \tag{2.3}$$

with

$$M(\lambda, \boldsymbol{\xi}) = \begin{pmatrix} 3\lambda & i\boldsymbol{\xi}^\top \\ i\boldsymbol{\xi} & \lambda I \end{pmatrix} + \begin{pmatrix} \hat{\chi}(\lambda^2 + |\boldsymbol{\xi}|^2) & 0 \\ 0 & N(\lambda, \boldsymbol{\xi}) \end{pmatrix}. \tag{2.4}$$

with

$$N(\lambda, \boldsymbol{\xi}) = \left(\hat{\sigma}\lambda^2 + \hat{\eta}|\boldsymbol{\xi}|^2\right) I + \left(\hat{\zeta} + \frac{1}{3}\hat{\eta}\right) \boldsymbol{\xi}\boldsymbol{\xi}^\top. \tag{2.5}$$

Note that, remarkably, the dispersion relation does not depend on the base state.

For any fixed $\boldsymbol{\xi}$, the left hand side of (2.3) is a polynomial of degree 8 in λ whose roots $\lambda_i(\xi)$, $i = 1, ..., 8$ are continuous functions we refer to as *characteristic rates*, which determine subluminality and dissipativity. For convenience, we set $\hat{\chi} = 1$, and to simplify notation we now write η, σ instead of $\hat{\eta}, \hat{\sigma}$. (The former can be achieved via a uniform scaling of space and time variables.)

Lemma 1. *The dispersion relation* (2.3) *factors as*

$$\Pi_L^\sigma(\lambda, |\boldsymbol{\xi}|)(\Pi_T^{\eta;\sigma}(\lambda, |\boldsymbol{\xi}|))^2 = 0$$

and thus decomposes into

$$0 = \Pi_L^\sigma(\lambda, |\boldsymbol{\xi}|) = (3\lambda + \lambda^2 + |\boldsymbol{\xi}|^2)(\lambda + \sigma(\lambda^2 + |\boldsymbol{\xi}|^2)) + |\boldsymbol{\xi}|^2 \tag{2.6}$$

and

$$0 = \Pi_T^{\sigma;\eta}(\lambda, |\boldsymbol{\xi}|) = \lambda + \sigma\lambda^2 + \eta|\boldsymbol{\xi}|^2. \tag{2.7}$$

For any $\boldsymbol{\xi} \in \mathbb{R}^3 \setminus \{0\}$, we decompose

$$\mathbb{C} \times \mathbb{C}^3 = L(\boldsymbol{\xi}) \oplus L^{\perp}(\boldsymbol{\xi}) \quad \text{with} \quad L(\boldsymbol{\xi}) \equiv \mathbb{C} \times \mathbb{C}\boldsymbol{\xi}, \quad L^{\perp}(\boldsymbol{\xi}) \equiv \{0\} \times \{\boldsymbol{\xi}\}^{\perp}.$$

A mode (2.2) is called longitudinal if $(\hat{\rho}, \hat{\mathbf{v}}) \in L(\boldsymbol{\xi})$; it is called transverse if $(\hat{\rho}, \hat{\mathbf{v}}) \in L^{\perp}(\boldsymbol{\xi})$. Relations (2.6),(2.7) are called the longitudinal and transverse dispersion relations, respectively.

Proof of Lemma 1: If $\boldsymbol{\xi} = 0$, this is immediate. Assume then that $\boldsymbol{\xi} \neq 0$. The restrictions of $M(\lambda, \boldsymbol{\xi})$ to its invariant spaces $L(\boldsymbol{\xi})$ and $L^{\perp}(\boldsymbol{\xi})$ are given by the 2×2 matrices

$$\begin{pmatrix} 3\lambda + \lambda^2 + |\boldsymbol{\xi}|^2 & i|\boldsymbol{\xi}| \\ i|\boldsymbol{\xi}| & \lambda + \sigma(\lambda^2 + |\boldsymbol{\xi}|^2) \end{pmatrix}$$

and

$$\begin{pmatrix} \lambda + \sigma\lambda^2 + \eta|\boldsymbol{\xi}|^2 & 0 \\ 0 & \lambda + \sigma\lambda^2 + \eta|\boldsymbol{\xi}|^2 \end{pmatrix},$$

respectively. To confirm this, let

$$\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix},$$

where R is a 3×3 rotation taking $R\boldsymbol{\xi} = (\xi, 0, 0)$ with $\xi = |\boldsymbol{\xi}| > 0$, and note that $\mathbf{R}M(\lambda, \xi)\mathbf{R}^{-1}$ is

$$\begin{pmatrix} 3\lambda & i\xi & 0 & 0 \\ i\xi & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} \lambda^2 + \xi^2 & 0 & 0 & 0 \\ 0 & \sigma(\lambda^2 + \xi^2) & 0 & 0 \\ 0 & 0 & \sigma\lambda + \eta\xi^2 & 0 \\ 0 & 0 & 0 & \sigma\lambda + \eta\xi^2 \end{pmatrix}. \quad \square$$

We start by showing that all transverse modes are strictly dissipative and subluminal. The letter ξ continues to denote $|\boldsymbol{\xi}|$.

Lemma 2. *If $\Pi_T^{\eta;\sigma}(\lambda, \xi) = 0$ for some $\xi > 0$, then $Re\{\lambda\} < 0$ and $|Im\{\lambda\}| < \xi$.*

Proof: In that case,

$$\lambda = \frac{1}{2\sigma}(-1 \pm \sqrt{1 - 4\sigma\eta\xi^2}),$$

from which $\mathrm{Re}\{\lambda\} < 0$ is obvious, and

$$(\mathrm{Im}\{\lambda\}/\xi)^2 < \eta/\sigma \leq 1.$$

The next two lemmas state that no longitudinal mode can be luminal or neutral.

Lemma 3. *For any $\sigma > 0$ and $\xi > 0$, $\Pi_L^{\sigma}(\alpha \pm i\xi, \xi) \neq 0$ for all $\alpha \in \mathbb{R}$.*

Lemma 4. *For any $\sigma > 0$ and $\xi > 0$, $\Pi_L^{\sigma}(i\beta, \xi) \neq 0$ for all $\beta \in \mathbb{R}$.*

Proof of Lemma 3: As $\Pi_L^\sigma(.,\xi)$ has real coefficients, it suffices to show that

$$\Pi_L^\sigma(\alpha+i\xi,\xi)=0 \tag{2.8}$$

is not possible. Substituting $\alpha+i\xi$ for λ in (2.6) and multiplying out gives

$$\begin{aligned}\Pi_L^\sigma(\alpha+i\xi,\xi) &= \sigma(\alpha+i\xi)^4+(1+3\sigma)(\alpha+i\xi)^3+(3+2\sigma\xi^2)(\alpha+i\xi)^2 \\ &\qquad +(1+3\sigma)\xi^2(\alpha+i\xi)+\sigma\xi^4+\xi^2 \\ &= \sigma(\alpha^4+4\alpha^3 i\xi-6\alpha^2\xi^2-4\alpha i\xi^3+\xi^4) \\ &\qquad +(1+3\sigma)(\alpha^3+3i\alpha^2\xi-3\alpha^2\xi-3\alpha\xi^2-i\xi^3) \\ &\qquad +(3+2\sigma\xi^2)(\alpha^2+2i\alpha\xi-\xi^2)+(1+3\sigma)\xi^2(\alpha+i\xi) \\ &\qquad +\sigma\xi^4+\xi^2 \\ &= \{\cdot\}_{Re}+i\,\{\cdot\}_{Im}\end{aligned} \tag{2.9}$$

where

$$\{\cdot\}_{Im} = \left\{(4\sigma\alpha^3\xi-4\sigma\alpha\xi^3)+(1+3\sigma)3\alpha^2\xi+(3+2\sigma\xi^2)2\alpha\xi\right\}_{Im} \tag{2.10}$$

$$\begin{aligned}\{\cdot\}_{Re} &= \big\{\sigma(\alpha^4-6\alpha^2\xi^2+\xi^4)+(1+3\sigma)(\alpha^3-3\alpha\xi^2) \\ &\qquad +(3+2\sigma\xi^2)(\alpha^2-\xi^2)+(1+3\sigma)\xi^2\alpha+\sigma\xi^4+\xi^2\big\}_{Re}.\end{aligned} \tag{2.11}$$

Noticing the ξ^3 term cancels in (2.11) and the ξ^4 term cancels in (2.10) we obtain after simplification

$$\{\cdot\}_{Im} = \alpha\xi(4\sigma\alpha^2+(1+3\sigma)3\alpha+6). \tag{2.12}$$

$$\{\cdot\}_{Re} = -\xi^2\left(4\sigma\alpha^2+2(1+3\sigma)\alpha+2\right)+\alpha^2\left(\sigma\alpha^2+(1+3\sigma)\alpha+3\right). \tag{2.13}$$

Thus, (2.8) is equivalent to $\{\cdot\}_{Im}=0$ and $\{\cdot\}_{Re}=0$. Setting (2.12) equal to zero leads to the condition

$$4\sigma\alpha^2+(1+3\sigma)3\alpha+6=0. \tag{2.14}$$

Setting (2.13) equal to zero and using (2.14) gives

$$4\xi^2((1+3\sigma)\alpha+4)+\alpha^2((1+3\sigma)\alpha+6)=0. \tag{2.15}$$

Letting

$$\gamma=(1+3\sigma)\alpha, \quad \delta=\frac{4}{9}\frac{3\sigma}{(1+3\sigma)^2}$$

and observing that

$$\max_{\sigma>0} \frac{3\sigma}{(1+3\sigma)^2} = \frac{1}{4},$$

we see that (2.8) is equivalent to the existence of $\gamma \in \mathbb{R}, \delta \in (0, \frac{1}{9}]$ such that

$$p_\delta(\gamma) \quad \equiv \quad \delta\gamma^2 + \gamma + 2 = 0 \quad \text{and} \quad -6 < \gamma < -4. \tag{2.16}$$

However, as

$$p_\delta(-4) = 16\delta - 2 \leq \frac{16}{9} - 2 < 0,$$

and

$$p_\delta(-6) = 36\delta - 4 \leq \frac{36}{9} - 4 = 0,$$

(2.16), and thus (2.8), is impossible. □

Proof of Lemma 4: Substituting $i\beta$ for λ in (2.6) and multiplying out gives

$$\begin{aligned} \Pi_L^\sigma(i\beta,\xi) &= \sigma\beta^4 - (1+3\sigma)i\beta^3 - (3+2\sigma\xi^2)\beta^2 + (1+3\sigma)\xi^2 i\beta + \sigma\xi^4 + \xi^2 \\ &= \left\{\sigma\beta^4 - (3+2\sigma\xi^2)\beta^2 + (\sigma\xi^4+\xi^2)\right\}_I + i\left\{-(1+3\sigma)\beta^2 + (1+3\sigma)\xi^2\right\}_{Im}. \end{aligned} \tag{2.17}$$

Now $\Pi_L^\sigma(i\beta,\xi) = 0$ requires $\{\cdot\}_{Re} = \{\cdot\}_{Im} = 0$, and we see $\{\cdot\}_{Im} = 0$ if and only if $\beta^2 = \xi^2$. Using this in $\{\cdot\}_{Re}$ gives

$$\{\cdot\}_{Re} = \sigma\beta^4 - (3+2\sigma\beta^2)\beta^2 + \sigma\xi^4 + \beta^2 = -2\beta^2 \neq 0, \tag{2.18}$$

because $\beta^2 = \xi^2 \neq 0$. Thus, $\Pi_L^\sigma(i\beta,\xi) = 0$ is impossible. □

The next two lemmas state that at least for certain valuse of the wave number ξ and the parameter σ, all longitudinal modes are subluminal and strictly dissipative.

Lemma 5. *There exist $\sigma > 0$ and $\xi > 0$ such that $\Pi_L^\sigma(\lambda,\xi) = 0$ implies $|Im\{\lambda\}| < \xi$.*

Lemma 6. *There exist $\sigma > 0$ and $\xi > 0$ such that $\Pi_L^\sigma(\lambda,\xi) = 0$ implies $Re\{\lambda\} < 0$.*

Proof of Lemma 5: Fix $\sigma = 1$. As

$$\Pi_L^1(\lambda,\xi) = \lambda^4 + 4\lambda^3 + (3+2\xi^2)\lambda^2 + 4\xi^2\lambda + \xi^4 + \xi^2, \tag{2.19}$$

the condition

$$\Pi_L^1(\lambda,\xi) = 0 \tag{2.20}$$

reduces for $\xi = 0$ to

$$0 = \lambda^4 + 4\lambda^3 + 3\lambda^2 = \lambda^2(\lambda^2 + 4\lambda + 3)$$

with roots

$$\lambda_{1,2}^0 = 0, \quad \lambda_3^0 = -1, \quad \lambda_4^0 = -3. \tag{2.21}$$

For sufficiently small $\xi \geq 0$, the latter two perturb smoothly as simple real roots

$$\lambda_3(\xi),\ \lambda_4(\xi)\ <\ 0. \tag{2.22}$$

To understand the perturbation behaviour of the double root $\lambda^0_{1,2}$, note that for $\xi > 0$, a number λ solves (2.20) if and only if

$$\hat{\lambda} \equiv \lambda/\xi \tag{2.23}$$

solves

$$0 = \hat{\Pi}(\hat{\lambda}, \xi) \equiv \hat{\lambda}^4\xi^2 + 4\hat{\lambda}^3\xi + (3 + 2\xi^2)\hat{\lambda}^2 + 4\hat{\lambda}\xi + \xi^2 + 1. \tag{2.24}$$

For $\xi = 0$, equation (2.24) has the two roots

$$\hat{\lambda}^0_{1,2} = \pm\frac{i}{\sqrt{3}}. \tag{2.25}$$

Since

$$\frac{\partial\hat{\Pi}}{\partial\hat{\lambda}}(\hat{\lambda}^0_j, 0) = 6\hat{\lambda}^0_j \neq 0, \quad j = 1, 2,$$

they perturb smoothly as simple zeros $\hat{\lambda}^0_j(\xi)$ for small $\xi \geq 0$. As

$$\frac{\partial\hat{\Pi}}{\partial\xi}(\hat{\lambda}^0_j, 0) = \frac{8}{3}\hat{\lambda}^0_j, \quad j = 1, 2,$$

we find

$$(\hat{\lambda}^0_j)'(0) = \frac{\partial\hat{\Pi}}{\partial\xi}(\hat{\lambda}^0_j, 0)\ /\ \frac{\partial\hat{\Pi}}{\partial\hat{\lambda}}(\hat{\lambda}^0_j, 0) = -\frac{4}{9} < 0, \quad j = 1, 2,$$

and thus

$$\mathrm{Re}(\hat{\lambda}_{1,2}(\xi)) < 0 \quad \text{for small } \xi > 0. \tag{2.26}$$

Undoing the scaling (2.23), we find two smooth continuations

$$\lambda_{1,2}(\xi) = \xi\hat{\lambda}_{1,2}(\xi)$$

of the double root $\lambda^0_{1,2}$, with

$$\mathrm{Re}(\lambda_{1,2}(\xi)) < 0 \quad \text{for small } \xi > 0. \tag{2.27}$$

Inequalities (2.22), (2.27) imply the assertion. □

Proof of Lemma 6: Keep $\sigma = 1$ and consider the four complex rates $\lambda_j(\xi), j = 1, 2, 3, 4$ established for small $\xi > 0$ in the last proof. The corresponding speeds

$$s_j(\xi) = -\frac{\mathrm{Im}\{\lambda_j\}}{\xi}, \quad j = 1, 2, 3, 4,$$

have limits

$$s_{1,2}(0) = \pm\frac{1}{\sqrt{3}}, \quad s_{3,4}(0) = 0.$$

This implies

$$s_j^2(\xi) < 1$$

for small $\xi > 0$. □

As the reader will have noticed, the rates $\lambda_{1,2}(\xi)$ correspond, in the large-wavelength limit $\xi \to 0$, to pure acoustics.

Proof of Theorem 1: Consider the simply connected parameter regime

$$\Omega \equiv \{(\sigma, \eta, \xi) \in (0, \infty)^3 : \eta \le \sigma\},$$

and on it the property

$$P(\sigma, \eta, \xi) : \text{ for all } \lambda \in \mathbb{C}, \ \Pi_L^\sigma(\lambda, \xi)\Pi_T^{\sigma,\eta}(\lambda, \xi) = 0 \text{ implies } Re\,\{\lambda\} < 0 \ \text{ and } |Im\,\{\lambda\}| < \xi.$$

Lemmas 2 through 6 together with the continuous dependence of the solution set

$$\Lambda^{\sigma,\eta} \equiv \{\lambda \in \mathbb{C} : \Pi_L^\sigma(\lambda, \xi)\Pi_T^{\sigma,\eta}(\lambda, \xi) = 0\}$$

on (σ, η), imply that the set

$$\tilde{\Omega} \equiv \{(\sigma, \eta, \xi) \in \Omega : P(\sigma, \eta, \xi) \text{ holds}\}$$

is actually identical with Ω. Now, Lemma 1 yields that for any $\boldsymbol{\xi} \neq 0$ and $\lambda \in \mathbb{C}$, (2.3) implies

$$Re\,\{\lambda\} < 0 \ \text{ and } |Im\,\{\lambda\}| < |\boldsymbol{\xi}|.$$

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