

# Causal dissipation and shock profiles in the relativistic fluid dynamics of pure radiation

Heinrich Freistühler and Blake Temple

*Proc. R. Soc. A* 2014 **470**, 20140055, published 19 March 2014

---

## References

**This article cites 18 articles**

<http://rspa.royalsocietypublishing.org/content/470/2166/20140055.full.html#ref-list-1>

## Subject collections

Articles on similar topics can be found in the following collections

[applied mathematics](#) (328 articles)  
[differential equations](#) (102 articles)  
[mathematical physics](#) (156 articles)

## Email alerting service

Receive free email alerts when new articles cite this article - sign up in the box at the top right-hand corner of the article or click [here](#)

## Research



**Cite this article:** Freistühler H, Temple B. 2014 Causal dissipation and shock profiles in the relativistic fluid dynamics of pure radiation. *Proc. R. Soc. A* **470**: 20140055. <http://dx.doi.org/10.1098/rspa.2014.0055>

Received: 20 January 2014

Accepted: 17 February 2014

### Subject Areas:

applied mathematics, mathematical physics, differential equations

### Keywords:

causal, dissipation, relativistic, Navier–Stokes, relativistic viscosity, pure radiation

### Authors for correspondence:

Heinrich Freistühler

e-mail: [heinrich.freistuehler@uni-konstanz.de](mailto:heinrich.freistuehler@uni-konstanz.de)

Blake Temple

e-mail: [temple@math.ucdavis.edu](mailto:temple@math.ucdavis.edu)

# Causal dissipation and shock profiles in the relativistic fluid dynamics of pure radiation

Heinrich Freistühler<sup>1</sup> and Blake Temple<sup>2</sup>

<sup>1</sup>Department of Mathematics and Statistics, University of Konstanz, 78457 Konstanz, Germany

<sup>2</sup>Department of Mathematics, University of California, Davis, CA 95616, USA

Current theories of dissipation in the relativistic regime suffer from one of two deficits: either their dissipation is not causal or no profiles for strong shock waves exist. This paper proposes a relativistic Navier–Stokes–Fourier-type viscosity and heat conduction tensor such that the resulting second-order system of partial differential equations for the fluid dynamics of pure radiation is symmetric hyperbolic. This system has causal dissipation as well as the property that all shock waves of arbitrary strength have smooth profiles. Entropy production is positive both on gradients near those of solutions to the dissipation-free equations and on gradients of shock profiles. This shows that the new dissipation stress tensor complies to leading order with the principles of thermodynamics. Whether higher order modifications of the ansatz are required to obtain full compatibility with the second law far from the zero-dissipation equilibrium is left to further investigations. The system has exactly three *a priori* free parameters  $\chi$ ,  $\eta$ ,  $\zeta$ , corresponding physically to heat conductivity, shear viscosity and bulk viscosity. If the bulk viscosity is zero (as is stated in the literature) and the total stress–energy tensor is trace free, the entire viscosity and heat conduction tensor is determined to within a constant factor.

## 1. Introduction

How to incorporate finite speed of propagation into theories of dissipation is a central theme of Applied Mathematics. In particular, the second-order terms that model dissipation in the classical Navier–Stokes–Fourier (NSF) equations make them parabolic, which leads to unbounded speeds in the limit of unbounded

wavenumbers. The most natural context in which to address this issue is Relativity, where causality is a principle of physics, meaning that no signals propagate faster than light. The purpose of this paper is to prove that a causal relativistic version of the NSF equations exists for the fluid dynamics of pure radiation and to derive these equations from first principles. Different from other proposals, we do not start from thermodynamics but from the requirements of hyperbolicity, causality *and* shock structure, and then demonstrate compatibility, to significant order, of the evolution equations with the second law of thermodynamics. A crucial ingredient of our argument at a technical level is the use of the Hughes–Kato–Marsden notion of symmetric hyperbolicity for second-order systems.

It has generally been believed for some time now that a causal version of the classical NSF equations that describes viscosity and heat conduction in Relativity does not exist. As is well known, the standard derivation of the NSF equations is based on augmenting the compressible Euler equations by dissipation terms linear in velocity and temperature gradients and consistent with isotropy and then imposing consistency with the second law of thermodynamics by requiring that entropy production has the correct sign on all gradients. This then determines the dissipation tensor to within three free parameters, namely the coefficients of heat conductivity and shear and bulk viscosity, the three dissipation parameters of the NSF equations (cf. [1]). When this argument is generalized to Relativity, one is led directly to the well-known Eckart–Landau–Weinberg (ELW) viscosity and heat conduction tensor, first derived by Eckart [2], with equivalent versions derived by Landau & Lifshitz [1] and Weinberg [3]. Like the classical NSF tensor, the ELW dissipation tensor again is linear in velocity and temperature gradients and contains three free parameters corresponding to heat conductivity and shear and bulk viscosity. Generally regarded as the relativistic version of NSF, the ELW theory is the starting point for the relativistic theory of dissipation. But like the classical NSF equations, the ELW equations are parabolic in nature, and hence they admit infinite speed of propagation. This was recognized early on as a flaw in the ELW theory, and the goal to repair the lack of causality in the ELW theory has stimulated significant further research, including deep and important work by Israel, Stewart, Geroch and Lindblom, among others, [4–6] and references therein. Most celebrated among them is the causal theory of relativistic dissipation developed by Israel and Stewart. Building on a relativistic version of Grad’s theory of moments, the Israel–Stewart Theory introduces additional state variables and augments the conservation laws of mass, momentum and energy with additional evolution equations. The net result is a theory of relativistic dissipation that is regarded as correct to leading order asymptotically near equilibrium. On the other hand, in [7], it was shown that the Israel–Stewart equations do not admit shock profiles for sufficiently strong shocks. The situation is thus similar to that with non-relativistic dissipative fluids, for which Grad Theory and its extensions [8,9] are particularly well justified near homogeneous states while the classical NSF model is known to also capture very nonlinear behaviour with steep gradients, such as shock waves [10].

Now it is a guiding principle of continuum physics that evolution equations should be *symmetric hyperbolic*. Based on this, we started from the idea to use the theory of the second-order symmetric hyperbolic systems to construct an NSF theory that would reconcile the requirements of causality and existence of profiles for shocks of arbitrary strength. On the other hand, all of the above-mentioned theories are based on the assumption that entropy production be positive on all gradients, including gradients far from the inviscid limit,<sup>1</sup> and early on we realized that we could not, simultaneously, also meet this assumption. However, our view is that this assumption is too stringent. Indeed, dissipative fluid dynamics is a perturbative theory, intended for small dissipation, the case when gradients are close to those that occur in the inviscid limit. That is, entropy production need not necessarily be positive on gradients far from this case. In this paper, we introduce a *general* covariant ansatz for a symmetric dissipation tensor which, like ELW, is linear in velocity and temperature gradients. We then impose the selection criterion that the resulting second-order system of equations with dissipation be symmetric hyperbolic in the sense

<sup>1</sup>By inviscid limit, we mean the limit of vanishing viscosity and vanishing heat conduction.

of Hughes, Kato and Marsden (HKM) [11]. In this most natural way, the equations are uniquely determined to within the three free parameters of NSF. Remarkably, we find that for the resulting dissipation tensor shocks of arbitrary strength *do* admit smooth profiles, and entropy production is positive on gradients near the inviscid limit, including those profiles.

Symmetric hyperbolicity is a condition that should hold for some choice of dependent variables and discovering the transformation to such variables is part of the challenge of verifying symmetric hyperbolicity for a given system of equations. For this purpose, we incorporate the classical Godunov variables [12] into our relativistic NSF setting. In theorem 2.1, we prove that, for the equations of pure radiation, imposing the condition that the equations with dissipation, written in Godunov variables, should be symmetric hyperbolic in the HKM sense in some Lorentz coordinate system, determines a unique causal dissipation tensor. That is, this selection criterion determines all coefficients in the ansatz in terms of the three free physical parameters of NSF, the heat conductivity  $\chi$ , the shear viscosity  $\eta$  and the bulk viscosity  $\zeta$ . The difference between the resulting equations and the ELW equations only involves terms that would be negligible near a classical limit but are significant for highly relativistic flows. Moreover, being tensorial, this dissipation operator is consistent with General Relativity. In theorem 2.2, we demonstrate that the resulting equations admit smooth profiles for all shock waves. Theorem 2.3 shows that entropy production goes the right way sufficiently close to the inviscid limit, including shock profiles. Finally, in theorem 2.4, we prove that all Fourier modes of the full, i.e. second plus first order, linearized equations move at speeds less than the speed of light, and decay in forward time, for all positive wavenumbers. This means that our system is causal and dissipative in the precise sense of these terms. The decay of all modes in the same direction provides an arrow of time, another expression of irreversibility. Taken together, these results demonstrate that, in the case of pure radiation, *a causal theory of dissipative fluid dynamics is possible as a hyperbolic analogue of NSF theory that is consistent for shock waves and the vanishing-dissipation limit.*<sup>2</sup>

The relation of our hyperbolic NSF theory to ELW theory is clarified in §3. The comparison with other frameworks, notably the Israel–Stewart theory, is the subject of ongoing investigations. Our theory is a second-order *hyperbolic regularization* of a first-order hyperbolic system of conservation laws. See [13] for a prototypical example of such regularization.

As with classical NSF, our theory is a continuum mechanical framework for dissipation, and as such does not determine the actual values of the free parameters  $\chi, \eta, \zeta$ . As usual, these must be determined from additional symmetries and/or physics on small scales. A particularly compelling choice of values considered in theorem 8.1 would induce implications for cosmology.

## 2. Statement of results

The relativistic Euler equations<sup>3</sup> take the covariant form

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = 0, \quad (2.1)$$

where  $T$  is the  $4 \times 4$  stress tensor

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + p\eta^{\alpha\beta}. \quad (2.2)$$

We write  $\rho, p, \theta$  and  $u^\alpha$  for energy density, pressure, temperature and velocity, respectively. It is with the Stefan–Boltzmann law

$$\rho = 3p = a\theta^4 \quad (2.3)$$

<sup>2</sup>We actually wonder whether these equations might not be valid also far from the inviscid limit, but this raises questions that must be postponed to future work.

<sup>3</sup>In general contexts, where the pressure  $p$  is a function of both energy  $\rho$  and particle number density  $n$ , the particle conservation equation  $\partial(nu^\beta)/\partial x^\beta = 0$  is needed to close system (2.1). We do not discuss this equation at all, as in the context of pure radiation, it uncouples, both in the inviscid case and in our proposed model with dissipation (2.4) and (2.5). (It is implicitly used in the derivation [3] of (7.1), however.)

that equations (2.1) provide the fluid dynamic representation of pure radiation.<sup>4</sup> We have chosen this particular fluid model because of its relative simplicity and its importance. This is, in particular, the setting for the radiation phase of the Standard Model of Cosmology, which lasts from very shortly after the Big Bang up until the time when radiation does not dominate matter anymore, some  $10^5$  years after the Big Bang [3]. During the radiation phase, the frames of isotropy of the energy evolve like the particle paths of a perfect fluid with constant sound speed  $s = c/\sqrt{3}$ , and thus one must wonder about the mechanisms for viscosity and heat conduction in this fluid dynamical model. These mechanisms may have played a crucial role regarding isotropy of the universe [14,15]. In addition, the compressible Euler equations are a highly nonlinear system of conservation laws, and so shock waves form. Thus, the details of the dissipative mechanisms must be known in order to correctly determine the internal structure of shock waves. While significant, both at the fundamental level of the Stefan–Boltzmann law and for the understanding of shock waves, the dissipation of pure radiation is difficult to measure directly, so its nature must be deduced. In this paper, we derive, from first principles, a new causal relativistic dissipation tensor for the fluid dynamics of pure radiation that incorporates viscosity and heat conduction in a naturally unifying, genuinely covariant manner and has the property that all shock waves possess a unique corresponding internal structure.

Motivated by the classical NSF description of non-relativistic fluid dynamics, the dissipative effects owing to a positive mean free path of radiation quanta are incorporated by adding a correction  $\Delta T^{\alpha\beta}$ , linear in velocity and temperature gradients, to the perfect fluid stress tensor  $T^{\alpha\beta}$ , thus modifying (2.1) to

$$\frac{\partial}{\partial x^\beta} \{T^{\alpha\beta} + \Delta T^{\alpha\beta}\} = 0. \quad (2.4)$$

The starting point of our argument is the observation that if such a dissipation stress tensor were assumed only to be symmetric and covariant, it could be given by<sup>5</sup>

$$-\Delta T^{\alpha\beta}|_0 = \begin{pmatrix} \kappa\dot{\theta} + \sigma\nabla \cdot \mathbf{v} & \chi \frac{\partial\theta}{\partial x_j} + \mu\dot{v}^j \\ \chi \frac{\partial\theta}{\partial x_i} + \mu\dot{v}^i & \eta(S\mathbf{v})^{ij} + (\zeta\nabla \cdot \mathbf{v} + \omega\dot{\theta})\delta^{ij} \end{pmatrix}, \quad (2.5)$$

where  $\mathbf{v} = (v_1, v_2, v_3)$  denotes 3-velocity and

$$S\mathbf{v} = D\mathbf{v} + (D\mathbf{v})^T - \frac{2}{3}\nabla \cdot \mathbf{v}I, \quad (D\mathbf{v})^{ij} = \frac{\partial v^i}{\partial x_j}, \quad (2.6)$$

and the coefficients  $\chi, \eta, \zeta, \kappa, \sigma, \omega, \mu$  are, at this stage, arbitrary functions of  $\theta$ . Equation (2.4) generalizes the ELW ansatz.

The idea is now to leave the three coefficients  $\chi, \eta, \zeta$ , which correspond to heat conduction, shear viscosity and bulk viscosity, free and determine  $\kappa, \sigma, \omega, \mu$  as functions of  $\chi, \eta, \zeta$ . The first purpose of the paper is to show that appropriate choices of  $\kappa, \sigma, \omega, \mu$  lead to consistency with both the requirement of causality and the theory of second-order hyperbolic systems.

Introducing the Godunov variable  $\psi^\alpha = \theta^{-1}u^\alpha$  [12,16], we view our dissipation stress tensor (2.5) as

$$-\Delta T^{\alpha\beta} = B^{\alpha\beta\gamma\delta}(\psi) \frac{\partial\psi_\gamma}{\partial x^\delta} \quad (2.7)$$

and write (2.4) as

$$B^{\alpha\beta\gamma\delta}(\psi) \frac{\partial^2\psi_\gamma}{\partial x^\beta \partial x^\delta} = R^\alpha(\psi, \partial\psi). \quad (2.8)$$

<sup>4</sup>For all what follows we set the Stefan–Boltzmann constant  $a$  to 1 by scaling.

<sup>5</sup>We use the symbol  $|_0$  to denote the representation of a tensor in the fluid's rest-frame.

Our main theorem is the following:

**Theorem 2.1.** Assume that the coefficients  $\chi, \eta, \zeta$  satisfy

$$\chi > 0, \quad \eta > 0 \quad \text{and} \quad \zeta \geq -\frac{1}{3}\eta. \quad (2.9)$$

Then, the second-order part of dissipative Euler equations (2.8),

$$B^{\alpha\beta\gamma\delta}(\psi) \frac{\partial^2 \psi_\gamma}{\partial x^\beta \partial x^\delta}, \quad (2.10)$$

is symmetric hyperbolic in the sense of Hughes–Kato–Marsden and causal if and only if

$$\kappa = \omega = -\chi \quad \text{and} \quad \sigma = -\mu = \frac{4}{3}\eta + \zeta. \quad (2.11)$$

In this case, equations (2.8) are well posed. If moreover  $\zeta = -\eta/3$  exactly, then (2.10) is even sharply causal.<sup>6</sup>

(See §4 for the precise definitions of *causal* and *sharply causal*.)

The second purpose of this note is to show that in the presence of appropriate dissipation, shock waves can be represented as smooth travelling waves. With  $x \equiv (x_\alpha)$ , a planar discontinuity

$$({}^0\rho, {}^0u)(x) = (\rho^\pm, u^\pm), \quad \pm x_\alpha N^\alpha > 0, \quad N_\alpha N^\alpha > 0 \quad (2.12)$$

is an inviscid shock wave if it satisfies the relativistic Rankine–Hugoniot conditions

$$[T^{\alpha\beta}]N_\beta = 0 \quad (2.13)$$

and is supersonic with respect to its upstream while subsonic with respect to its downstream state [17].<sup>7</sup> We show:

**Theorem 2.2.** Assume that the coefficients  $\chi, \eta, \zeta, \kappa, \sigma, \omega, \mu$  satisfy (2.9) and (2.11). Then, any inviscid shock wave has a dissipative structure, i.e. there exists a unique smooth shock profile

$$(\rho, u)(x) = (R, U)(x_\alpha N^\alpha) \quad \text{with} \quad (R, U)(\pm\infty) = (\rho^\pm, u^\pm) \quad (2.14)$$

solving (2.4).

Thirdly, we prove that entropy production goes the right way sufficiently close to the inviscid limit, including shock waves.

**Theorem 2.3.** Assume that the coefficients  $\chi, \eta, \zeta, \kappa, \sigma, \omega, \mu$  satisfy (2.9) and (2.11) and have the natural temperature dependence (cf. [3, p. 57])

$$\chi = \hat{\chi}\theta^3, \quad \eta = \hat{\eta}\theta^4 \quad \text{and} \quad \zeta = \hat{\zeta}\theta^4. \quad (2.15)$$

Then (i) The entropy production associated with (2.4) is strictly positive on all gradients  $(R'N, U'N)$  of shock profiles (2.14), and (ii) on all gradients of solutions to dissipation-free equations (2.1), entropy production is non-negative to leading order in the dissipation coefficients  $\chi, \eta, \zeta$  if and only if

$$\hat{\chi} + 3\hat{\zeta} - 2\hat{\eta} \geq 0. \quad (2.16)$$

Inequality (2.16) is well within the range of anticipated values of the dissipation parameters, cf. [3] and §8.

Before we developed the perspective taken in this paper, we came to similar conclusions by studying, instead of only the hyperbolicity of the second-order part, the dispersion relation of the full linearization of (2.4) that takes account also of the first-order terms. This has led to the following theorem that in the general case of assuming (2.9) and (2.11) (but not requiring (2.15) or (2.16)), all linear plane waves with non-vanishing wavenumbers travel at subluminal speeds and are damped in the sense of having strictly negative growth rates in time.

<sup>6</sup>We mention this case as an interesting limit, cf., however, §4.

<sup>7</sup>For results on inviscid shock waves in relativistic fluids we refer the reader to [18–20] and references therein.

**Theorem 2.4.** Let  $\tilde{\Delta}T$  denote (2.5) under the assumptions of (2.9) and (2.11). Then, each Fourier–Laplace mode for the linearization of (2.4) about any constant state travels at a speed strictly slower than the speed of light, and decays with time at a non-vanishing rate, for all wavenumbers  $\xi \neq 0$ .

Theorem 2.4 completes our justification of  $\tilde{\Delta}T$  by establishing that full system (3.2) is *causal* and *dissipative* in the proper technical sense of these words. In particular, this demonstrates that our evolution equations have an irreversible arrow of time.

We believe that theorems 2.1–2.4 of this paper<sup>8</sup> together provide a convincing demonstration of a causal hyperbolic analogue of NSF theory.

A comparison of our new theory of dissipation with the ELW theory, together with the general covariant expression of our dissipation tensor (theorem 3.1), is presented in §3. Theorems 2.1 and 2.2 are proved in §§4 and 5, theorem 2.4 is proved in §6 and theorem 2.3 is postponed until §7. In §8, theorem 8.1 introduces special choices of ratios among  $\chi, \eta, \zeta$  that seem particularly compelling.

The use of symmetric hyperbolic systems of first order is common in classical and relativistic continuum mechanics (cf. [6,9,16,21,22] and references therein). The second-order symmetric hyperbolic approach has been applied by Hughes, Kato and Marsden to the vacuum Einstein equations and to non-relativistic elasticity theory, both without dissipation [11]. As far as we know, this paper is the first to use symmetric hyperbolicity as a selection criterion to identify physically acceptable second-order differential operators that represent dissipation and to establish well posedness of the resulting system of PDEs.

The essence of our argument, however, lies in the general ansatz (3.6) for the dissipation stress tensor, which includes the terms with the ‘new’ coefficients  $\kappa, \sigma, \mu, \omega$ . Early on, Thomas [23] and Weinberg [15] did consider some terms of this sort, but only the ‘ $\mu$ -term’ appears in the classical ELW theory (cf. [3, p. 55]). In our above notation, the ELW theory corresponds to the case  $\kappa = \sigma = \omega = \chi - \mu = 0$ . For our choice,  $\kappa, \sigma, \mu, \omega$  are all determined via (2.11), and we consider the corresponding parts of (3.6) as relativistic corrections that express new higher order physical effects. Note that these terms are precisely the ones that are negligible near the classical limit.

The approaches taken in this paper enable analogous findings for other fluids. The authors are working this out in ongoing investigations.

### 3. Comparison with Eckart–Landau–Weinberg theory

Let

$$\Delta T = \tilde{\Delta}T = \tilde{L}(\partial U, \partial \theta) \quad (3.1)$$

denote our derived dissipation stress tensor (2.5) under assumptions (2.9) and (2.11), expressing that  $\tilde{L}$  is linear in velocity and temperature gradients. The tensor  $\tilde{\Delta}T$  is determined by the same three free parameters  $\chi, \eta$  and  $\zeta$  for heat conductivity, shear viscosity and bulk viscosity, respectively, as ELW, but it is composed differently, and we are proposing the second-order system of four equations

$$\nabla \cdot (T + \tilde{\Delta}T) = 0, \quad (3.2)$$

as the proper, causal, relativistic counterpart of the classical NSF theory. To compare with ELW theory, we now explicitly give  $T, \tilde{\Delta}T, \Delta T_{\text{ELW}}$  and the resulting equations  $\nabla \cdot (T + \tilde{\Delta}T) = 0$  and  $\nabla \cdot (T + \Delta T_{\text{ELW}}) = 0$  by their representation in the fluid’s rest frame.

In the particle rest frame, the inviscid part  $T$  reduces to

$$T|_0 = \begin{pmatrix} \rho & \frac{4}{3}\rho v^i \\ \frac{4}{3}\rho v^j & \frac{1}{3}\rho \delta^{ij} \end{pmatrix},$$

where  $\mathbf{v} = (v^1, v^2, v^3)$  is the 3-velocity and  $i, j = 1, 2, 3$ . Moreover, the general form of a covariant dissipation stress tensor that is linear in velocity and temperature gradients and treats the

<sup>8</sup>The paper here was preceded by preliminary unpublished results on a special class of tensors (2.5) (H. Freistühler & B. Temple 2011, unpublished note).

temporal derivatives at the same level as the spatial ones is given in the particle rest frame by (2.5) (cf. theorem 3.1). From (2.5), one recovers ELW theory by specifying  $\kappa = \sigma = \omega = 0$ ,  $\mu = \chi$ :

$$-\Delta T_{\text{ELW}|_0} = \begin{pmatrix} 0 & \chi \left( \frac{\partial \theta}{\partial x_j} + \theta v^j \right) \\ \chi \left( \frac{\partial \theta}{\partial x_i} + \theta v^i \right) & \eta (S\mathbf{v})^{ij} + \zeta (\nabla \cdot \mathbf{v}) \delta^{ij} \end{pmatrix}.$$

Our own proposal (2.11) yields

$$-\tilde{\Delta} T|_0 = \begin{pmatrix} -\chi \dot{\theta} + \sigma \nabla \cdot \mathbf{v} & \chi \frac{\partial \theta}{\partial x_j} - \sigma v^j \\ \chi \frac{\partial \theta}{\partial x_i} - \sigma v^i & \eta (S\mathbf{v})^{ij} + (\zeta \nabla \cdot \mathbf{v} - \chi \dot{\theta}) \delta^{ij} \end{pmatrix}, \quad \sigma = \frac{4}{3} \eta + \zeta. \quad (3.3)$$

Taking the divergence, and treating  $\chi, \eta, \zeta$  as arbitrary functions of the temperature, we conclude that, written in the fluid's rest frame at any given space-time point, the ELW equations reduce to

$$\begin{pmatrix} \chi \left( \Delta \theta + \frac{1}{c} \theta \frac{\partial^2}{\partial t \partial x^j} v^j \right) \\ -(\eta \nabla \cdot (S\mathbf{v}) + \zeta \nabla (\nabla \cdot \mathbf{v})) \end{pmatrix} = R_{\text{ELW}}, \quad (3.4)$$

while our equations (3.2) reduce at that point to their simplest form,

$$\begin{pmatrix} \chi \left( \frac{1}{c^2} \frac{\partial^2 \theta}{\partial t^2} - \Delta \theta \right) \\ \sigma \frac{1}{c^2} \frac{\partial^2 \mathbf{v}}{\partial t^2} - (\eta \nabla \cdot (S\mathbf{v}) + \zeta \nabla (\nabla \cdot \mathbf{v})) \end{pmatrix} = \tilde{R}, \quad (3.5)$$

where the right-hand sides  $R_{\text{ELW}}$  and  $\tilde{R}$  contain no higher than first derivatives. On the left, we have written  $c$  instead of 1 for the speed of light  $c$ , to show that in the limit  $c \rightarrow \infty$  one recovers the classical NSF dissipation. The reader can easily check that the 'separation' on the left-hand side of (3.5) into one hyperbolic operator acting only on  $\theta$  and another hyperbolic operator acting only on  $\mathbf{v}$  is due to cancellations of mixed derivatives involving the terms accompanied by  $\sigma$  in the first row of (3.3) and the terms accompanied by  $\chi$  in the second row. These terms thus do not influence the leading order part of the equations. But as  $\chi$  and  $\sigma$  depend on the temperature, these terms do give rise to quadratic gradient terms in the nonlinear problem, and therefore would indeed, if our theory is correct, correspond to new physical effects.

We end this section with the following theorem showing that the general dissipation tensor (2.5) (and hence  $\tilde{\Delta} T$ ) is fully covariant, and as such can be incorporated naturally into general relativity.

**Theorem 3.1.** *The tensor  $\Delta T$  given in the particle's rest frame by (2.5), takes the general covariant form*

$$-\Delta T^{\alpha\beta} \equiv u^\alpha u^\beta P + (H^{\alpha\gamma} u^\beta + H^{\beta\gamma} u^\alpha) Q_\gamma + H^{\alpha\beta} R + H^{\alpha\gamma} H^{\beta\delta} W_{\gamma\delta} \quad (3.6)$$

with

$$P = \kappa u^\gamma \frac{\partial \theta}{\partial x^\gamma} + \sigma \frac{\partial u^\gamma}{\partial x^\gamma}, \quad R = \omega u^\gamma \frac{\partial \theta}{\partial x^\gamma} + \zeta \frac{\partial u^\gamma}{\partial x^\gamma}$$

and

$$Q_\alpha \equiv \chi \frac{\partial \theta}{\partial x^\alpha} + \mu u^\beta \frac{\partial u_\alpha}{\partial x^\beta}, \quad W_{\alpha\beta} \equiv \eta \left( \frac{\partial u_\alpha}{\partial x^\beta} + \frac{\partial u_\beta}{\partial x^\alpha} - \frac{2}{3} \eta_{\alpha\beta} \frac{\partial u^\gamma}{\partial x^\gamma} \right),$$

where

$$H^{\alpha\beta} \equiv \eta^{\alpha\beta} + u^\alpha u^\beta.$$

*Proof.* The theorem is verified directly by expressing the above tensors in the fluid's rest frame. ■

## 4. Causal dissipation

Symmetric hyperbolicity, first introduced for first-order equations, is the most natural criterion to guarantee finite speed of propagation and well posedness for systems of partial differential equations. This property has indeed been regarded as a principle of continuum physics. We now apply this principle to (2.4) and (2.5). Remarkably, the notion of symmetric hyperbolicity for second-order systems introduced by Hughes *et al.* [11] turns out to be tailored for this purpose.

Building directly on [11], we call a tensorial differential operator

$$B^{\alpha\beta\gamma\delta}(\psi) \frac{\partial^2 \psi_\gamma}{\partial x^\beta \partial x^\delta} \quad (4.1)$$

symmetric hyperbolic in the sense of Hughes, Kato and Marsden ('HKM hyperbolic') if it satisfies the symmetry condition

$$\tilde{B}^{\alpha\beta\gamma\delta} = \tilde{B}^{\gamma\delta\alpha\beta}, \quad \text{with } \tilde{B}^{\alpha\beta\gamma\delta} \equiv \frac{1}{2}(B^{\alpha\beta\gamma\delta} + B^{\alpha\delta\gamma\beta}), \quad (4.2)$$

as well as the definiteness conditions

$$B^{\alpha\beta\gamma\delta} H_\beta H_\delta V_\alpha V_\gamma < 0 \quad \text{for all } V_\alpha \neq 0, \quad (4.3)$$

and

$$B^{\alpha\beta\gamma\delta} N_\beta N_\delta V_\alpha V_\gamma > 0 \quad \text{for all } V_\alpha \neq 0, \quad (4.4)$$

for

$$\text{some } H_\beta \text{ with } H_\beta H^\beta < 0 \text{ and all } N_\beta \neq 0 \text{ with } N_\beta H^\beta = 0. \quad (4.5)$$

In that case, we also call any covariant system (2.8) of partial differential equations HKM hyperbolic; such a system is symmetric hyperbolic in the sense of [11], when written in coordinates obtained via a Lorentz transform that makes  $H_\beta$  the direction of time.

We call an HKM hyperbolic operator (4.1) *causal* if (4.3) holds for all time-like  $H_\beta$ ; we call it *sharply causal* if (4.3) holds for all time-like  $H_\beta$  and (4.4) holds for all space-like  $N_\beta$ . Causality and sharp causality express the additional requirements that the propagation speeds of the second-order part are no larger than or are all identical to the speed of light, respectively. (Only) in the latter case, (2.8) is symmetric hyperbolic in the sense of [11] in all Lorentz frames.

**Lemma 4.1.** *For the dissipative Euler equations (2.8) with (2.5), symmetry condition (4.2) holds if*

$$\sigma = -\mu \quad \text{and} \quad \omega = -\chi. \quad (4.6)$$

*Proof.* Starting with covariant form (3.6), and expressing derivatives as

$$\frac{\partial \theta}{\partial x^\delta} = \theta^2 u^\gamma \frac{\partial \psi_\gamma}{\partial x^\delta}$$

and

$$\frac{\partial u^\sigma}{\partial x^\delta} = \theta H^{\sigma\gamma} \frac{\partial \psi_\gamma}{\partial x^\delta},$$

we compute

$$\begin{aligned} B^{\alpha\beta\gamma\delta} &= +u^\alpha u^\beta (\kappa \theta^2 u^\gamma u^\delta + \sigma \theta H^{\gamma\delta}) + H^{\alpha\beta} (\omega \theta^2 u^\gamma u^\delta + \zeta \theta H^{\gamma\delta}) + \chi \theta^2 (H^{\alpha\delta} u^\beta + H^{\beta\delta} u^\alpha) u^\gamma \\ &\quad + \mu \theta (H^{\alpha\gamma} u^\beta + H^{\beta\gamma} u^\alpha) u^\delta + \eta \theta (H^{\alpha\gamma} H^{\beta\delta} + H^{\alpha\delta} H^{\beta\gamma} - (\frac{2}{3}) H^{\alpha\beta} H^{\gamma\delta}) \\ &= \check{B}^{\alpha\beta\gamma\delta} + \hat{B}^{\alpha\beta\gamma\delta} \end{aligned}$$

with

$$\begin{aligned} \check{B}^{\alpha\beta\gamma\delta} &= ((\kappa + \omega + 2\chi)\theta^2 + (\sigma + 2\mu)\theta) u^\alpha u^\beta u^\gamma u^\delta + \chi \theta^2 u^\alpha u^\gamma \eta^{\beta\delta} + \mu \theta u^\beta u^\delta \eta^{\alpha\gamma} \\ &\quad + \eta \theta (H^{\alpha\gamma} H^{\beta\delta} + H^{\alpha\delta} H^{\beta\gamma} - (\frac{2}{3}) H^{\alpha\beta} H^{\gamma\delta}) + \zeta \theta H^{\alpha\beta} H^{\gamma\delta} \end{aligned}$$

and

$$\hat{B}^{\alpha\beta\gamma\delta} = \sigma \theta u^\alpha u^\beta \eta^{\gamma\delta} + \mu \theta u^\alpha u^\delta \eta^{\beta\gamma} + \omega \theta^2 u^\gamma u^\delta \eta^{\alpha\beta} + \chi \theta^2 u^\beta u^\gamma \eta^{\alpha\delta}.$$

We note that in any case

$$\check{B}^{\alpha\beta\gamma\delta} = \check{B}^{\gamma\delta\alpha\beta}.$$

If (4.6) holds, we also have

$$\hat{B}^{\alpha\beta\gamma\delta} + \hat{B}^{\alpha\delta\gamma\beta} = 0,$$

and thus

$$\tilde{B}^{\alpha\beta\gamma\delta} = \tilde{B}^{\gamma\delta\alpha\beta}.$$

■

**Lemma 4.2.** *Assume (2.9) and (2.11). Then the operator (4.1) is causal. It is sharply causal if and only if  $\zeta = -\eta/3$ .*

*Proof.* Writing, by virtue of (2.11),

$$\begin{aligned} \check{B}^{\alpha\beta\gamma\delta} &= \kappa\theta^2 u^\alpha u^\beta u^\gamma u^\delta + \chi\theta^2 u^\alpha u^\gamma H^{\beta\delta} + \mu\theta u^\beta u^\delta H^{\alpha\gamma} \\ &\quad + \eta\theta(H^{\alpha\gamma}H^{\beta\delta} + H^{\alpha\delta}H^{\beta\gamma} - (\frac{2}{3})H^{\alpha\beta}H^{\gamma\delta}) + \zeta\theta H^{\alpha\beta}H^{\gamma\delta}, \end{aligned}$$

and contracting with  $N$  twice, we find that the directional dissipation tensor

$${}^N B^{\alpha\gamma} \equiv B^{\alpha\beta\gamma\delta} N_\beta N_\delta = \check{B}^{\alpha\beta\gamma\delta} N_\beta N_\delta$$

has the rest-frame matrix representation

$${}^N B^{\alpha\gamma}|_0 = \begin{pmatrix} \theta^2 \chi N_\beta N^\beta & 0 \\ 0 & \theta(\eta N_\beta N^\beta \delta^{ij} + (\zeta + \frac{\eta}{3})(N^i N^j - N_0^2 \delta^{ij})) \end{pmatrix}. \quad (4.7)$$

Having eigenvalues

$$\theta^2 \chi N_\beta N^\beta =: \lambda_0, \quad \theta((\frac{4}{3})\eta + \zeta)N_\beta N^\beta =: \lambda_1, \quad \theta(-((\frac{4}{3})\eta + \zeta)N_0^2 + \eta N_i N^i), \quad (4.8)$$

this matrix is negative for all time-like  $N_\beta$ . It is positive for all space-like  $N_\beta$  if and only if  $\zeta = -\eta/3$ . ■

Lemmas 4.1 and 4.2 obviously imply theorem 2.1.

Lemmas 4.1 and 4.2 obviously imply that choices (2.9) and (2.11) of theorem 2.1 determine a symmetric hyperbolic system in the sense of HKM. Reversing the above argument, it is not difficult to prove the converse is true as well. The well posedness assertion follows immediately from the fundamental result of Hughes *et al.* [11].

## 5. Shock profiles

Remarkably, the symmetry and definiteness properties of  $B^{\alpha\beta\gamma\delta}$  are sufficient to show that all shock waves have profiles.

From (2.4) and (2.7), we find that the possible profile  $\Psi$  of a shock wave  $\psi(x) = \Psi(x_\alpha N^\alpha)$  is governed by the ODE

$$N_\beta N_\delta (B^{\alpha\beta\gamma\delta}(\Psi)\Psi'_\gamma)' = N_\beta (T^{\alpha\beta}(\Psi))' \quad (5.1)$$

or

$$N_\beta N_\delta B^{\alpha\beta\gamma\delta}(\Psi)\Psi'_\gamma = N_\beta T^{\alpha\beta}(\Psi) - q^\alpha, \quad (5.2)$$

with  $q^\alpha$  a constant of integration. Assume now without loss of generality that  $N^\beta = \delta^{\beta 1}$  and

$$q^\alpha = \psi^\alpha = 0 \quad \text{for } \alpha = 2, 3.$$

Then it suffices to consider (5.2) with all indices running only from 0 to 1. Using (4.7) and (4.8), we find that (5.2) reads

$$\left. \begin{aligned} \lambda_0 \Psi'_0 &= \frac{4}{3} \rho u^0 u^1 - q_0 = +\frac{4}{3} (\Psi_\beta \Psi^\beta)^{-3} \Psi_0 \Psi_1 - q_0 \equiv f_0(\Psi_0, \Psi_1) \\ \text{and } \lambda_1 \Psi'_1 &= \frac{4}{3} \rho u^1 u^1 + \frac{1}{3} \rho - q_1 = -\frac{4}{3} (\Psi_\beta \Psi^\beta)^{-3} \Psi_1 \Psi_1 + \frac{1}{3} (\Psi_\beta \Psi^\beta)^{-2} - q_1 \equiv f_1(\Psi_0, \Psi_1). \end{aligned} \right\} \quad (5.3)$$

We analyse the phase portraits of this family of planar dynamical system on its natural domain of definition  $\Omega = \{(\Psi_0, \Psi_1) \in \mathbb{R}^2 : \Psi_0 > |\Psi_1|\}$ .

**Lemma 5.1.** *System (5.3) has at most two rest points. It has exactly two if and only if  $q$  is timelike. In that case, the two rest points are connected by a heteroclinic orbit.*

*Proof.* The requirement  $f_0(\Psi_0, \Psi_1) = f_1(\Psi_0, \Psi_1) = 0$  is equivalent to

$$\rho^2 = \frac{9}{16} \frac{q_0^2}{u_1^2(1+u_1^2)}, \quad \text{sgn } u_1 = \text{sgn } q_0, \quad u_0 = (1+u_1^2)^{1/2} \quad (5.4)$$

and

$$16(-q_0^2 + q_1^2)u_1^4 + (8q_0^2 + 16q_1^2)u_1^2 - q_0^2 = 0. \quad (5.5)$$

It is easy to see that polynomial (5.5) in  $u_1^2$  has two positive roots indeed if and only if  $q_0^2 > q_1^2$ . We assume this now. We also suppose that  $q_0 < 0$ , i.e. the shock under consideration is a ‘1-shock’. (The situation for  $q_0 > 0$ , the situation is completely analogous; cf. [18].) One could now consider the finite region bounded by the two nullclines  $\mathcal{N}_j \equiv \{(\Psi_0, \Psi_1) : f_j(\Psi_0, \Psi_1) = 0\}$ ,  $j = 0, 1$ , and finish the proof analogously to Gilbarg’s in the case of classical fluid dynamics [10].

As an alternative, we note that the right-hand side of (5.3) is the gradient of  $L : \Omega \rightarrow \mathbb{R}$ ,

$$L(\Psi_0, \Psi_1) = \frac{1}{3}(\Psi_\beta \Psi^\beta)^{-2} \Psi_1 - (q_0 \Psi_0 + q_1 \Psi_1).$$

Lax’s inequalities [24] imply that at one of the two rest points, a supersonic state  $\Psi^-$ , the Hessian

$$\frac{\partial^2 L}{\partial \psi_\alpha \partial \psi_\gamma}(\psi)$$

has two positive eigenvalues, while at the other rest point, a subsonic state  $\Psi^+$ , this matrix has one positive and one negative eigenvalue. Now, the bounded closed-level curves of  $L$  that surround  $\Psi^-$  cover a simply connected region  $\tilde{\Omega} \subset \Omega$  that has  $\Psi^+$  as a boundary point.  $\Psi^-$  is an unstable node, and thus the  $\alpha$ -limit of that branch of the stable manifold of  $\Psi^+$  that lies in  $\tilde{\Omega}$ . ■

Owing to the Rankine–Hugoniot conditions, every inviscid shock wave appears as a pair of rest points of (5.3) for some value of  $q^\alpha$ . Theorem 2.2 is thus a direct consequence of lemma 5.1.

## 6. Subluminality and damping of plane waves

In this section, we give a proof of theorem 2.4. The purpose of theorem 2.4 is to provide an independent justification of new equations (3.5) by demonstrating causality and dissipativity at the level of modes. For this we now focus on linear plane waves, the Fourier–Laplace modes associated with full linearizations, at any constant state, of our proposed new PDE system (3.2). Here, *full* means that we do not restrict attention to the leading second-order part of the equations but include the first-order *acoustic* part as well. We prove the subluminality and decay of all modes for this mixed-order combination. Note that this is not an automatic consequence of theorem 2.1. The proof consists in a careful analysis of the system’s dispersion relation.

Written out, the linearized equations read

$$\left\{ 3 \frac{\partial \theta}{\partial t} + \bar{\theta} \nabla \cdot \mathbf{v} \right\} + \left\{ \hat{\chi} \left( \frac{\partial^2 \theta}{\partial t^2} - \Delta \theta \right) \right\} = 0 \quad (6.1)$$

and

$$\left\{ \frac{1}{\bar{\theta}} \nabla \theta + \frac{\partial \mathbf{v}}{\partial t} \right\} + \left\{ \hat{\sigma} \frac{\partial^2 \mathbf{v}}{\partial t^2} - (\hat{\eta} \nabla \cdot \mathbf{S} \mathbf{v} + \hat{\zeta} \nabla (\nabla \cdot \mathbf{v})) \right\} = 0,$$

where  $\bar{\theta}$  denotes the temperature at the constant state at which the linearization is taken, and  $\hat{\sigma} = (4\hat{\eta}/3) + \hat{\zeta}$ .

A Fourier–Laplace mode

$$\begin{pmatrix} \hat{\rho} \\ \hat{\mathbf{v}} \end{pmatrix} e^{\lambda t + i\boldsymbol{\xi} \cdot \mathbf{x}} = \begin{pmatrix} \hat{\rho} \\ \hat{\mathbf{v}} \end{pmatrix} e^{\operatorname{Re}\{\lambda\}t} e^{i(\operatorname{Im}\{\lambda\}t + \boldsymbol{\xi} \cdot \mathbf{x})}, \quad (\lambda, \boldsymbol{\xi}) \in \mathbb{C} \times \mathbb{R}^3 \quad (6.2)$$

solving (6.1) is called *dissipative* if  $\operatorname{Re}\{\lambda\} \leq 0$ ; it is called *strictly dissipative* if  $\operatorname{Re}\{\lambda\} < 0$  and *dissipation-free* or *neutral* if  $\operatorname{Re}\{\lambda\} = 0$ . The speed of a mode is given by

$$s = -\frac{\operatorname{Im}\{\lambda\}}{|\boldsymbol{\xi}|};$$

it is called *subluminal* if  $s^2 < c^2$ ; it is called *luminal* if  $s^2 = c^2$ .

The existence of a Fourier–Laplace mode (6.2) is equivalent to the dispersion relation

$$\det M(\lambda, |\boldsymbol{\xi}|) = 0 \quad (6.3)$$

with

$$M(\lambda, \boldsymbol{\xi}) = \begin{pmatrix} 3\lambda & i\boldsymbol{\xi}^\top \\ i\boldsymbol{\xi} & \lambda I \end{pmatrix} + \begin{pmatrix} \hat{\chi}(\lambda^2 + |\boldsymbol{\xi}|^2) & 0 \\ 0 & N(\lambda, \boldsymbol{\xi}) \end{pmatrix} \quad (6.4)$$

with

$$N(\lambda, \boldsymbol{\xi}) = (\hat{\sigma}\lambda^2 + \hat{\eta}|\boldsymbol{\xi}|^2)I + (\hat{\zeta} + \frac{1}{3}\hat{\eta})\boldsymbol{\xi}\boldsymbol{\xi}^\top. \quad (6.5)$$

Note that, remarkably, the dispersion relation does not depend on the base state.

For any fixed  $\boldsymbol{\xi}$ , the left-hand side of (6.3) is a polynomial of degree 8 in  $\lambda$  whose roots  $\lambda_i(\boldsymbol{\xi})$ ,  $i = 1, \dots, 8$  are continuous functions we refer to as *characteristic rates*, which determine subluminality and dissipativity. For convenience, we set  $\hat{\chi} = 1$ , and to simplify notation we now write  $\eta, \sigma$  instead of  $\hat{\eta}, \hat{\sigma}$ . (The former can be achieved via a uniform scaling of space and time variables.)

**Lemma 6.1.** *The dispersion relation (6.3) factors as*

$$\Pi_L^\sigma(\lambda, |\boldsymbol{\xi}|)(\Pi_T^{\eta, \sigma}(\lambda, |\boldsymbol{\xi}|))^2 = 0$$

and thus decomposes into

$$0 = \Pi_L^\sigma(\lambda, |\boldsymbol{\xi}|) = (3\lambda + \lambda^2 + |\boldsymbol{\xi}|^2)(\lambda + \sigma(\lambda^2 + |\boldsymbol{\xi}|^2)) + |\boldsymbol{\xi}|^2 \quad (6.6)$$

and

$$0 = \Pi_T^{\sigma, \eta}(\lambda, |\boldsymbol{\xi}|) = \lambda + \sigma\lambda^2 + \eta|\boldsymbol{\xi}|^2. \quad (6.7)$$

For any  $\boldsymbol{\xi} \in \mathbb{R}^3 \setminus \{0\}$ , we decompose

$$\mathbb{C} \times \mathbb{C}^3 = L(\boldsymbol{\xi}) \oplus L^\perp(\boldsymbol{\xi}) \quad \text{with } L(\boldsymbol{\xi}) \equiv \mathbb{C} \times \mathbb{C}\boldsymbol{\xi}, \quad L^\perp(\boldsymbol{\xi}) \equiv \{0\} \times \{\boldsymbol{\xi}\}^\perp.$$

A mode (6.2) is called *longitudinal* if  $(\hat{\rho}, \hat{\mathbf{v}}) \in L(\boldsymbol{\xi})$ ; it is called *transverse* if  $(\hat{\rho}, \hat{\mathbf{v}}) \in L^\perp(\boldsymbol{\xi})$ . Relations (6.6) and (6.7) are called the longitudinal and transverse dispersion relations, respectively.

*Proof of lemma 6.1.* If  $\boldsymbol{\xi} = 0$ , this is immediate. Assume then that  $\boldsymbol{\xi} \neq 0$ . The restrictions of  $M(\lambda, \boldsymbol{\xi})$  to its invariant spaces  $L(\boldsymbol{\xi})$  and  $L^\perp(\boldsymbol{\xi})$  are given by the  $2 \times 2$  matrices

$$\begin{pmatrix} 3\lambda + \lambda^2 + |\boldsymbol{\xi}|^2 & i|\boldsymbol{\xi}| \\ i|\boldsymbol{\xi}| & \lambda + \sigma(\lambda^2 + |\boldsymbol{\xi}|^2) \end{pmatrix}$$

and

$$\begin{pmatrix} \lambda + \sigma\lambda^2 + \eta|\boldsymbol{\xi}|^2 & 0 \\ 0 & \lambda + \sigma\lambda^2 + \eta|\boldsymbol{\xi}|^2 \end{pmatrix},$$

respectively. To confirm this, let

$$\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix},$$

where  $R$  is a  $3 \times 3$  rotation taking  $R\xi = (\xi, 0, 0)$  with  $\xi = |\xi| > 0$ , and note that  $\mathbf{R}M(\lambda, \xi)\mathbf{R}^{-1}$  is

$$\begin{pmatrix} 3\lambda & i\xi & 0 & 0 \\ i\xi & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} \lambda^2 + \xi^2 & 0 & 0 & 0 \\ 0 & \sigma(\lambda^2 + \xi^2) & 0 & 0 \\ 0 & 0 & \sigma\lambda + \eta\xi^2 & 0 \\ 0 & 0 & 0 & \sigma\lambda + \eta\xi^2 \end{pmatrix}.$$

We start by showing that all transverse modes are strictly dissipative and subluminal. The letter  $\xi$  continues to denote  $|\xi|$ .

**Lemma 6.2.** *If  $\Pi_T^{\eta, \sigma}(\lambda, \xi) = 0$  for some  $\xi > 0$ , then  $\text{Re}\{\lambda\} < 0$  and  $|\text{Im}\{\lambda\}| < \xi$ .*

*Proof.* In that case,

$$\lambda = \frac{1}{2\sigma}(-1 \pm \sqrt{1 - 4\sigma\eta\xi^2}),$$

from which  $\text{Re}\{\lambda\} < 0$  is obvious, and

$$\left(\frac{\text{Im}\{\lambda\}}{\xi}\right)^2 < \frac{\eta}{\sigma} \leq 1.$$

The next two lemmas state that no longitudinal mode can be luminal or neutral.

**Lemma 6.3.** *For any  $\sigma > 0$  and  $\xi > 0$ ,  $\Pi_L^\sigma(\alpha \pm i\xi, \xi) \neq 0$  for all  $\alpha \in \mathbb{R}$ .*

**Lemma 6.4.** *For any  $\sigma > 0$  and  $\xi > 0$ ,  $\Pi_L^\sigma(i\beta, \xi) \neq 0$  for all  $\beta \in \mathbb{R}$ .*

*Proof of lemma 6.3.* As  $\Pi_L^\sigma(\cdot, \xi)$  has real coefficients, it suffices to show that

$$\Pi_L^\sigma(\alpha + i\xi, \xi) = 0 \tag{6.8}$$

is not possible. Substituting  $\alpha + i\xi$  for  $\lambda$  in (6.6) and multiplying out gives

$$\begin{aligned} \Pi_L^\sigma(\alpha + i\xi, \xi) &= \sigma(\alpha + i\xi)^4 + (1 + 3\sigma)(\alpha + i\xi)^3 + (3 + 2\sigma\xi^2)(\alpha + i\xi)^2 \\ &\quad + (1 + 3\sigma)\xi^2(\alpha + i\xi) + \sigma\xi^4 + \xi^2 \\ &= \sigma(\alpha^4 + 4\alpha^3i\xi - 6\alpha^2\xi^2 - 4\alpha i\xi^3 + \xi^4) \\ &\quad + (1 + 3\sigma)(\alpha^3 + 3i\alpha^2\xi - 3\alpha^2\xi - 3\alpha\xi^2 - i\xi^3) \\ &\quad + (3 + 2\sigma\xi^2)(\alpha^2 + 2i\alpha\xi - \xi^2) + (1 + 3\sigma)\xi^2(\alpha + i\xi) + \sigma\xi^4 + \xi^2 \\ &= \{\cdot\}_{\text{Re}} + i\{\cdot\}_{\text{Im}}, \end{aligned} \tag{6.9}$$

where

$$\{\cdot\}_{\text{Im}} = \{(4\sigma\alpha^3\xi - 4\sigma\alpha\xi^3) + (1 + 3\sigma)3\alpha^2\xi + (3 + 2\sigma\xi^2)2\alpha\xi\}_{\text{Im}} \tag{6.10}$$

and

$$\begin{aligned} \{\cdot\}_{\text{Re}} &= \{\sigma(\alpha^4 - 6\alpha^2\xi^2 + \xi^4) + (1 + 3\sigma)(\alpha^3 - 3\alpha\xi^2) \\ &\quad + (3 + 2\sigma\xi^2)(\alpha^2 - \xi^2) + (1 + 3\sigma)\xi^2\alpha + \sigma\xi^4 + \xi^2\}_{\text{Re}}. \end{aligned} \tag{6.11}$$

Noticing the  $\xi^3$  term cancels in (6.11) and the  $\xi^4$  term cancels in (6.10), we obtain after simplification

$$\{\cdot\}_{\text{Im}} = \alpha\xi(4\sigma\alpha^2 + (1 + 3\sigma)3\alpha + 6) \tag{6.12}$$

and

$$\{\cdot\}_{\text{Re}} = -\xi^2(4\sigma\alpha^2 + 2(1 + 3\sigma)\alpha + 2) + \alpha^2(\sigma\alpha^2 + (1 + 3\sigma)\alpha + 3). \tag{6.13}$$

Thus, (6.8) is equivalent to  $\{\cdot\}_{\text{Im}} = 0$  and  $\{\cdot\}_{\text{Re}} = 0$ . Setting (6.12) equal to zero leads to the condition

$$4\sigma\alpha^2 + (1 + 3\sigma)3\alpha + 6 = 0. \tag{6.14}$$

Setting (6.13) equal to zero and using (6.14) gives

$$4\xi^2((1+3\sigma)\alpha+4) + \alpha^2((1+3\sigma)\alpha+6) = 0. \quad (6.15)$$

Letting

$$\gamma = (1+3\sigma)\alpha \quad \text{and} \quad \delta = \frac{4}{9} \frac{3\sigma}{(1+3\sigma)^2}$$

and observing that

$$\max_{\sigma>0} \frac{3\sigma}{(1+3\sigma)^2} = \frac{1}{4},$$

we see that (6.8) is equivalent to the existence of  $\gamma \in \mathbb{R}, \delta \in (0, 1/9]$  such that

$$p_\delta(\gamma) \equiv \delta\gamma^2 + \gamma + 2 = 0 \quad \text{and} \quad -6 < \gamma < -4. \quad (6.16)$$

However, as

$$p_\delta(-4) = 16\delta - 2 \leq \frac{16}{9} - 2 < 0$$

and

$$p_\delta(-6) = 36\delta - 4 \leq \frac{36}{9} - 4 = 0,$$

(6.16), and thus (6.8), is impossible. ■

*Proof of lemma 6.4.* Substituting  $i\beta$  for  $\lambda$  in (6.6) and multiplying out gives

$$\begin{aligned} \Pi_L^\sigma(i\beta, \xi) &= \sigma\beta^4 - (1+3\sigma)i\beta^3 - (3+2\sigma\xi^2)\beta^2 + (1+3\sigma)\xi^2i\beta + \sigma\xi^4 + \xi^2 \\ &= \{\sigma\beta^4 - (3+2\sigma\xi^2)\beta^2 + (\sigma\xi^4 + \xi^2)\}_I + i\{-(1+3\sigma)\beta^2 + (1+3\sigma)\xi^2\}_I. \end{aligned} \quad (6.17)$$

Now  $\Pi_L^\sigma(i\beta, \xi) = 0$  requires  $\{\cdot\}_{\text{Re}} = \{\cdot\}_{\text{Im}} = 0$ , and we see  $\{\cdot\}_{\text{Im}} = 0$  if and only if  $\beta^2 = \xi^2$ . Using this in  $\{\cdot\}_{\text{Re}}$  gives

$$\{\cdot\}_{\text{Re}} = \sigma\beta^4 - (3+2\sigma\beta^2)\beta^2 + \sigma\xi^4 + \beta^2 = -2\beta^2 \neq 0, \quad (6.18)$$

because  $\beta^2 = \xi^2 \neq 0$ . Thus,  $\Pi_L^\sigma(i\beta, \xi) = 0$  is impossible. ■

The next two lemmas state that at least for a certain value of the wavenumber  $\xi$  and the parameter  $\sigma$ , all longitudinal modes are subluminal and strictly dissipative.

**Lemma 6.5.** *There exist  $\sigma > 0$  and  $\xi > 0$  such that  $\Pi_L^\sigma(\lambda, \xi) = 0$  implies  $|\text{Im}\{\lambda\}| < \xi$ .*

**Lemma 6.6.** *There exist  $\sigma > 0$  and  $\xi > 0$  such that  $\Pi_L^\sigma(\lambda, \xi) = 0$  implies  $\text{Re}\{\lambda\} < 0$ .*

*Proof of lemma 6.5.* Fix  $\sigma = 1$ . As

$$\Pi_L^1(\lambda, \xi) = \lambda^4 + 4\lambda^3 + (3+2\xi^2)\lambda^2 + 4\xi^2\lambda + \xi^4 + \xi^2, \quad (6.19)$$

the condition

$$\Pi_L^1(\lambda, \xi) = 0 \quad (6.20)$$

reduces for  $\xi = 0$  to

$$0 = \lambda^4 + 4\lambda^3 + 3\lambda^2 = \lambda^2(\lambda^2 + 4\lambda + 3)$$

with roots

$$\lambda_{1,2}^0 = 0, \quad \lambda_3^0 = -1 \quad \text{and} \quad \lambda_4^0 = -3. \quad (6.21)$$

For sufficiently small  $\xi \geq 0$ , the latter two perturb smoothly as simple real roots

$$\lambda_3(\xi), \lambda_4(\xi) < 0. \quad (6.22)$$

To understand the perturbation behaviour of the double root  $\lambda_{1,2}^0$ , note that for  $\xi > 0$ , a number  $\lambda$  solves (6.20) if and only if

$$\hat{\lambda} \equiv \frac{\lambda}{\xi} \quad (6.23)$$

solves

$$0 = \hat{\Pi}(\hat{\lambda}, \xi) \equiv \hat{\lambda}^4\xi^2 + 4\hat{\lambda}^3\xi + (3+2\xi^2)\hat{\lambda}^2 + 4\hat{\lambda}\xi + \xi^2 + 1. \quad (6.24)$$

For  $\xi = 0$ , equation (6.24) has the two roots

$$\hat{\lambda}_{1,2}^0 = \pm \frac{i}{\sqrt{3}}. \quad (6.25)$$

As

$$\frac{\partial \hat{\Pi}}{\partial \hat{\lambda}}(\hat{\lambda}_j^0, 0) = 6\hat{\lambda}_j^0 \neq 0, \quad j = 1, 2,$$

they perturb smoothly as simple zeros  $\hat{\lambda}_j^0(\xi)$  for small  $\xi \geq 0$ . As

$$\frac{\partial \hat{\Pi}}{\partial \xi}(\hat{\lambda}_j^0, 0) = \frac{8}{3}\hat{\lambda}_j^0, \quad j = 1, 2,$$

we find

$$(\hat{\lambda}_j^0)'(0) = \frac{(\partial \hat{\Pi} / \partial \xi)(\hat{\lambda}_j^0, 0)}{(\partial \hat{\Pi} / \partial \hat{\lambda})(\hat{\lambda}_j^0, 0)} = -\frac{4}{9} < 0, \quad j = 1, 2,$$

and thus

$$\operatorname{Re}(\hat{\lambda}_{1,2}(\xi)) < 0 \quad \text{for small } \xi > 0. \quad (6.26)$$

Undoing the scaling (6.23), we find two smooth continuations

$$\lambda_{1,2}(\xi) = \xi \hat{\lambda}_{1,2}(\xi)$$

of the double root  $\lambda_{1,2}^0$ , with

$$\operatorname{Re}(\lambda_{1,2}(\xi)) < 0 \quad \text{for small } \xi > 0. \quad (6.27)$$

Inequalities (6.22) and (6.27) imply the assertion. ■

*Proof of lemma 6.6.* Keep  $\sigma = 1$  and consider the four complex rates  $\lambda_j(\xi)$ ,  $j = 1, 2, 3, 4$  established for small  $\xi > 0$  in the last proof. The corresponding speeds

$$s_j(\xi) = -\frac{\operatorname{Im}\{\lambda_j\}}{\xi}, \quad j = 1, 2, 3, 4$$

have limits

$$s_{1,2}(0) = \pm \frac{1}{\sqrt{3}} \quad \text{and} \quad s_{3,4}(0) = 0.$$

This implies that

$$s_j^2(\xi) < 1$$

for small  $\xi > 0$ . ■

As the reader will have noticed, the rates  $\lambda_{1,2}(\xi)$  correspond, in the large-wavelength limit  $\xi \rightarrow 0$ , to pure acoustics.

*Proof of theorem 2.1.* Consider the simply connected parameter regime

$$\Omega \equiv \{(\sigma, \eta, \xi) \in (0, \infty)^3 : \eta \leq \sigma\},$$

and on it the property

$$P(\sigma, \eta, \xi) : \text{for all } \lambda \in \mathbb{C}, \Pi_L^\sigma(\lambda, \xi)\Pi_T^{\sigma,\eta}(\lambda, \xi) = 0 \text{ implies } \operatorname{Re}\{\lambda\} < 0 \quad \text{and} \quad |\operatorname{Im}\{\lambda\}| < \xi.$$

Lemmas 4.2–6.5 together with the continuous dependence of the solution set

$$\Lambda^{\sigma,\eta} \equiv \{\lambda \in \mathbb{C} : \Pi_L^\sigma(\lambda, \xi)\Pi_T^{\sigma,\eta}(\lambda, \xi) = 0\}$$

on  $(\sigma, \eta)$ , imply that the set

$$\tilde{\Omega} \equiv \{(\sigma, \eta, \xi) \in \Omega : P(\sigma, \eta, \xi) \text{ holds}\}$$

is actually identical with  $\Omega$ . Now, lemma 4.1 yields that for any  $\xi \neq 0$  and  $\lambda \in \mathbb{C}$ , (6.3) implies

$$\operatorname{Re}\{\lambda\} < 0 \quad \text{and} \quad |\operatorname{Im}\{\lambda\}| < |\xi|. \quad \blacksquare$$

## 7. Entropy production

We now study our new ansatz from a thermodynamic point of view. To prove theorem 2.3, note that for pure radiation, the local entropy production rate [3, p. 54],

$$\frac{\partial S^\alpha}{\partial x^\alpha} = -\frac{1}{\theta} \frac{\partial u_\alpha}{\partial x^\beta} \Delta T^{\alpha\beta} + \frac{1}{\theta^2} \frac{\partial \theta}{\partial x^\beta} u_\alpha \Delta T^{\alpha\beta}, \quad (7.1)$$

is given by the quadratic form

$$\begin{aligned} \mathcal{Q}(\partial\theta, \partial\mathbf{v}) &\equiv -\frac{1}{\theta^2} \frac{\partial \theta}{\partial t} \Delta T^{00}|_0 - \frac{1}{\theta^2} \left( \frac{\partial \theta}{\partial x^i} + \theta \frac{\partial v_i}{\partial t} \right) \Delta T^{i0}|_0 - \frac{1}{\theta} \frac{\partial v_i}{\partial x^j} \Delta T^{ij}|_0 \\ &= \frac{1}{\theta^2} \frac{\partial \theta}{\partial t} \left( -\chi \frac{\partial \theta}{\partial t} + \left( \frac{4}{3} \eta + \zeta \right) \frac{\partial v^l}{\partial x^l} \right) + \frac{1}{\theta^2} \left( \frac{\partial \theta}{\partial x^i} + \theta \frac{\partial v_i}{\partial t} \right) \left( \chi \frac{\partial \theta}{\partial x_i} + \mu \frac{\partial v^i}{\partial t} \right) \\ &\quad + \frac{1}{\theta} \frac{\partial v_i}{\partial x^j} \eta \left( \frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} \frac{\partial v^l}{\partial x^l} \delta^{ij} \right) + \frac{1}{\theta} \frac{\partial v_i}{\partial x^j} \left( \zeta \frac{\partial v^l}{\partial x^l} - \chi \frac{\partial \theta}{\partial t} \right) \delta^{ij} \\ &\equiv \mathcal{Q}_1(\partial\theta, \partial\mathbf{v}) + \mathcal{Q}_2(\partial\theta, \partial\mathbf{v}) + \mathcal{Q}_3(\partial\theta, \partial\mathbf{v}) + \mathcal{Q}_4(\partial\theta, \partial\mathbf{v}). \end{aligned}$$

$\mathcal{Q}$  is obviously not non-negative on all gradients of arbitrary functions  $(\theta, \mathbf{v})$ ; it directly follows that one can construct initial data leading to negative entropy production at least for short times. However, the latter does not necessarily rule our proposal out.

**Lemma 7.1.** *On gradients solving the inviscid Euler equations at a point,*

$$\mathcal{Q}(\partial\theta, \partial\mathbf{v}) \equiv \left( \frac{1}{2} \hat{\eta} \|\mathcal{S}\mathbf{v}\|^2 + \frac{2}{9} (\hat{\chi} + 3\hat{\zeta} - 2\hat{\eta})(\nabla \cdot \mathbf{v})^2 \right) \theta^3.$$

*Proof.* As always (cf. [3]),

$$\mathcal{Q}_3(\partial\theta, \partial\mathbf{v}) = \frac{1}{2} \hat{\eta} \theta^3 \|\mathcal{S}\mathbf{v}\|^2.$$

In the rest frame, the inviscid Euler equations read, at a point,

$$\left. \begin{aligned} \frac{1}{\theta} \frac{\partial \theta}{\partial t} + \frac{1}{3} \nabla \cdot \mathbf{v} &= 0 \\ \frac{1}{\theta} \nabla \theta + \frac{\partial \mathbf{v}}{\partial t} &= 0. \end{aligned} \right\} \quad (7.2)$$

and

Using them, we find

$$\mathcal{Q}_2(\partial\theta, \partial\mathbf{v}) = 0$$

and

$$\mathcal{Q}_1(\partial\theta, \partial\mathbf{v}) + \mathcal{Q}_4(\partial\theta, \partial\mathbf{v}) = \frac{2}{9} (\hat{\chi} + 3\hat{\zeta} - 2\hat{\eta}) \theta^3 (\nabla \cdot \mathbf{v})^2. \quad \blacksquare$$

Part (ii) of theorem 2.3 is an easy corollary of this; part (i) follows directly from

**Lemma 7.2.**  *$\mathcal{Q}$  is positive on the gradient of any plane-wave function  $\psi(x) = \Psi(x_\alpha N^\alpha)$  with space-like direction of propagation  $N^\alpha$ .*

*Proof.*

$$\frac{\partial S^\alpha}{\partial x^\alpha} = B^{\alpha\beta\gamma\delta} \frac{\partial \psi_\alpha}{\partial x^\beta} \frac{\partial \psi_\gamma}{\partial x^\delta} = N B^{\alpha\gamma} \Psi'_\alpha \Psi'_\gamma > 0. \quad \blacksquare$$

Recall now that linear-dissipation ansatz (3.6) is based on a smallness assumption. This is often expressed by the concept of vanishing viscosity, which is here the limit for  $\epsilon \searrow 0$  in

$$\frac{\partial}{\partial x^\beta} \{T^{\alpha\beta} + \epsilon \Delta T^{\alpha\beta}\} = 0. \quad (7.3)$$

Especially, inviscid shock waves (2.12) appear as limits of viscous shock waves,

$$({}^0\rho, {}^0u)(x) = \lim_{\epsilon \rightarrow 0} ({}^\epsilon\rho, {}^\epsilon u)(x) \quad \text{with } ({}^\epsilon\rho, {}^\epsilon u)(x) = (R, U) \left( \frac{x_\alpha N^\alpha}{\epsilon} \right). \quad (7.4)$$

Because of theorem 2.3, we expect that for our dissipation stress tensor (3.6), (2.9) and (2.11), the vanishing-viscosity limit should be well behaved near any ‘entropy solution’ of the inviscid Euler equations, with, in particular, corresponding nearby solutions of the dissipative Euler equations having non-negative entropy production everywhere as soon as  $\epsilon > 0$  is small enough.

## 8. The viscosity of radiation?

The requirement (2.16) seems natural; cf. [3, p. 57]. Noting that

$$T^\alpha{}_\alpha + \Delta T^\alpha{}_\alpha = \Delta T^\alpha{}_\alpha|_0 = \frac{2}{3}(\hat{\chi} + 3\hat{\zeta} - 2\hat{\eta})\nabla \cdot \mathbf{v},$$

we end with the following now immediate observations.

### Theorem 8.1.

(i) If  $\Delta T^{\alpha\beta}$  is trace-free on inviscid gradients, i.e. if

$$\hat{\chi} + 3\hat{\zeta} - 2\hat{\eta} = 0, \quad (8.1)$$

the entropy production is given by

$$\mathcal{Q} = \frac{1}{2}\hat{\eta}\theta^3\|\mathcal{S}\mathbf{v}\|^2.$$

(ii) If moreover  $\zeta = 0$ , then heat conduction  $\chi$  and shear viscosity  $\eta$  are linked as

$$\hat{\chi} = 2\hat{\eta}.$$

In case (ii), all coefficients in (3.6) are uniquely determined up to one natural common constant scale factor. Note that the literature provides strong reasons for the assumption  $\zeta = 0$  (see again [3, p. 57]). To motivate (i), note that as the total stress–energy tensor  $T^{\alpha\beta} + \Delta T^{\alpha\beta}$  of pure radiation should ultimately derive from Maxwell tensors, and the inviscid stress tensor  $T^{\alpha\beta}$  is trace-free, the assumption that  $\Delta T^{\alpha\beta}$  be trace-free seems natural. Identity (8.1) implies, in particular, that exactly pure rotations,

$$D\mathbf{v} = -(D\mathbf{v})^\top,$$

and isotropic expansion/contraction,

$$D\mathbf{v} = kI,$$

are non-dissipative, and thus these would be preferred in the course of time. Theorem 3.1 would thus appear to have implications for Cosmology, cf. [14,15].

**Acknowledgement.** The work leading to this paper was originally initiated by a joint one-month stay of the authors at the Institut des Hautes Etudes Scientifiques, Bur-sur-Yvette, in the year 2000. We are grateful for hospitality and financial support.

**Funding statement.** H.F. supported by DFG through its Excellence Grant to University of Konstanz. B.T. supported by NSF Applied Mathematics grant no. DMS-070-7532.

## References

1. Landau LD, Lifshitz EM. 1959 *Fluid mechanics*, Section 127. London, UK: Pergamon Press.
2. Eckart C. 1940 The thermodynamics of irreversible processes. 3: Relativistic theory of the simple fluid. *Phys. Rev.* **58**, 919–924. (doi:10.1103/PhysRev.58.919)
3. Weinberg S. 1972 *Gravitation and cosmology: principles and applications of the general theory of relativity*. New York, NY: John Wiley & Sons.
4. Israel W. 1976 Nonstationary irreversible thermodynamics: a causal relativistic theory. *Ann. Phys.* **100**, 310–331. (doi:10.1016/0003-4916(76)90064-6)
5. Israel W, Stewart J. 1979 Transient relativistic thermodynamics and kinetic theory. *Ann. Phys.* **118**, 341–372. (doi:10.1016/0003-4916(79)90130-1)
6. Geroch R, Lindblom L. 1990 Dissipative relativistic fluid theories of divergence type. *Phys. Rev. D* **41**, 1855–1861. (doi:10.1103/PhysRevD.41.1855)
7. Olson TS, Hiscock WA. 1990 Plane steady shock waves in Israel–Stewart fluids. *Ann. Phys.* **204**, 331–350. (doi:10.1016/0003-4916(90)90393-3)
8. Grad H. 1949 On the kinetic theory of rarefied gases. *Comm. Pure Appl. Math.* **2**, 331–407. (doi:10.1002/cpa.3160020403)
9. Müller I, Ruggeri T. 1993 Extended thermodynamics. In *Springer tracts in natural philosophy*, vol. 37. New York, NY: Springer.
10. Gilbarg D. 1951 The existence and limit behavior of the one-dimensional shock layer. *Am. J. Math.* **73**, 256–274. (doi:10.2307/2372177)
11. Hughes T, Kato T, Marsden J. 1976 Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Rational Mech. Anal.* **63**, 273–294.
12. Godunov SK. 1961 An interesting class of quasi-linear systems. *Dokl. Akad. Nauk SSSR* **139**, 521–523.
13. Jin S, Xin Z. 1995 The relaxation schemes for systems of conservation laws in arbitrary space dimensions. *Comm. Pure Appl. Math.* **XLVIII**, 235–276. (doi:10.1002/cpa.3160480303)
14. Misner C. 1968 The isotropy of the universe. *Astrophys. J.* **151**, 4310–4457. (doi:10.1086/149448)
15. Weinberg S. 1971 Entropy generation and the survival of proto-galaxies in an expanding universe. *Astrophys. J.* **168**, 175–194. (doi:10.1086/151073)
16. Anile AM. 1989 *Relativistic fluids and magneto-fluids*. With Applications in Astrophysics and Plasma Physics. Cambridge, UK: Cambridge University Press.
17. Taub AH. 1948 Relativistic Rankine–Hugoniot equations. *Phys. Rev.* **74**, 328–334. (doi:10.1103/PhysRev.74.328)
18. Smoller J, Temple B. 1993 Global solutions to the relativistic Euler equations. *Comm. Math. Phys.* **156**, 67–99. (doi:10.1007/BF02096733)
19. Smoller J, Temple B. 2001 Shock wave solutions of the Einstein equations: a general theory with examples. In *Advances in the theory of shock waves* (eds H Freistühler, A Szepessy). Progress in Nonlinear Differential Equations and their Applications, no. 47, pp. 105–258. Boston, MA: Birkhäuser Boston.
20. Freistühler H, Raoofi M. 2007 Stability of perfect-fluid shock waves in special and general relativity. *Class. Quantum Grav.* **24**, 4439–4455. (doi:10.1088/0264-9381/24/17/011)
21. Friedrichs KO. 1954 Symmetric hyperbolic linear differential equations. *Comm. Pure Appl. Math.* **7**, 345–392. (doi:10.1002/cpa.3160070206)
22. Freistühler H, Trakhinin Y. 2013 Symmetrizations of RMHD equations and stability of relativistic current-vortex sheets. *Class. Quantum Grav.* **30**, 085012. (doi:10.1088/0264-9381/30/8/085012)
23. Thomas LH. 1930 The radiation field for a fluid in motion. *Q. J. Math.* **1**, 239–251. (doi:10.1093/qmath/os-1.1.239)
24. Lax P. 1957 Hyperbolic systems of conservation laws II. *Comm. Pure Appl. Math.* **10**, 537–566. (doi:10.1002/cpa.3160100406)