NUMERICAL ANALYSIS OF A CANONICAL SHOCK WAVE INTERACTION PROBLEM IN GENERAL RELATIVITY

BLAKE TEMPLE AND ZEKE VOGLER

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Abstract. We present the analysis of convergence of the *locally inertial Godunov method with dynamical time dilation* applied to a canonical initial data set which is arguably the simplest initial data that creates a point of shock wave interaction in General Relativity. New applications include the analysis of convergence in the presence of new boundary conditions which enables one to test the validity of the Einstein constraint equations numerically in new Lipschitz continuous space-time metrics. The numerical method, introduced in [14, 15], is an algorithm for simulating general relativistic shock-waves in spherically symmetric spacetimes, and the analysis here rigorously establishes claims made in authors' PRSA article [15].

1. Introduction

In this paper we supply the proof of the convergence theorem stated without proof in Section 3 of authors PRSA article [15]. It provides conditions for convergence of the *locally inertial Godunov method with dynamic time dilation*, the numerical method for computing shock-wave solutions of the Einstein-Euler equations in Standard Schwarzschild Coordinates (SSC), the topic of [15]. The analysis gives convergence of the residual assuming a total variation bound uniform in time, a result perfectly suited to the simulation because the total variation bound is easily demonstrated numerically. The convergence theorem here together with the simulation in [14] are the basis for [15], arguably the first careful simulation of a point of shock-wave interaction in General Relativity (GR). The point of departure for this work is [2], but new features must be addressed because the conservation laws are coupled to equations for the metric, and the initial data set imposes new boundary conditions different from those addressed in [2]. Thus the standard connections between the Glimm scheme and the Godunov scheme do not apply directly. In particular, the analysis here is general enough to apply to the GR initial data set introduced in [14, 15], obtained by matching a critical ($k = 0$) Friedman-Robertson-Walker (FRW) spacetime in self-similar variables, to a static Tolman-Oppenheimer-Volkoff (TOV) spacetime, Lipschitz continuously at a given time in SSC coordinates. The resulting simulation produces gravitational metrics that are *only* Lipschitz continuous $C^{0,1}$ at shocks, and $C^{0,1}$ solutions of the Einstein equations only solve the Einstein equations $G = \kappa T$ weakly, not strongly. Thus convergence is no mute point in light of the fact that second derivatives of a $C^{0,1}$ metric contain delta function sources, and a convergence proof is required to rule out the possibility that delta function sources

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appear in the curvature tensor G, (even though it is second order in metric derivatives), and hence the fluid stress tensor T, as well. The numerics in [15] also establish the consistency of imposing two boundary conditions, one on the FRW side and one on the TOV side, even though the numerical method involves first order ODE's in the metric, making two boundary conditions appear over-constrained. The convergence theory here thus anchors the numerical verification of the Einstein constraint equations in [15].

Metrics that are only $C^{0,1}$ at shock waves do not admit locally inertial coordinate frames within the smooth $C^{2,1}$ atlas of coordinate transformations, and it is an open problem as to whether locally inertial coordinate systems can be reached within the larger atlas of $C^{1,1}$ coordinate transformations. The $C^{1,1}$ atlas admits Jacobians J that can be only Lipschitz continuous as well, and so such transformations hold the possibility that a transformed metric $\bar{g} = J^T g J^{-T}$ could be in $C^{1,1}$ even though g is only $C^{0,1}$. If the metric cannot be so smoothed from $C^{0,1}$ by a coordinate transformation within the $C^{1,1}$ atlas in a neighborhood of a point p, then we call p a *regularity singularity*. It is an open problem as to whether regularity singularities can be created by shock wave interaction in GR, [9]. In celebrated work from the 1960's, Israel proved that a metric Lipschitz continuous across a smooth shock surface can always be smoothed from $C^{0,1}$ to $C^{1,1}$ by the $C^{1,1}$ transformation to Gaussian Normal Coordinates in a neighborhood of the shock surface. But since Israel's work, it has been unknown whether such coordinate transformations exist at general points of shock wave interaction. The first step forward since the Israel result for smooth single shock surfaces was established in a nontrivial new proof by Reintjes, (announced in [8]). The proof establishes that such coordinates exist at points of shock wave interaction exhibiting the properties of the shock interaction simulated in [14, 15]. Thus the proof given here which establishes the convergence properties of the locally inertial Godunov method used in the simulation in [15], is important in anchoring the mathematics which has established this first extension of Israel's theorem. Moreover, the proof is interesting in its own right. In particular, to establish the consistency of the method, the proof must account for an interesting leading order cancellation between errors arising from discontinuities of the metric at the boundaries of the grid cells that impose the Riemann problem step of the fractional step Godunov method, with corresponding errors arising from the ODE step, c.f. (1.3). Whether or not regularity singularities exist in more complicated points of shock wave interaction, where either the symmetry of the space-time is broken, or the wave interactions are more complicated, remains an open problem. The issues regarding the conjecture of regularity singularities will be set out in the forthcoming paper [9]. For brevity, we refer the reader to [15], [8] for an introduction to the methods described in this paper, and [15] for the notation we assume at the start.

The theorem applies to a spherically symmetric gravitational metric in SSC

$$ds^2 = -B(r,t)dt^2 + \frac{dr^2}{A(r,t)} + r^2 d\Omega^2 \equiv g_{ij}dx^i dx^j, \tag{1.1}$$

where r is the radial variable, $d\Omega$ is the line element on the unit sphere, and we write

$$\mathbf{A} \equiv (\mathbf{A}, \mathbf{B}),$$

to denote the two components of the metric in vector form. In [**?**], the Einstein equations for a perfect fluid in SSC are shown to be equivalent to the system of equations

$$\begin{aligned} u_t + f(\mathbf{A}, u)_x &= g(\mathbf{A}, u, x), \\ \mathbf{A}' &= h(\mathbf{h}, u, x). \end{aligned} \tag{1.2}$$

where $u = (T_M^{00}, T_M^{01})$ are the so called Minkowski energy and momentum densities, c.f. [15, 2]. The main conclusion of the theorem stated and proved below is that a sequence of approximate solutions $(u_{\Delta x}, \mathbf{A}_{\Delta x}) \to (u, \mathbf{A})$ generated by the *locally inertial Godunov scheme with dynamic time-dilation* starting from explicit initial boundary data defined in [], must converge to a weak solution of the Einstein-Euler equations (1.2) under the assumption that the approximations are bounded with bounded oscillation. The theorem reduces the proof that a numerical approximation really represents an exact solution containing no delta functions sources in the curvature tensors, to the two things most easily established: namely, that the numerical simulation *converges* without *oscillations*. The proof here is a modification of the Groah and Temple argument used for the locally inertial Glimm scheme [2], with several differences. First, in the theorem here, we allow variable time steps, and address a different boundary value problem which includes both right and left boundary data which reflect the limited extent in space of a computer simulation. Both the Glimm scheme and the Godunov methods employ Riemann problem approximations, but the Godunov method employs averaging rather than random sampling at each time step. The theorem proved here assumes *convergence* and a *total variation bound*, (it is still an open problem to prove such bounds analytically, but for our purposes here, it is easy to verify these bounds numerically), while the Groah and Temple theorem establishes both bounds by an argument using wave strengths to bound the total variation of waves in the Riemann problem step, followed by an application of Helly's theorem to prove convergence.

There are two main parts to the proof. The first is to show the discontinuities in the metric $\mathbf{A}$ along the boundary of Riemann cells are accounted for by the inclusion of the term

$$\mathbf{A}' \cdot \nabla_{\mathbf{A}} f(\mathbf{A}_{ij}, \hat{u}, x) \tag{1.3}$$

in the ODE step [15]. The second part is to prove the jumps in the approximate solution $u_{\Delta x}$ along the time steps are of order Δx. We reiterate that Groah and Temple [2] did not assume, but proved, the convergence and total variation bounds which we take as assumptions here. For our purposes, the assumed convergence and total variation bounds are natural assumptions that can be verified numerically. In particular, our main theorem is perfectly suited to the numerical simulation of points of shock-wave interaction in [14, 15]: Once one establishes convergence and a total variation bounds numerically, the theorem here implies convergence of $u_{\Delta x}$ to a weak solution of the Einstein equations. For the shock wave interaction problem created in [15] from initial data obtained by matching an FRW metric to a TOV metric at fixed time, the boundary data consists of the values of these metric solutions restricted to the two boundaries. The Einstein constraint equations then implies the consistency of imposing the two conditions. We refer the reader to [15] for details and numerical printouts.

Again, we refer to [15] for the construction of the initial data, and the notation we employ in the convergence proof below.

2. The Convergence Theorem

The main theorem of this paper is the following:

Theorem 2.1. *Let $u_{\Delta x}(t,x)$ and $\mathbf{A}_{\Delta x}(t,x)$ be the approximate solution generated by the locally inertial Godunov method starting from the initial data $u_{\Delta x}(t_0,x)$ and $\mathbf{A}_{\Delta x}(t_0,x)$ for $t_0>0$. Assume these approximate solutions exist up to some time $t_{end}>t_0$ and converge to a solution $(u_{\Delta x},\mathbf{A}_{\Delta x})\to(u,\mathbf{A})$ as $\Delta x\to 0$ along with a total variation bound at each time step t_j*

$$T.V._{[r_{min},r_{max}]}\{u_{\Delta x}(t_j,\cdot)\}<V, \tag{2.1}$$

where $T.V._{[r_{min},r_{max}]}\{u_{\Delta x}(t_j,\cdot)\}$ represents the total variation of the function $u_{\Delta x}(t_j,x)$ on the interval $[r_{min},r_{max}]$. Assume the total variation is independent of the time step t_j and the mesh length Δx. Then the solution $(u,\mathbf{A})$ is a weak solution to the Einstein equations (1.26)-(1.29) in [2].

Proof. Suppose we have approximate solutions $(u_{\Delta x},\mathbf{A}_{\Delta x})$ obtained by the locally inertial Godunov method that satisfy the hypothesis of the theorem. Having a total variation bound at each time t_j places a total variation bound on the inputs to all the Riemann problems posed at that time. In [2], Groah and Temple show a total variation bound on the inputs implies a total variation bound on the solution to the Riemann problem for any time t such that $t_j\le t<t_{j+1}$. By the self similarity of the solution to the Riemann problem, this result also implies a total variation bound for any space coordinate within the Riemann cell. Specifically, we have the following bounds, which supply our starting point here:

$$T.V._{[x_{i-1},x_i]}\{u_{\Delta x}(t,\cdot)\}<V, \tag{2.2}$$

and

$$T.V._{[t_j,t_{j+1})}\{u_{\Delta x}(\cdot,x)\}<V, \tag{2.3}$$

for any x and t within the Riemann cell R_{ij}.

All the functions f, G, and g derived in [2] are smooth, and it is the metric that is only Lipschitz continuous. The smoothness of these functions is used throughout this proof.

Let $T=t_{end}-t_0$ be the overall time of the solution, and for each mesh length Δx define the minimum time length

$$\Delta t\equiv\min_j\{\Delta t_j\} \tag{2.4}$$

as the minimum over all the time lengths defined in [15]. By definition, this time length is proportional to the mesh length, $\Delta t\propto\Delta x$, implying $O(\Delta t)=O(\Delta x)$, and there exists a constant C bounding all the time lengths, $\Delta t_j<C\Delta t$ for all j. Throughout this proof, let C be a generic constant only depending on the bounds for the solution $[t_0,t_{end}]\times[r_{min},r_{max}]$. This variable is created to unify all the time steps, and more importantly, used to calculate the maximum number of time steps needed to go from t_0 to t_{end}.

We now follow the development of Groah and Temple in [2]. Recall from [15], $u^{RP}_{\Delta x}(t,x)$ denotes the collection of the exact solutions in all the Riemann cells R_{ij} for the Riemann problem of the homogenous system

$$u_t + f(\mathbf{A}_{ij}, u)_x = 0. \tag{2.5}$$

So $u^{RP}_{\Delta x}(t,x)$ satisfies the weak form of this conservation law in each Riemann cell

$$\begin{aligned}0 = &\int\int_{R_{ij}} \left\{-u^{RP}_{\Delta x}\varphi_t - f(\mathbf{A}_{ij}, u^{RP}_{\Delta x})\varphi_x\right\} dxdt \\ &+ \int_{R_i} \left\{u^{RP}_{\Delta x}(t_{j+1},x)\varphi(t_{j+1},x) - u^{RP}_{\Delta x}(t_j^+,x)\varphi(t_j,x)\right\} dx \\ &+ \int_{R_j} \left\{f(\mathbf{A}_{ij}, u^{RP}_{\Delta x}(t,x_i))\varphi(t,x_i)\right. \\ &\left.- f(\mathbf{A}_{ij}, u^{RP}_{\Delta x}(t,x_{i-1}))\varphi(t,x_{i-1})\right\} dt,\end{aligned} \tag{2.6}$$

where φ is a smooth test function with $Supp(\varphi) \subset [t_0, t_{end}) \times [a,b]$ for $a < r_{min} < r_{max} < b$.

Remember from [15], $\hat{u}(t,u_0)$ denotes the solution to the ODE

$$\begin{gathered}\hat{u}_t = G(\mathbf{A}_{ij}, \hat{u}, x) = g(\mathbf{A}_{ij}, \hat{u}, x) - \mathbf{A}' \cdot \nabla_{\mathbf{A}} f(\mathbf{A}_{ij}, \hat{u}, x), \\ \hat{u}(0) = u_0.\end{gathered} \tag{2.7}$$

Therefore,

$$\hat{u}(t,u_0) = u_0 + \int_0^t \{g(\mathbf{A}_{ij}, \hat{u}(\xi,u_0), x) - \mathbf{A}' \cdot \nabla_{\mathbf{A}} f(\mathbf{A}_{ij}, \hat{u}(\xi,u_0), x)\} d\xi. \tag{2.8}$$

Also, recall from [15] $u_{\Delta x}$ denotes the approximate solution obtained using the fractional step method. Since our fractional method takes the Riemann problem solution and feeds it into the ODE step, $u_{\Delta x}$ is defined on every Riemann cell R_{ij} as

$$\begin{aligned}u_{\Delta x}(t,x) = u^{RP}_{\Delta x}(t,x) + \int_{t_j}^t &\left\{g(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u^{RP}_{\Delta x}(t,x)), x)\right. \\ &\left.- \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u^{RP}_{\Delta x}(t,x))) \cdot \mathbf{A}'_{\Delta x}\right\} d\xi.\end{aligned} \tag{2.9}$$

This expression implies the error between the approximate solution and the Riemann problem solution is on the order of Δx; a fact that is repeatedly used throughout the proof.

Define the residual $\varepsilon = \varepsilon(u_{\Delta x}, \mathbf{A}_{\Delta x}, \varphi)$ of $u_{\Delta x}$ and $\mathbf{A}_{\Delta x}$ as the error of the solution in satisfying the weak form of the conservation law (1.2) by

$$\begin{aligned}\varepsilon(u_{\Delta x}, \mathbf{A}_{\Delta x}, \varphi) \equiv &\int_{r_{min}}^{r_{max}} \int_{t_0}^{t_{end}} \{-u_{\Delta x}\varphi_t - f(\mathbf{A}_{\Delta x}, u_{\Delta x})\varphi_x - g(\mathbf{A}_{\Delta x}, u_{\Delta x}, x)\varphi\} dxdt \\ &- I_1 - I_2 \\ &\sum_{i=1,j}^{i=n+1} \int_{R_{ij}} \{-u_{\Delta x}\varphi_t - f(\mathbf{A}_{ij}, u_{\Delta x})\varphi_x - g(\mathbf{A}_{ij}, u_{\Delta x}, x)\varphi\} dxdt \\ &- I_1 - I_2,\end{aligned} \tag{2.10}$$

where

$$I_1 \equiv \int_{r_{min}}^{r_{max}} u_{\Delta x}(t_0^+, x)dx = \sum_{i=1}^{n+1} \int_{R_i} u_{\Delta x}(t_0^+, x)dx, \tag{2.11}$$

and

$$\begin{aligned} I_2 \equiv & \int_{t_0}^{t_{end}} \left\{ f(\mathbf{A}_{ij}, u_{\Delta x}(t, r_{min}^+))\varphi(t, r_{min}^+) - f(\mathbf{A}_{ij}, u_{\Delta x}(t, r_{max}^+))\varphi(t, r_{max}^+) \right\} dt \\ = & \sum_j \int_{R_j} \left\{ f(\mathbf{A}_{ij}, u_{\Delta x}(t, r_{min}^+))\varphi(t, r_{min}^+) - f(\mathbf{A}_{ij}, u_{\Delta x}(t, r_{max}^+))\varphi(t, r_{max}^+) \right\} dt, \end{aligned} \tag{2.12}$$

The expression $\sum_{i=1,j}^{i=n+1}$ denotes a double sum where the index i runs across all the spatial gridpoints, and the index j runs across all the temporal gridpoints. Recall from [15], n is the number of spatial gridpoints, and there are $n+1$ Riemann cells. Our goal is to show $\varepsilon(u_{\Delta x}, \mathbf{A}_{\Delta x}, \varphi) = O(\Delta x)$ because if the approximation converges $(u_{\Delta x}, \mathbf{A}_{\Delta x}) \to (u, \mathbf{A})$ as $\Delta x \to 0$, then the limit function satisfies the condition of being a weak solution to the Einstein equations $\varepsilon(u, \mathbf{A}, \varphi) = 0$.

Substituting (2.9) into (2.10) gives us

$$\begin{aligned} \varepsilon = \sum_{i=1,j}^{i=n+1} \int\int_{R_{ij}} & \left\{ -u_{\Delta x}^{RP}\varphi_t - f(\mathbf{A}_{ij}, u_{\Delta x})\varphi_x - g(\mathbf{A}_{ij}, u_{\Delta x}, x)\varphi \right. \\ & - \varphi_t \int_{t_j}^{t} \left[g(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u_{\Delta x}^{RP}(t,x)), x) \right. \\ & \left. \left. - \frac{\partial f}{\partial \mathbf{A}} (\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u_{\Delta x}^{RP}(t,x))) \cdot \mathbf{A}'_{\Delta x} \right] d\xi \right\} dxdt - I_1 - I_2. \end{aligned} \tag{2.13}$$

Define

$$\begin{aligned} I_{ij}^1(t,x) \equiv \int_{t_j}^{t} & \left[g(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u_{\Delta x}^{RP}(t,x)), x) \right. \\ & \left. - \frac{\partial f}{\partial \mathbf{A}} (\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u_{\Delta x}^{RP}(t,x))) \cdot \mathbf{A}'_{\Delta x} \right] d\xi \end{aligned} \tag{2.14}$$

Plugging the weak form of the conservation law (2.6) of each grid rectangle into (2.13) gives us

$$\begin{aligned} \varepsilon = & \sum_{i=1,j}^{i=n+1} \int\int_{R_{ij}} \left\{ \varphi_x [f(\mathbf{A}_{ij}, u_{\Delta x}^{RP}) - f(\mathbf{A}_{ij}, u_{\Delta x})] - g(\mathbf{A}_{ij}, u_{\Delta x}, x)\varphi \right. \\ & \qquad \left. - \varphi_t I_{ij}^1(t,x) \right\} dxdt \\ & - I_1 - \sum_{i=1,j}^{i=n+1} \int_{R_i} \left\{ u_{\Delta x}^{RP}(t_{j+1}^-, x)\varphi(t_{j+1}, x) - u_{\Delta x}^{RP}(t_j^+, x)\varphi(t_j, x) \right\} dx \\ & - I_2 - \sum_{i=1,j}^{i=n+1} \int_{R_j} \left\{ f(\mathbf{A}_{ij}, u_{\Delta x}^{RP}(t, x_i))\varphi(t, x_i) - f(\mathbf{A}_{ij}, u_{\Delta x}^{RP}(t, x_{i-1}))\varphi(t, x_{i-1}) \right\} dt. \end{aligned} \tag{2.15}$$

Note

$$\left|f(\mathbf{A}_{ij}, u_{\Delta x}^{RP}) - f(\mathbf{A}_{ij}, u_{\Delta x})\right| \leq C\Delta t \tag{2.16}$$

which implies

$$\begin{aligned}\left|\sum_{i=1,j}^{i=n+1} \int\int_{R_{ij}} \varphi_x[f(\mathbf{A}_{ij}, u_{\Delta x}^{RP}) - f(\mathbf{A}_{ij}, u_{\Delta x})]dxdt\right| \\ \leq C\left\|\varphi\right\|_\infty \Delta t^2 \Delta x \left(\frac{T}{\Delta t}\right)(n+1) = O(\Delta x)\end{aligned} \tag{2.17}$$

where the number of time steps is proportional to $T/\Delta t$ and the number of space steps is $O(1/\Delta x)$ [15].

Since $u_{\Delta x}^{RP}(t_j^+, x) = u_{\Delta x}(t_j^+, x)$, the following sum is rearranged to become

$$\begin{aligned}-I_1 - &\sum_{i=1,j}^{i=n+1} \int_{R_i} \left\{u_{\Delta x}^{RP}(t_{j+1}^-, x)\varphi(t_{j+1}, x) - u_{\Delta x}^{RP}(t_j^+, x)\varphi(t_j, x)\right\} dx \\ &= \sum_{j\neq 0} \int_{r_{min}}^{r_{max}} \left\{u_{\Delta x}(t_j^+, x) - u_{\Delta x}^{RP}(t_j^-, x)\right\} \varphi(t_j, x)dx \\ &= \sum_{j\neq 0} \int_{r_{min}}^{r_{max}} \varphi(t_j, x)\left\{u_{\Delta x}(t_j^+, x) - u_{\Delta x}(t_j^-, x)\right\} dx \\ &+ \sum_{j\neq 0} \int_{r_{min}}^{r_{max}} \varphi(t_j, x)\left\{u_{\Delta x}(t_j^-, x) - u_{\Delta x}^{RP}(t_j^-, x)\right\} dx,\end{aligned} \tag{2.18}$$

where the term $u_{\Delta x}(t_j, x)$ is added and subtracted to isolate the jump in the solution $u_{\Delta x}$ across the time step t_j. We define this jump $\varepsilon_1 = \varepsilon_1(u_{\Delta x}, \mathbf{A}_{\Delta x}, \varphi)$ as

$$\varepsilon_1(u_{\Delta x}, \mathbf{A}_{\Delta x}, \varphi) \equiv \sum_{j\neq 0} \int_{r_{min}}^{r_{max}} \varphi(t_j, x)\left\{u_{\Delta x}(t_j^+, x) - u_{\Delta x}(t_j^-, x)\right\} dx, \tag{2.19}$$

and this definition allows us to rewrite (2.15) as

$$\begin{aligned}\varepsilon = \ &O(\Delta x) + \varepsilon_1 + \sum_{i=1,j}^{i=n+1} \int\int_{R_{ij}} \left\{-g(\mathbf{A}_{ij}, u_{\Delta x}, x)\varphi - \varphi_t I_{ij}^1(t, x)\right\} dxdt \\ &+ \sum_{j\neq 0} \int_{r_{min}}^{r_{max}} \varphi(t, x)\left\{u_{\Delta x}(t_j^-, x) - u_{\Delta x}^{RP}(t_j^-, x)\right\} dx \\ &- I_2 - \sum_{i=1,j}^{i=n+1} \int_{R_j} \left\{f(\mathbf{A}_{ij}, u_{\Delta x}^{RP}(t, x_i))\varphi(t, x_i) - f(\mathbf{A}_{ij}, u_{\Delta x}^{RP}(t, x_{i-1}))\varphi(t, x_{i-1})\right\} dt\end{aligned} \tag{2.20}$$

But the last sum is rearranged to cancel the boundary conditions as follows:

$$
\begin{aligned}
-I_2 - \sum_{i=1,j}^{i=n+1} \int_{R_j} &\left\{ f(\mathbf{A}_{ij}, u_{\Delta x}^{RP}(t,x_i))\varphi(t,x_i) - f(\mathbf{A}_{ij}, u_{\Delta x}^{RP}(t,x_{i-1}))\varphi(t,x_{i-1}) \right\} dt \\
= \sum_{i=1,j}^{i=n} \int_{R_j} &\left\{ f(\mathbf{A}_{i+1,j}, u_{\Delta x}^{RP}(t,x_i)) - f(\mathbf{A}_{ij}, u_{\Delta x}^{RP}(t,x_i)) \right\} \varphi(t,x_i) dt \\
+ \sum_j \int_{R_j} &\left\{ f(\mathbf{A}_{1,j}, u_{\Delta x}^{RP}(t,x_0)) - f(\mathbf{A}_{1,j}, u_{\Delta x}(t,x_0)) \right\} \varphi(t,x_0) dt \\
+ \sum_j \int_{R_j} &\left\{ f(\mathbf{A}_{n+1,j}, u_{\Delta x}^{RP}(t,x_{n+1})) - f(\mathbf{A}_{n+1,j}, u_{\Delta x}(t,x_{n+1})) \right\} \varphi(t,x_{n+1}) dt,
\end{aligned}
\tag{2.21}
$$

where

$$
\begin{aligned}
&\left| \sum_j \int_{R_j} \left\{ f(\mathbf{A}_{1,j}, u_{\Delta x}^{RP}(t,x_0)) - f(\mathbf{A}_{1,j}, u_{\Delta x}(t,x_0)) \right\} \varphi(t,x_0) dt \right| \\
&\leq \|\varphi\|_\infty C \Delta t^2 \left(\frac{T}{\Delta t} \right) = O(\Delta x),
\end{aligned}
\tag{2.22}
$$

and similarly

$$
\left| \sum_j \int_{R_j} \left\{ f(\mathbf{A}_{n+1,j}, u_{\Delta x}^{RP}(t,x_{n+1})) - f(\mathbf{A}_{n+1,j}, u_{\Delta x}(t,x_{n+1})) \right\} \varphi(t,x_{n+1}) dt \right| = O(\Delta x).
\tag{2.23}
$$

Note that the resulting double sum in (2.21) lost a term, resulting in only n terms.

To simplify the I_{ij}^1 term, we add and subtract a term deviating from it by an order of Δx, use integration by parts on the new term, and with the result add and subtract another term to reduce the expression further. To this end, let

$$
\begin{aligned}
I_{\Delta S} \equiv \sum_{i=1,j}^{i=n+1} \int\int_{R_{ij}} \varphi_t \int_{t_j}^{t} &\left\{ g(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u_{\Delta x}^{RP}(\xi,x)), x) - g(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u_{\Delta x}^{RP}(t,x)), x) \right. \\
&- \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij}, \hat{u}(\xi - t, u_{\Delta x}^{RP}(\xi,x))) \cdot \mathbf{A}'_{\Delta x} \\
&\left. + \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij}, \hat{u}(\xi - t, u_{\Delta x}^{RP}(t,x))) \cdot \mathbf{A}'_{\Delta x} \right\} d\xi dx dt.
\end{aligned}
\tag{2.24}
$$

From the total variation bound on the Riemann problems and the smoothness of f, this term is bounded by

$$\begin{aligned} |I_{\Delta S}| &\le \sum_{i=1,j}^{i=n+1} \int\int_{R_{ij}} \|\varphi_t\|_\infty \int_{t_j}^{t} C\ T.V._{[x_{i-1},x_i]}\{u_{\Delta x}(\cdot,t_j)\}\,d\xi dx dt \\ &\le \|\varphi_t\|_\infty C\Delta t^2 \Delta x \sum_j T.V._{[r_{min},r_{max}]}\{u_{\Delta x}(\cdot,t_j)\} \\ &\le CV\,\|\varphi_t\|_\infty \Delta x \Delta t^2 \frac{T}{\Delta t} = O(\Delta x^2), \end{aligned} \tag{2.25}$$

and the above procedure reduces the term to

$$\begin{aligned} -\int\int_{R_{ij}} \varphi_t I^1_{ij}(t,x)dxdt &= I_{\Delta S} - \sum_{i=1,j}^{i=n+1} \int\int_{R_{ij}} \varphi_t \int_{t_j}^{t} \Big\{ g(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u^{RP}_{\Delta x}(\xi,x)),x) \\ &\qquad - \frac{\partial f}{\partial \mathbf{A}}\left(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u^{RP}_{\Delta x}(\xi,x))\right)\cdot \mathbf{A}'_{\Delta x}\Big\}\, dxdt \\ &= O(\Delta x^2) - \sum_{i=1,j}^{i=n+1} \int_{R_i} \Bigg\{ \varphi(t_{j+1},x) \int_{t_j}^{t_{j+1}} \Big[g(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u^{RP}_{\Delta x}(\xi,x)),x) \\ &\qquad - \frac{\partial f}{\partial \mathbf{A}}\left(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u^{RP}_{\Delta x}(\xi,x))\right)\cdot \mathbf{A}'_{\Delta x}\Big]\, d\xi \\ &\qquad - \int_{t_j}^{t_{j+1}} \varphi[g(\mathbf{A}_{ij}, u_{\Delta x}, x) - \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij}, u_{\Delta x})\cdot \mathbf{A}'_{\Delta x}]d\xi \Bigg\}\, dx \\ &= O(\Delta x^2) - \sum_{i=1,j}^{i=n+1} \int_{R_i} \Bigg\{ \varphi(t_{j+1},x) \int_{t_j}^{t_{j+1}} \Big[g(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u^{RP}_{\Delta x}(t_{j+1},x)),x) \\ &\qquad - \frac{\partial f}{\partial \mathbf{A}}\left(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u^{RP}_{\Delta x}(t_{j+1},x))\right)\cdot \mathbf{A}'_{\Delta x}\Big]\, d\xi \Bigg\}\, dt + I_4 + I_5, \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} I_4 \equiv \sum_{i=1,j}^{i=n+1} \int_{R_i} \Bigg\{ &\varphi(t_{j+1},x) \int_{t_j}^{t_{j+1}} \Big[g(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u^{RP}_{\Delta x}(t_{j+1},x)),x) \\ &- g(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u^{RP}_{\Delta x}(\xi,x)),x) - \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u^{RP}_{\Delta x}(t_{j+1},x)))\cdot \mathbf{A}'_{\Delta x} \\ &+ \frac{\partial f}{\partial \mathbf{A}}\left(\mathbf{A}_{ij}, \hat{u}(\xi - t_j, u^{RP}_{\Delta x}(\xi,x))\right)\cdot \mathbf{A}'_{\Delta x}\Big]\, d\xi \Bigg\}\, dx, \end{aligned} \tag{2.27}$$

and

$$I_5 \equiv \sum_{i=1,j}^{i=n+1} \int\int_{R_{ij}} \varphi \left[g(\mathbf{A}_{ij}, u_{\Delta x}, x) - \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij}, u_{\Delta x})\cdot \mathbf{A}'_{\Delta x}\right] dxdt. \tag{2.28}$$

Again by smoothness and the total variation bound, we have

$$
\begin{aligned}
|I_4| &\leq \|\varphi\|_\infty \sum_{i=1,j}^{i=n+1} C \ \ T.V._{[x_{i-1},x_i]}\left\{u_{\Delta x}(\cdot,t_j)\right\}\Delta x\Delta t \\
&\leq \|\varphi\|_\infty C\Delta x\Delta t\sum_j T.V._{[r_{min},r_{max}]}\left\{u_{\Delta x}(\cdot,t_j)\right\} = \|\varphi\|_\infty CV\Delta x\Delta t\frac{T}{\Delta t} = O(\Delta x).
\end{aligned}
\tag{2.29}
$$

Substituting (2.21) and (2.26) into (2.20) along with using (2.9) as an identity leaves us with

$$
\begin{aligned}
\varepsilon = \ & O(\Delta x) + \varepsilon_1 - \sum_{i=1,j}^{i=n+1}\int\int_{R_{ij}}\varphi\frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij},u_{\Delta x})\cdot\mathbf{A}'_{\Delta x}dxdt \\
&+\sum_{i=1,j}^{i=n}\int_{R_j}\varphi(t,x_i)\left\{f(\mathbf{A}_{i+1,j},u^{RP}_{\Delta x}(t,x_i)) - f(\mathbf{A}_{ij},u^{RP}_{\Delta x}(t,x_i))\right\}dt
\end{aligned}
\tag{2.30}
$$

The second sum represents the jump in the flux function f, resulting from the discontinuities in the metric $\mathbf{A}$, and the first sum is the addition to the ODE step (2.7) specifically designed to cancel these jumps in the flux.

To see how the cancelation works, we perform a Taylor expansion on the test function, and we add and subtract terms deviating by order Δx. The first sum in (2.30) is expanded as

$$
\begin{aligned}
&\sum_{i=1,j}^{i=n+1}\int\int_{R_{ij}}\varphi\frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij},u_{\Delta x})\cdot\mathbf{A}'_{\Delta x}dxdt \\
&\quad= \sum_{i=1,j}^{i=n+1}\int\int_{R_{ij}}\varphi(x_i,t)\frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij},u_{\Delta x})\cdot\mathbf{A}'_{\Delta x}dxdt + O(\Delta x) \\
&\quad= \sum_{i=1,j}^{i=n+1}\int_{R_j}\varphi(x_i,t)\int_{R_i}\left\{\frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij},u_{\Delta x})\cdot\mathbf{A}'_{\Delta x} - \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij},u^{RP}_{\Delta x})\cdot\mathbf{A}'_{\Delta x}\right\}dxdt \\
&\quad+ \sum_{i=1,j}^{i=n+1}\int_{R_j}\varphi(x_i,t)\int_{R_i}\left\{\frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij},u^{RP}_{\Delta x})\cdot\mathbf{A}'_{\Delta x} - \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij},u^{RP}_{\Delta x}(x_i,t))\cdot\mathbf{A}'_{\Delta x}\right\}dxdt \\
&\quad+ \sum_{i=1,j}^{i=n+1}\int_{R_j}\varphi(x_i,t)\int_{R_i}\left\{\frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij},u^{RP}_{\Delta x}(x_i,t))\cdot\mathbf{A}'_{\Delta x}\right. \\
&\qquad\qquad\left. - \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{\Delta x}(x+\frac{\Delta x}{2},t_j),u^{RP}_{\Delta x}(x_i,t))\cdot\mathbf{A}'_{\Delta x}\right\}dxdt \\
&\quad+ \sum_{i=1,j}^{i=n+1}\int_{R_j}\varphi(x_i,t)\int_{x_{i-1}}^{x_i}\frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{\Delta x}(x+\frac{\Delta x}{2},t_j),u^{RP}_{\Delta x}(x_i,t))\cdot\mathbf{A}'_{\Delta x}dxdt + O(\Delta x)
\end{aligned}
\tag{2.31}
$$

From the smoothness of f, each of the first three sums in equation (2.31) are $O(\Delta x)$ for the following reasons: the first sum is order Δx from the ODE step in the definition of the approximate solution $u_{\Delta x}$ (2.9), the second sum is order Δx^2 by the total variation bound on solutions to the Riemann problems, and the third sum is order Δx by the Lipschitz continuity of the metric $\mathbf{A}$. After these bounds are established, (2.31) reduces to

$$\begin{aligned}
&\sum_{i=1,j}^{i=n+1} \int\int_{R_{ij}} \varphi \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{ij}, u_{\Delta x}) \cdot \mathbf{A}'_{\Delta x} dx dt \\
&\quad = \sum_{i=1,j}^{i=n+1} \int_{R_j} \varphi(x_i, t) \int_{x_{i-1}}^{x_i} \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}_{\Delta x}(x + \frac{\Delta x}{2}, t_j), u^{RP}_{\Delta x}(x_i, t)) \cdot \mathbf{A}'_{\Delta x} dx dt + O(\Delta x) \\
&\quad = \sum_{i=1,j}^{i=n+1} \int_{R_j} \varphi(x_i, t) \int_{x_{i-1}}^{x_i} \frac{\partial f}{\partial x}(\mathbf{A}_{\Delta x}(x + \frac{\Delta x}{2}, t_j), u^{RP}_{\Delta x}(x_i, t)) dx dt + O(\Delta x) \\
&\quad = \sum_{i=1,j}^{i=n+1} \int_{R_j} \varphi(t, x_i) \left\{ f(\mathbf{A}_{i+1,j}, u^{RP}_{\Delta x}(t, x_{i+})) - f(\mathbf{A}_{ij}, u^{RP}_{\Delta x}(t, x_{i+})) \right\} dt + O(\Delta x).
\end{aligned} \tag{2.32}$$

Plugging this result (2.32) into (2.30) gives us

$$\begin{aligned}
\varepsilon = \; & O(\Delta x) + \varepsilon_1 \\
& - \sum_j \int_{R_j} \varphi(t, x_{n+1}) \left\{ f(\mathbf{A}_{n+2,j}, u^{RP}_{\Delta x}(t, x_{n+1})) - f(\mathbf{A}_{n+1,j}, u^{RP}_{\Delta x}(t, x_{n+1})) \right\} dt,
\end{aligned} \tag{2.33}$$

where one term remains due to the mismatch in the number of terms in the spatial sum. Clearly, this last term is $O(\Delta x)$.

So the residual boils down to

$$\varepsilon(u_{\Delta x}, \mathbf{A}_{\Delta x}, \varphi) = \varepsilon_1(u_{\Delta x}, \mathbf{A}_{\Delta x}, \varphi) + O(\Delta x), \tag{2.34}$$

hence all that remains to show is

$$\varepsilon_1 = \sum_{j \neq 0} \int_{r_{min}}^{r_{max}} \varphi(t_j, x) \left\{ u_{\Delta x}(t_j^+, x) - u_{\Delta x}(t_j^-, x) \right\} dx = O(\Delta x). \tag{2.35}$$

To estimate ε_1, we break up the sum by each time step t_j and define

$$\begin{aligned}
\varepsilon_1^j &\equiv \int_{r_{min}}^{r_{max}} \varphi(t_j, x) \left\{ u_{\Delta x}(t_j^+, x) - u_{\Delta x}(t_j^-, x) \right\} dx \\
&= \sum_i \int_{x_{i-}}^{x_{i+}} \varphi(t_j, x) \left\{ u_{\Delta x}(t_j^+, x) - u_{\Delta x}(t_j^-, x) \right\} dx,
\end{aligned} \tag{2.36}$$

with $x_{i+} \equiv x_{i+\frac{1}{2}}$ and $x_{i-} \equiv x_{i-\frac{1}{2}}$.

Recall from [15], the approximate solution for the new time step t_j^+ is computed by the Godunov step, using averages at the top of each Riemann cell R_{ij}. In particular, the solution at each new time step is

$$u_{\Delta x}(t_j^+, x) \equiv \hat{u}(t_j - t_{j-1}, \bar{u}(t_j), x)) \tag{2.37}$$

where

$$\bar{u}(t_j) \equiv \frac{1}{\Delta x} \int_{x_{i-}}^{x_{i+}} u_{\Delta x}^{RP}(t_j, x) dx \tag{2.38}$$

To finish the proof, a lemma is needed, which is proven at the end. This lemma states that the difference between the solution of the ODE step starting at the cell average average minus the solution starting at a state along the top of the Riemann cell, is bounded by the total variation of the Riemann problem.

Lemma 2.1. *Let $u_{\Delta x}^{RP}$ represent the solution of the Riemann problem in the Riemann cell $R_{i,j-1}$ and $\bar{u}_{\Delta x}(t)$ denote the average of the Riemann problem solution across Riemann cell. Let $\hat{u}$ be the solution obtained by the ODE step (2.7) and φ be a smooth test function. Then the following bound holds*

$$\begin{aligned} &\left| \int_{x_{i-}}^{x_{i+}} \left\{ \hat{u}(t_j - t_{j-1}, \bar{u}_{\Delta x}(t_j), x) - \hat{u}(t_j - t_{j-1}, u_{\Delta x}^{RP}(t_j, x), x) \right\} \varphi(t_j, x) dx \right| \\ &\qquad \leq \; C \left\| \varphi \right\|_\infty \Delta x \Delta t \;\, T.V._{[x_i, x_{i+1}]}\{u_{\Delta x}(t_j, \cdot)\} \end{aligned} \tag{2.39}$$

for some constant C.

Using Lemma 2.1, (2.36) is rewritten as a solution to the ODE step (2.7) to obtain the bound

$$\begin{aligned} \varepsilon_1^j &= \sum_i \int_{x_{i-}}^{x_{i+}} \varphi(t_j, x) \left\{ \hat{u}(t_j - t_{j-1}, \bar{u}(t_j), x) - \hat{u}(t_j - t_{j-1}, u_{\Delta x}^{RP}(t_j, x), x) \right\} dx \\ &\leq C \left\| \varphi \right\|_\infty \Delta x \Delta t \sum_i T.V._{[x_{i-}, x_{i+}]}\{u_{\Delta x}^{RP}(\cdot, t_j)\} \\ &= C \left\| \varphi \right\|_\infty \Delta x \Delta t \;\, T.V._{[r_{min}, r_{max}]}\{u_{\Delta x}^{RP}(\cdot, t_j)\} \end{aligned} \tag{2.40}$$

By the total variation bound on $u_{\Delta x}(t_j, \cdot)$, the residual is bounded by

$$\varepsilon_1 \leq \sum_{j \neq 0} C \left\| \varphi \right\|_\infty \Delta x \Delta t \;\, T.V._{[r_{min}, r_{max}]}\{u_{\Delta x}^{RP}(\cdot, t_j)\} \leq C \frac{T}{\Delta t} \Delta x \Delta t V = O(\Delta x). \tag{2.41}$$

Therefore, $\varepsilon = O(\Delta x)$, as claimed. This completes the proof of Theorem 2.1, once we supply the proof of Lemma 2.1. □

Proof of Lemma 2.1

We use the following preliminary result: given a function defined on a set of points, the difference between the average value of the function, and its value at any other point is bounded by the total variation of the function on the set. We record this in the next lemma:

Lemma 2.2. *Let $u(x)$ be a function on the set $[x_{i-}, x_{i+}]$ and*

$$\bar{u} = \frac{1}{\Delta x} \int_{x_{i-}}^{x_{i+}} u(x) dx \tag{2.42}$$

be the average of u on this set. Then we have

$$|\bar{u} - u(x)| \leq \sup_{x_1, x_2 \in [x_i, x_{i+1}]} |u(x_1) - u(x_2)| \leq T.V._{[x_{i-}, x_{i+}]}\{u(\cdot)\}. \tag{2.43}$$

Now recall the solution to the ODE step has the form:

$$\hat{u}(t_j - t_{j-1}, u^{RP}_{\Delta x}(t_j, x), x) = u^{RP}_{\Delta x}(t, x) + \int_{t_{j-1}}^{t_j} G(\mathbf{A}_{ij}, u^{RP}_{\Delta x}(t, x), x)dt. \tag{2.44}$$

Substituting this into (2.39) gives

$$\begin{aligned}
&\left|\int_{x_{i-}}^{x_{i+}} \left\{\hat{u}(t_j - t_{j-1}, \bar{u}_{\Delta x}(t_j), x) - \hat{u}(t_j - t_{j-1}, u^{RP}_{\Delta x}(t_j, x), x)\right\} \varphi(t_j, x)dx\right| \\
=&\left|\int_{x_{i-}}^{x_{i+}} \left\{(\bar{u}_{\Delta x}(t_j) - u^{RP}_{\Delta x}(t_j, x))\right.\right. \\
&\left.\left. + \int_{t_{j-1}}^{t_j} (G(\mathbf{A}_{ij}, \bar{u}_{\Delta x}(t), x) - G(\mathbf{A}_{ij}, u^{RP}_{\Delta x}(t, x), x))dt\right\} \varphi(t_j, x)dx\right| \\
=&\left|\int_{x_{i-}}^{x_{i+}} \left\{\bar{u}_{\Delta x}(t_j) - u^{RP}_{\Delta x}(t_j, x)\right\} \varphi(t_j, x_i)dx\right. \\
&\left. + \int_{x_{i-}}^{x_{i+}} \int_{t_{j-1}}^{t_j} \left\{G(\mathbf{A}_{ij}, \bar{u}_{\Delta x}(t_j), x) - G(\mathbf{A}_{ij}, u^{RP}_{\Delta x}(t_j, x), x)\right\} dt \;\; \varphi(t_j, x_i)dx\right| \\
&+ O(\Delta x^2),
\end{aligned} \tag{2.45}$$

where the test function in the first term is approximated by a Taylor expansion. By the definition of the average $\bar{u}$, the first term is zero. Thus, by the smoothness of G, the bound (2.39) follows from

$$\begin{aligned}
&\left|\int_{x_{i-}}^{x_{i+}} \left\{\hat{u}(t_j - t_{j-1}, \bar{u}_{\Delta x}(t_j), x) - \hat{u}(t_j - t_{j-1}, u^{RP}_{\Delta x}(t_j, x), x)\right\} \varphi(t_j, x)dx\right| \\
&\qquad \leq \;\; C \left\|\varphi\right\|_\infty \Delta x \Delta t \sup_{x_{i-}<x<x_{i+}} \left\{\left|\bar{u}_{\Delta x}(t_j) - u^{RP}_{\Delta x}(t_j, x)\right|\right\} \\
&\qquad \leq \;\; C \left\|\varphi\right\|_\infty \Delta x \Delta t \;\; T.V._{[x_{i-}, x_{i+}]}\{u_{\Delta x}(t_j, \cdot)\},
\end{aligned} \tag{2.46}$$

where Lemma 2.2 is used to bound the difference between the average and any state in the solution to the Riemann problem. This completes the proof of Lemma 2.1, and therefore the proof of our main result, Theorem 2.1, is complete.

□

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Department of Mathematics, University of California, Davis, CA 95616, USA
E-mail address: temple@math.ucdavis.edu

Department of Mathematics, University of California, Davis, CA 95616, USA
E-mail address: zekius@math.ucdavis.edu