An Instability of the Standard Model of Cosmology Creates the Anomalous Acceleration Without Dark Energy

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Abstract: We clarify and identify the condition for smoothness at the center of spherically symmetric solutions of the Einstein equations, and use the new condition to introduce a new asymptotic ansatz that describes general smooth solutions near the center of symmetry. Applying the ansatz to the $k = 0$, $p = 0$ Friedman approximation to the Standard Model of Cosmology (SM), we prove that smooth perturbations trigger an instability in the SM on the scale of the supernova data. The instability creates a large, under-dense region of accelerated uniform expansion which produces *precisely* the same range of corrections to redshift vs luminosity as are produced by the cosmological constant in the theory of Dark Energy. A universal behavior is exhibited because all spherically symmetric solutions that are smooth at $r = 0$ in Standard Schwarzschild Coordinates (SSC) are characterized by the two dimensional phase portrait of the instability, and according to this phase portrait, all sufficiently small under-dense perturbations evolve to a unique stable rest-point. The instability is triggered by the one parameter family of self-similar waves which the authors previously proposed as possible local time-asymptotic wave patterns for perturbations of the SM at the end of the radiation epoch. A numerical simulation determines a unique wave in the family that accounts for the same Hubble constant and quadratic correction to redshift vs luminosity as in a universe with seventy percent Dark Energy, and the third order correction, .58 larger than that produced by Dark Energy, distinguishes the two theories.

1. Introduction

In this announcement we accomplish the program set out by the first two authors in [25, 20, 21], to evolve a one parameter family of

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GR simple-waves[1] which the authors identified as canonical perturbations of the Standard Model of cosmology (SM)[2] during the radiation epoch of the Big Bang, up through the $p = 0$ epoch to present time. Our purpose is to investigate a possible connection with the observed anomalous acceleration of the galaxies, [20, 13]. The analysis of these waves led us to identify a condition for smoothness at the center of spherically symmetric spacetimes that respects the Einstein evolution equations. The constraint of smoothness provides a new closed ansatz for Taylor expanding smooth spherically symmetric solutions about the center of symmetry.[3] Applying this ansatz to perturbations of the SM when $p = 0$, we discover that smooth spherical perturbations of the SM evolve near the center according to a universal phase portrait in which the SM appears as an unstable saddle rest point, (see Figure 1). The instability is triggered when the pressure drops to $p = 0$, and this mechanism creates large regions of accelerated uniform expansion within Einstein's original theory without the cosmological constant. We prove that these accelerated regions introduce *precisely* the same range of corrections to redshift vs luminosity as are produced by the cosmological constant in the theory of Dark Energy. A universal behavior is exhibited because all sufficiently small perturbations tend time asymptotically to a unique stable rest point where the spacetime is Minkowski. Based on this, we accomplish our initial program by proving that these perturbations are consistent with, and the instability is triggered by, the one parameter family of self-similar waves proposed by the authors in [20], as possible time-asymptotic wave patterns for perturbations of the SM at the end of the radiation epoch. By numerical simulation we identify a unique wave in the family that accounts for the same values of the Hubble constant and quadratic correction to redshift vs luminosity as are implied by the theory of Dark Energy

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[1] By *simple-wave* we mean a perturbation of the Friedman space-time on which the Einstein equations reduce to ODE's, c.f. [6].

[2] Assuming the so-called Cosmological Principle, that the universe is uniform on the largest scale, the evolution of the universe on that scale is described by a Friedman spacetime, [27], which is determined by the equation of state in each epoch. In this paper we let SM denote the approximation to the Standard Model of cosmology without Dark Energy given by the critical $k = 0$ Friedman universe with equation of state $p = \frac{c^2}{3}\rho$ during the radiation epoch, and $p = 0$ thereafter, (c.f. the ΛCDM model with $\Lambda = 0$, [13]). We consider here spherical perturbations from SM, a simplification in which large scale aspherical effects and small scale inhomogeneities are neglected.

[3] This analysis makes no assumptions about solutions far from the center of symmetry. Author's work in [8] shows that solutions with positive velocity can be extended beyond a given radius with arbitrary initial density and velocity profiles.

with $\Omega_\Lambda \approx .7$. A numerical simulation of the third order correction associated with that unique wave establishes a testable prediction that distinguishes this theory from the theory of Dark Energy. Note that earlier attempts to identify an instability in the SM were inconclusive, c.f. [29, 38]. Here we characterize the sought after instability, show it is triggered by a family of simple wave perturbations from the radiation epoch, and use it to identify an alternative, testable mathematical explanation for the anomalous acceleration of the galaxies that does not invoke Dark Energy.

Most of the expansion of the universe before the pressure drops to $p \approx 0$, is governed by the radiation epoch, a period in which the large scale evolution is approximated by the equations of pure radiation. These equations take the form of the relativistic p-system of shock wave theory, and for such highly nonlinear equations, one expects complicated solutions to become simpler. Solutions of the p-system typically decay to a concatenation of *simple waves*, solutions along which the equations reduce to ODE's, [12, 6, 21]. Based on this, together with the fact that large fluctuations from the radiation epoch (like the baryonic acoustic oscillations) are typically spherical, [13], the authors began the program in [25] by looking for a family of spherically symmetric solutions that perturb the SM during the radiation epoch when the equation of state $p = \frac{c^2}{3}\rho$ holds, and on which the Einstein equations reduce to ODE's. In [20, 21], we identified a unique family of such solutions which we refer to as a-waves, parameterized by the so called *acceleration parameter* $a > 0$, normalized so that $a = 1$ is the SM[4], and is the only known family of solutions which both (1) perturb Friedman spacetimes, and (2) reduce the Einstein equations to ODEs, [1, 21, 2]. Since when $p = 0$, *under-densities* relative to the SM are a natural mechanism for creating anomalous *accelerations*, (less matter present to slow the expansion implies a larger expansion rate, [13]), we restrict to the perturbations $a < 1$ which induce under-densities relative to the SM, [20, 21]. Thus our starting hypothesis was that the anomalous acceleration of the galaxies is due to a local under-density relative to the SM, on the scale of the supernova data [4], created by a perturbation that has decayed (locally near the center) to an a-wave, $a < 1$,

[4]This family of waves was first discovered from a different point of view in the fundamental paper [1]. C.f. also the *self-similarity hypothesis* in [2]. As far as we know, our's is the first attempt to connect this family of waves with the anomalous acceleration.

by the end of the radiation epoch.[5] From this starting hypothesis we discovered more.

In this paper we prove the following: (i) The $k = 0$, $p = 0$ Friedman spacetime is *unstable*, and smooth spherical perturbations evolve near the center, according to a *universal phase portrait*, in which the SM appears as an unstable saddle rest point; (ii) A small under-density created by an a-wave at the end of radiation, triggers the formation of a large region of accelerated expansion which extends further and further outward from the center, becoming more flat and more uniform, as time evolves; (iii) Neglecting errors in the measurable quantities at fourth order in fractional distance to the Hubble radius, (c.f. footnote 8 below), this extended region moving outward from the center evolves according to an autonomous system of two ODE's, and is described by a solution trajectory that starts near the unstable rest point corresponding to the SM at the end of radiation, and evolves along the unstable manifold to a unique stable rest point M where the metric is Minkowski. All solutions within the entire domain of attraction, evolve to the rest point M. (iv) We identify the Euclidean coordinate systems in a neighborhood of $r = 0$ which naturally impose smoothness at the center of solutions in SSC. We show that *all* spherically symmetric solutions of the Einstein equations for $p = 0$ that are smooth and locally inertial at $r = 0$ in SSC, are gauge equivalent to solutions characterized by the two dimensional phase portrait of the leading order equations near the center. During the evolution from SM to M, the quadratic correction to redshift vs luminosity (as measured near the center) assumes *precisely the same range of values* as Dark Energy theory. That is, letting

$$H\,d_\ell = \mathrm{z} + Q\mathrm{z}^2 + C\mathrm{z}^3 + O(\mathrm{z}^4) \tag{1.1}$$

denote the relation between redshift factor z and luminosity distance d_ℓ at a given value of the Hubble constant H as measured at the center[6], the value of the quadratic correction Q increases from the SM value $Q = .25$ at the end of radiation, to the value $Q = .5$ as $t \to \infty$. This is precisely the same range of values Q takes on in Dark Energy theory as the fraction Ω_Λ of Dark Energy to classical energy increases from its value of $\Omega_\Lambda \approx 0$ at the end of radiation, to $\Omega_\Lambda = 1$ as $t \to \infty$. This

[5] Since time asymptotic wave patterns typically involve multiple simple waves, we make no hypothesis regarding the space-time far from the center of the a-wave, taking the secondary waves as unknown.

[6] For FRW, Q is determined by the value of the so-called *deceleration parameter* q_0, and C is determined by the *jerk* j, c.f., [13]. The deceleration parameter gives Q through $H_0 d_\ell = \mathrm{z} - \frac{3+q_0}{2}\mathrm{z}^2 + O(\mathrm{z}^3)$, with $q_0 = -10/3 < 0$ in SM.

holds for any $a < 1$ near $a = 1$, and for any value of the cosmological constant $\Lambda > 0$, assuming only that a and Λ both induce a negligibly small correction to the SM value $Q = .25$ at the end of radiation.[7] Indeed, this holds for *any* under-dense perturbation that follows the unstable trajectory of rest point SM into the rest point M, (c.f. Figure 1).

These results are recorded in the following theorem. Here we let present time in a given model denote the time at which the Hubble constant H (as defined in (1.1)) reaches its present measured value $H = H_0$, this time being different in different models.

Theorem 1. *Let $t = t_0$ denote present time in the wave model and $t = t_{DE}$ present time in the Dark Energy[8] model. Then there exists a unique value of the acceleration parameter $\underline{a} = 0.999999426 \approx 1-5.74\times 10^{-7}$ corresponding to an under-density relative to the SM at the end of radiation, such that the subsequent $p = 0$ evolution starting from this initial data evolves to time $t = t_0$ with $H = H_0$ and $Q = .425$, in agreement with the values of H and Q at $t = t_{DE}$ in the Dark Energy model. The cubic correction at $t = t_0$ in the wave theory is then $C = 0.359$, while Dark Energy theory gives $C = -0.180$ at $t = t_{DE}$. The times are related by $t_0 \approx .95\, t_{DE}$.*

In principle, adding acceleration to a model increases the expansion rate H and consequently the age of the universe because it then takes longer for the Hubble constant H to decrease to its present small value H_0. The numerics confirm that the age of the universe well approximates the age obtained by adding in Dark Energy.

We emphasize that t_0, Q and C in the wave model, are determined by a alone. Indeed, the initial data at the end of radiation, which determines the $p = 0$ evolution, depends, at the start, on two parameters: the acceleration parameter a of the self-similar waves, and the initial temperature T_* at which the pressure is assumed to drop to zero. But our numerics show that the dependence on the starting temperature is negligible for T_* in the range $3000^o K \leq T_* \leq 9000^o K$, (the range assumed in cosmology, [13]). Thus for the temperatures appropriate for cosmology, t_0, Q and C are determined by a alone.

[7] We qualify with this latter assumption only because, in Dark Energy theory, the value of Ω_Λ is small but not exactly equal to zero at the end of radiation; and in the wave theory, the value of Q jumps down slightly below $Q = .25$ at the end of radiation before it increases to $Q = .5$ from that value as $t \to \infty$.

[8] By the Dark Energy model we refer to the critical $k = 0$ Friedman universe with cosmological constant, taking the present value $\Omega_\Lambda = .7$ as the best fit to the supernova data among the two parameters (k, Λ),[15, 16].

A measure of the *severity* of the instability created by the $a = \underline{a}$ perturbation of the SM , is quantified by the numerical simulation. For example, comparing the initial density ρ_{wave} for $a = \underline{a}$ at the center of the wave, to the corresponding initial density ρ_{sm} in the SM at the end of radiation $t = t_*$, gives $\frac{\rho_{wave}}{\rho_{sm}} \approx 1 - (1.88) \times 10^{-6} \approx 1$. During the $p = 0$ evolution, this ratio evolves to a *seven-fold* under-density in the wave model relative to the SM by present time, i.e., $\frac{\rho_{wave}}{\rho_{sm}} = 0.144$ at $t = t_0$.

Our wave theory is based on the self-similarity variable $\xi = r/ct < 1$, which we introduce as a natural measure of the outward distance from the center of symmetry $r = 0$ in the inhomogeneous spacetimes we describe in SSC. We call ξ the *fractional distance to the Hubble radius* because $1/ct$ is the Hubble radius in the Friedman spacetime, and t is chosen to be proper time at $r = 0$ in our SSC gauge. Thus it is convenient to define $1/ct$ to be the Hubble radius in our inhomogeneous spacetimes as well. Moreover, the SSC radial variable approximately measures arclength distance at fixed time in our SSC spacetimes when $\xi << 1$, and exactly measures arclength at fixed time in the Friedman spacetime in co-moving coordinates. Thus when $\xi << 1$, ξ tells approximately how far out relative to the Hubble radius an observer at the center of our inhomogeneous spacetimes would conclude an object observed at ξ were positioned, if he mistakenly thought he were in a Friedman spacetime.[9] We show below (c.f. Section 4.2), that if we neglect errors $O(\xi^4)$, and then further neglect small errors between the wave metric and the Minkowski metric (which tend to zero, at that order, with approach to the stable rest point, c.f. (iii) above), and also neglect errors due to relativistic corrections in the velocities of the fluid relative to the center (where the velocity is zero), the resulting spacetime is, like a Friedman spacetime, *independent of the choice of center.* Thus the central region of approximate uniform density at present time $t = t_0$ in the wave model extends out from the center $r = 0$ at $t = 0$ in SSC, to radial values r small enough so that the fractional distance to the Hubble radius $\xi = r/ct_0$ satisfies $\xi^4 << 1$.

The cubic correction C to redshift vs luminosity is a *verifiable* prediction of the wave theory which distinguishes it from Dark Energy theory. In particular, $C > 0$ in the wave model and $C < 0$ in the Dark Energy model implies that the cubic correction *increases* the right hand side of (1.1), (i.e., increases the discrepancy between the observed redshifts and the predictions of the SM) far from the center in the wave theory,

[9]Here ξ is just a measure of distance in SSC, and need not have a precise physical interpretation for $\xi >> 1$, [27, 18, 21].

while it *decreases* the right hand side of (1.1) far from the center in the Dark Energy theory. Now the anomalous acceleration was originally derived from a collection of data points, and the $\Omega_\Lambda \approx .7$ critical FRW spacetime is obtained as the best fit to Friedman spacetimes among the parameters (k, Λ). We understand that the current data is sufficient to provide a value for Q, but not C, [10]. Presently it is not clear to the authors whether or not there are indications in the data that could distinguish $C < 0$ from $C > 0$.

Finally, we remark that the problems we posed and solved in this paper resulted from a self-contained line of reasoning stemming from questions that naturally arose from authors' earlier investigations on incorporating a shock wave into the SM of cosmology, [19, 20, 21]. Other interesting attempts to model the anomalous acceleration by under-density theories based on spherically symmetric solutions represented in Lemaitre-Tolman-Bondi (LTB) assuming $p = 0$ coordinates can be found in [4, 5], and references [37]-[65] of [26], including [28]-[37] listed below. In principle, this involves choosing initial data to match the observations, and proposing the LTB time reversal from this as the cosmological model. Our approach in SSC is significantly different in principle, because we identify a *mechanism*, (the instability), by which the redshift vs luminosity data is generically created. (C.f [29] for earlier attempts to identify an instability in a *long wavelength* limit.) We begin in the the next section by explaining our views on why we believe our under-density theory is clear and definitive in SSC, but would be at best problematic to accomplish in LTB. This is further discussed in the Appendix. For the analysis here, we assume the existence of a smooth solution of the Einstein equations representing a large underdense region that expands outward from the end of radiation out to present time, and obtain rigid constraints on such solutions by expanding about the center. No assumption except existence is made about the solution far from the center. In forthcoming work the authors will address the mathematical existence problem along the lines set out in [8].

In Section 2 we explain how to impose smoothness at the center $r = 0$ in SSC, and explain the difficulties this overcomes in LTB coordinates. This is further developed in the Appendix. In Section 3.1 we derive an alternative formulation of the $p = 0$ Einstein equations in spherical symmetry, and prove that the evolution preserves smoothness. In Section 4.1 we introduce our new asymptotic ansatz for corrections to the SM which are consistent with the condition at $r = 0$ for smooth solutions derived in Section 2. In Section 4.2 we use the exact equations together with our ansatz to derive asymptotic equations for the corrections, and use these to derive the universal phase portrait. In Section 4.3 we derive the correct redshift vs luminosity relation for the SM including the corrections. In Section 4.4 we introduce a gauge transformation that converts the a-waves at the end of radiation into initial data that is consistent with our ansatz. In Section 4.5 we present our numerics that identifies the unique a-wave $a = \underline{a}$ in the family that meets the conditions $H = H_0$ and $Q = .425$ at $t = t_0$, and explain our predicted cubic correction $C = 0.359$. In Section 4.6 we discuss the uniform space-time created at the center of the perturbation. Concluding remarks are given in Section 5. Details are omitted in this announcement. We use the convention $c = 1$ when convenient.

2. The Advantages of SSC over LTB

The discovery of the instability in SM in this paper relies on the derivation of a new phase portrait for smooth spherical perturbations of the critical $k = 0$, $p = 0$ Friedman spacetime near the center of symmetry. Earlier attempts to explain the anomalous acceleration by spherical under-dense perturbations of Friedman expressed in LTB coordinates failed to identify this universal phase portrait. We start by explaining why we were able to accomplish this in SSC coordinates, but not in LTB. The answer is that smoothness is imposed at the center by the condition that all odd order derivatives of metric components and functions vanish at $r = 0$ in SSC, and this is a complicated solution dependent condition in LTB. (This condition first emerged when we Taylor expanded a-waves at $r = 0$ in SSC, [20, 21].)

The problem centers around the validity of approximating solutions by finite Taylor expansions about the center of symmetry, so the main issue is to guarantee that our solutions are indeed smooth in a neighborhood of the center. Of course the universe is not smooth on small scales, so our assumption is simply that the center is not special regarding the level of smoothness assumed in the large scale approximation of the universe. Smoothness at a point P in a spacetime manifold is determined by the atlas of coordinate charts defined in a neighborhood of P, the smoothness of tensors being identified with the smoothness of the tensor components as expressed in the coordinate systems of the atlas. The problem with using spherical coordinates like LTB and SSC in GR is that $r = 0$ is a coordinate singularity, and functions are defined only for radial coordinate $r \geq 0$, but a coordinate system must be specified in a *neighborhood* of $r = 0$ to impose the conditions for smoothness at the center.

We begin by showing that this issue can be resolved relatively easily in SSC because the SSC coordinates are precisely the spherical coordinates associated with Euclidean *coordinate charts* defined in a neighborhood of $r = 0$. Based on this, we show below that the condition for smoothness of metric components and functions in SSC is simply that all odd derivatives should vanish at $r = 0$. Identifying these Euclidean coordinate charts and the associated condition within the LTB framework is problematic because the radial coordinate in LTB is taken to be co-moving with the fluid, and so unlike SSC, the LTB radial coordinate is not associated with any Euclidean coordinate chart defined in a neighborhood of $r = 0$. Thus unlike SSC, it is problematic to express a condition for smoothness in terms of derivatives at $r = 0$ in LTB coordinates, because such a criterion would be solution dependent. We believe that this has been the main obstacle to prior attempts to establish a connection between under-density models and the anomalous acceleration in LTB spacetimes. Indeed, previous attempts to model the anomalous acceleration in LTB have confronted the problem that models with a negative deceleration parameter consistent with the supernova data, all exhibit a "central weak singularity" at $r = 0$, c.f. [29, 38, 37]. Since no criterion is given to distinguish smooth from non-smooth solutions based on derivatives at $r = 0$ in LTB coordinates, it is almost impossible to untangle the essential from the removable singularities at $r = 0$ in LTB. Allowing "singularities" at $r = 0$ in the LTB spacetimes has the problem of introducing a plethora of new non-smooth solutions of the Einstein equations that obscure the existence of the simple phase portrait which applies only to the *actual* smooth perturbations which would appear without singularities if expressed in SSC instead of LTB.

Consider now in more detail the problem of representing a smooth, spherically symmetric perturbation of the Friedman spacetime in GR. To start, assume the existence of a solution of Einstein's equations representing a large, smooth under-dense region of spacetime that expands from the end of radiation out to present time. For smooth perturbations, there should exist a coordinate system in a neighborhood of the center of symmetry, in which the solution is represented as smooth. Assume we have such a coordinate system $(t, \mathbf{x}) \in \mathcal{R} \times \mathcal{R}^3$ with $\mathbf{x} = 0$ at the center, and use the notation $x = (x^0, x^1, x^2, x^3) \equiv (t, \mathbf{x})$, $\mathbf{x} \equiv (x, y, z)$, (there should be no confusion with the ambiguity in x). Spherical symmetry makes it convenient to represent the spatial Euclidean coordinates $\mathbf{x} \in \mathcal{R}^3$ in spherical coordinates (r, θ, ϕ), with $r = |\mathbf{x}|$. Since generically, any spherically symmetric metric can be transformed locally to SSC form, [18], we assume the spacetime represented in the coordinate system (t, r, θ, ϕ) takes the SSC form

$$ds^2 = -B(r,t)dt^2 + \frac{dr^2}{A(r,t)} + r^2 d\Omega^2. \tag{2.2}$$

This is equivalent to the metric in Euclidean coordinates $\mathbf{x}$ taking the form

$$ds^2 = -B(|\mathbf{x}|,t)dt^2 + \frac{dr^2}{A(|\mathbf{x}|,t)} + |\mathbf{x}|^2 d\Omega^2, \tag{2.3}$$

with

$$dr^2 = \frac{x^2dx^2 + y^2dy^2 + z^2dz^2}{\sqrt{x^2+y^2+z^2}}, \tag{2.4}$$

and

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\Omega^2. \tag{2.5}$$

To guarantee the smoothness of our perturbations of Friedman at the center, we assume a gauge in which

$$B(t,r) = 1 + O(r^2), \tag{2.6}$$
$$A(t,r) = 1 + O(r^2), \tag{2.7}$$

so also

$$\frac{1}{A(t,r)} = 1 + O(r^2). \tag{2.8}$$

This sets the SSC time gauge to proper geodesic time at $r = 0$, and makes the SSC coordinates locally inertial at $r = 0$ at each time $t > 0$, a first step in guaranteeing that the spherical perturbations of Friedman which we study, are smooth at the center. Keep in mind that the SSC form is invariant under arbitrary transformation of time, so we are free to choose geodesic time at $r = 0$; and the locally inertial condition at $r = 0$ simply imposes that the corrections to Minkowski at $r = 0$ are second order in r. (These assumptions make physical sense, and their consistency is guaranteed by reversing the steps in the argument to follow.) In particular, the SSC metric (2.2) tends to Minkowski at $r = 0$. We now ask what conditions on the metric functions A, B are imposed by assuming the SSC metric be *smooth* when expressed in our *original* Euclidean coordinate chart $(t, \mathbf{x})$ defined in a neighborhood of a point at $r = 0$, $t > 0$.

Transforming the SSC metric (2.2) to $(t, \mathbf{x})$ coordinates and using (2.3)-(2.7) gives

$$\begin{aligned} ds^2 &= -B(|\mathbf{x}|,t)dt^2 + dx^2 + dy^2 + dz^2 \\ & \frac{1}{r^2}\left(\frac{1}{A(|\mathbf{x}|,t)} - 1\right)\left(x^2dx^2 + y^2dy^2 + z^2dz^2 + 2xydxdy\right). \end{aligned} \tag{2.9}$$

Thus the smoothness of A and B guarantees the smoothness of the Euclidean spacetime metric (2.9) in $(t, \mathbf{x})$ coordinates everywhere except at $\mathbf{x} = 0$, and for smoothness at $\mathbf{x} = 0$, the $|\mathbf{x}|$ requires that the metric functions A and B satisfy the condition that all *odd* r-derivatives vanish at $r = 0$. To see this, observe that a function $f(r)$ represents a smooth spherically symmetric function of the Euclidean coordinates $\mathbf{x}$ at $r = |\mathbf{x}| = 0$ if and only if the function

$$g(x) = f(|\mathbf{x}|)$$

is smooth at $\mathbf{x} = 0$. Assuming f is smooth for $r \geq 0$, (by which we mean f is smooth for $r > 0$, and one sided derivatives exist at $r = 0$), and taking the n'th derivative of g from the left and right and setting them equal gives the smoothness condition $f^n(0) = (-1)^n f^n(0)$. We state this formally as:

Lemma 1. *A function $f(r)$ of the radial coordinate $r = |\mathbf{x}|$ represents a smooth function of the underlying Euclidean coordinates $\mathbf{x}$ if and only if f is smooth for $r \geq 0$, and all odd derivatives vanish at $r = 0$. Moreover, if any odd derivative $f^{(n+1)}(0) \neq 0$, then $f(|\mathbf{x}|)$ has a jump discontinuity in its $n+1$ derivative, and hence a kink singularity in its n'th derivative at $r = 0$.*

As an immediate consequence we obtain the condition for smoothness of SSC metrics at $r = 0$:

Corollary 1. *The SSC metric (2.2) is smooth at $r = 0$ in the sense that the metric components in (2.9) are smooth functions of the Euclidean coordinates $(t, \mathbf{x})$ if and only if the component functions $A(r,t)$, $B(r,t)$ are smooth in time and smooth for $r > 0$, all odd one-sided r-derivatives vanish at $r = 0$, and all even r-derivatives are bounded at $r = 0$.*

To conclude, solutions of the Einstein equation in SSC have four unknowns, the metric components A, B, the density ρ and the scalar velocity v. It is easy to show that if the SSC metric components satisfy the condition that all odd order r-derivatives vanish at $r = 0$, then the components of the unit 4-velocity vector u associated with smooth curves that pass through $r = 0$ will have the same property[10], and the

[10] This implies that the coordinates are smooth functions of arclength along curves passing through $r = 0$.

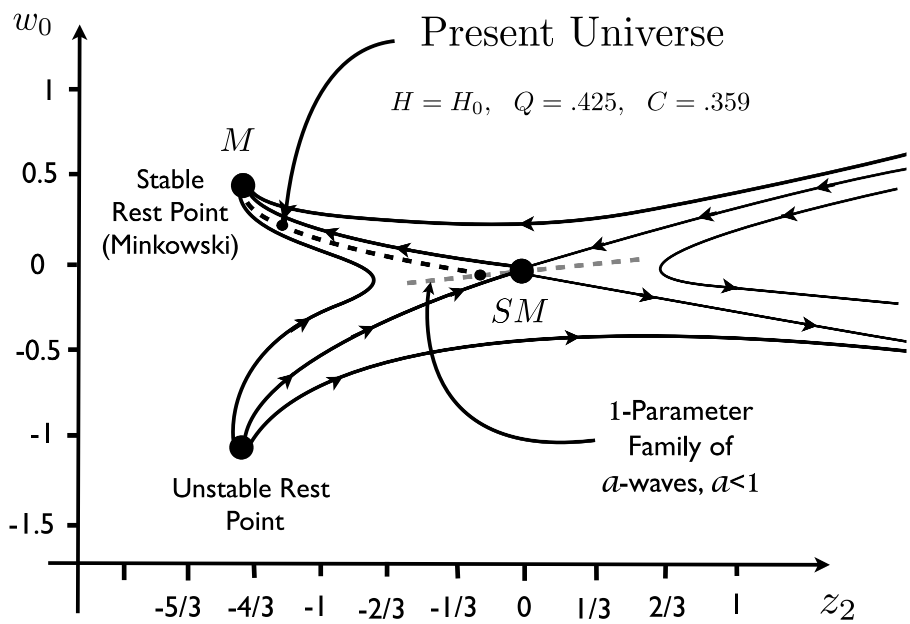
scalar velocity $v = \frac{1}{\sqrt{AB}}\frac{dr}{dt}$ will have the property that all even derivatives vanish at $r = 0$ (because v is an outward velocity which picks up a change of sign when represented in x). Thus smoothness of SSC solutions at $r = 0$ at fixed time is equivalent to requiring that the metric components satisfy the condition that all odd r-derivatives vanish at $r = 0$. These then give conditions on SSC solutions equivalent to the condition that the solutions are smooth in the ambient Euclidean coordinate systems x. Theorem 2 of Section 3.1 below proves that smoothness in the coordinate system x at $r = 0$ at each time in this sense is preserved by the Einstein evolution equations for SSC metrics when $p = 0$. In particular, this demonstrates that our condition for smoothness of SSC metrics at $r = 0$ is equivalent to the well-posedness of solutions in the ambient Euclidean coordinates defined in a neighborhood of $r = 0$. Thus we obtain the condition for smoothness of SSC metrics at $r = 0$ based on the Euclidean coordinate systems associated with SSC, and show this is preserved by the evolution of the Einstein equations. Since smoothness of the SSC metric components in this sense is equivalent to smoothness of the x-coordinates with respect to arclength along curves passing through $r = 0$, in this sense, our condition for smoothness is *geometric*.

Now note that we could just as well start with an SSC metric (2.2) satisfying (2.8), (2.7), and then take $\mathbf{x}$ to any Euclidean coordinate system $x = (x^0, x^1, x^2, x^3)$ which satisfies $r = |\mathbf{x}|$ and $dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\Omega^2$, so that $\mathbf{x}$ is determined to within a rotation. The point is that the connection between the Euclidean coordinate system and the spherical coordinate system employed in SSC is a *Euclidean* coordinate relationship, independent of the spacetime manifold represented in that coordinate system. In LTB coordinates, it is difficult to untangle the natural Euclidean coordinate system defined in a neighborhood of $r = 0$ because the radial LTB coordinate is not the radial coordinate of any Euclidean coordinate system, and the mapping to SSC depends on the solution. Thus the condition on derivatives of functions at $r = 0$ in LTB that guarantee the SSC condition for smoothness would depend on solutions as well. This might explain why it has proven difficult to express the condition of "smoothness at the center" in the literature on LTB spacetimes, and why SSC is crucial to our discovery of the instability of SM.

3. Presentation of Results

We begin by recalling that a-waves form a 1-parameter family of spherically symmetric solutions of the Einstein equations $G = \kappa T$ that

Figure 1. Phase Portrait for Central Region



depend only on the self-similarity variable $\xi = r/t$, and exist when $p = \frac{c^2}{3}\rho$. They reduce to the critical SM Friedman spacetime for pure radiation when $a = 1$. In contrast, no such family of self-similar perturbations of SM exist when $p = 0$, and only the SM $k = 0$ Friedman spacetime itself can be expressed in self-similar form when $p = 0$. Thus to evolve the self-similar waves into the $p = 0$ epoch, we Taylor expand the solutions in even powers of ξ about the center in SSC, using the fact of Lemma 1, (and Theorem 2 below), that the vanishing of odd powers is equivalent to the smoothness of solutions at the center. This ansatz is sufficiently general to incorporate initial data from the self-similar waves at the end of radiation, and we deduce the evolution of the corrections induced by a-waves at the end of radiation near the center from the phase portrait of the resulting asymptotic equations. In fact, our results apply not just to perturbations by a-waves, but to any perturbation consistent with our asymptotic ansatz at the end of radiation, so long as the perturbation lies within the domain of attraction of the stable rest point to which the perturbation $a = \underline{a}$ evolves.

3.1. **The $p = 0$ Einstein Equations in Coordinates Aligned with the Physics.** In this section we introduce a new formulation of the $p = 0$ Einstein equations that describe outwardly expanding spherically symmetric solutions. We do not employ co-moving coordinates, [4], but rather use ξ as a spacelike variable because it is better aligned with the physics. That is, our derivation starts with metrics in Standard Schwarzschild Coordinates (SSC), where the metric takes the canonical form,

$$ds^2 = -B(t,r)dt^2 + \frac{1}{A(t,r)}dr^2 + r^2 d\Omega^2, \tag{3.10}$$

but our subsequent analysis is done in (t, ξ) coordinates, where $\xi = r/t$. Our starting point is the observation that the SSC metric form is invariant under transformations of t, and there exists a time coordinate in which SM is self-similar in the sense that the metric components A, B, the velocity v and ρr^2 are functions of ξ alone. This self-similar form exists, but is different for $p = \frac{c^2}{3}\rho$ and $p = 0$, [2, 22]. Taking $p = 0$, letting v denote the SSC velocity and ρ the co-moving energy density, and eliminating all unknowns in terms of v and the Minkowski energy density $T^{00}_M = \frac{\rho}{1-\left(\frac{v}{c}\right)^2}$, (c.f. [8]), the locally inertial formulation of the Einstein equations $G = \kappa T$ introduced in [8] reduce to

$$\begin{aligned}
(\kappa T^{00}_M r^2)_t + \left\{ \sqrt{AB}\tfrac{v}{r}\left(\kappa T^{00}_M r^2\right)\right\}_r &= -2\sqrt{AB}\tfrac{v}{r}\left(\kappa T^{00}_M r^2\right), \\
\left(\tfrac{v}{r}\right)_t + r\sqrt{AB}\left(\tfrac{v}{r}\right)\left(\tfrac{v}{r}\right)_r &= -\sqrt{AB}\left\{\left(\tfrac{v}{r}\right)^2 + \tfrac{1-A}{2Ar^2}\left(1 - r^2\left(\tfrac{v}{r}\right)^2\right)\right\}, \\
r\tfrac{A'}{A} &= \left(\tfrac{1}{A} - 1\right) - \tfrac{1}{A}\kappa T^{00}_M r^2, \\
r\tfrac{B'}{B} &= \left(\tfrac{1}{A} - 1\right) + \tfrac{1}{A}\left(\tfrac{v}{c}\right)^2 \kappa T^{00}_M r^2,
\end{aligned}$$

where prime denotes d/dr. Note that the $1/r$ singularity is present in the equations because incoming waves can amplify without bound. We resolve this for outgoing expansions by assuming $w = v/\xi$ is positive and finite at $r = \xi = 0$. Making the substitution $D = \sqrt{AB}$, taking $z = \kappa T^{00}_M r^2$ as the dimensionless density, $w = \frac{v}{\xi}$ as the dimensionless velocity with $\xi = r/t$ and rewriting the equations in terms of (t, ξ), we obtain

$$tz_t + \xi \left\{(-1 + Dw)z\right\}_\xi = -Dwz, \tag{3.11}$$

$$tw_t + \xi \left(-1 + Dw\right) w_\xi = w - D \left\{w^2 + \tfrac{1-\xi^2 w^2}{2A} \left[\tfrac{1-A}{\xi^2}\right]\right\} \tag{3.12}$$

$$\xi A_\xi = (1 - A) - z \tag{3.13}$$

$$\tfrac{\xi D_\xi}{D} = \tfrac{1}{A} \left\{(1 - A) - \tfrac{\left(1-\xi^2 w^2\right)}{2} z\right\}. \tag{3.14}$$

That is, since the sound speed is zero when $p = 0$, $w(t, 0) > 0$ restricts us to expanding solutions in which all information from the fluid propagates outward from the center.

4. Smoothness of solutions in the ambient Euclidean coordinate system in a neighborhood of $r = 0$

In this section we prove that the ambient Euclidean coordinate system $x = (x^0, x^1, x^2, x^3) = (t, x, y, z)$ associated with spherical SSC coordinates is preserved by the evolution of the Einstein equations. By Lemma 1, smoothness of SSC solutions at $r = 0$ is imposed by the condition that odd order r-derivatives of the metric components and the density vanish at $r = 0$, and even derivatives of the velocity v vanish at $r = 0$. Since $\xi = r/t$, imposing this condition on r-derivatives at $t > 0$ is equivalent to imposing it on ξ-derivatives, and since $w = \xi v$, $D = \sqrt{AB}$, $z = \rho r^2$, smoothness at $r = 0$ is equivalent to the condition that all odd derivatives of (z, w, A, D) vanish at $\xi = 0$, $t > 0$. The following theorem establishes that smoothness in the ambient coordinate system x is preserved by the evolution of the Einstein equations in SSC, and hence that SSC solutions meeting this condition are well posed in x.

Theorem 2. *Assume $z(t, \xi), w(t, \xi), A(t, \xi), D(t, \xi)$ are a given smooth solution of our $p = 0$ equations (3.11)-(3.14) satisfying*

$$z = O(\xi^2), \quad w = w_0(t) + O(\xi^2), \tag{4.15}$$

$$A = 1 + O(\xi^2), \quad D = 1 + O(\xi^2), \tag{4.16}$$

for $0 < t_0 \le t < t_1$, and assume that at $t = t_0$ the solution agrees with initial data

$$z(t_0, \xi) = \bar{z}(\xi),\ w(t_0, \xi) = \bar{w}(\xi), \tag{4.17}$$

$$A(t_0, \xi) = \bar{A}(\xi),\ D(t_0, \xi) = \bar{D}(\xi) \tag{4.18}$$

such that each initial data function $\bar{z}(\xi)$, $\bar{w}(\xi)$, $\bar{A}(\xi)$, $\bar{D}(\xi)$ satisfies the condition that all odd ξ-derivates vanish at $\xi = 0$. Then all odd ξ-derivatives of the solultion $z(t,\xi), w(t,\xi), A(t,\xi), D(t,\xi)$ vanish at $\xi = 0$ for all $t_0 < t < t_1$.

Proof: Start with equations (3.11)-(3.14) in the form

$$\begin{aligned} tz_t &= -\xi \left\{(-1+Dw)z\right\}_\xi - Dwz, && (4.19)\\ tw_t &= -\xi\,(-1+Dw)\,w_\xi + w && (4.20)\\ &\qquad -D\left\{w^2 + \frac{1-A}{2A\xi^2}(1-\xi^2 w^2)\right\}, \\ \xi A_\xi &= (1-A) - z, && (4.21)\\ \xi D_\xi &= \frac{D}{2A}\left\{2(1-A) - z + \xi^2 w^2 z\right\}. && (4.22) \end{aligned}$$

First note that products and quotients of smooth functions that satisfy the condition that all odd derivatives vanish at $\xi = 0$, also have this property. Now for a function $F(t,\xi)$, let $F_\xi^{(n)}(t)$ denote the n'th partial derivative of F with respect to ξ at $\xi = 0$. We prove the theorem by induction on n. For this, assume $n \geq 1$ is odd, and make the induction hypothesis that for all odd $k < n$, $F_\xi^{(k)}(t) = 0$ for all $t \geq t_0$ and all functions $F = z, w, A, D$, (functions of (t,ξ)). We prove that $F_\xi^{(n)}(t) = 0$ for $t > t_0$. For this we employ the following simple observation: If n is odd, and the n'th derivative of the product of m functions

$$\frac{\partial^n}{\partial \xi^n}(F_1 \cdots F_m)$$

is expanded into a sum by the product rule, the only terms that will not have a factor containing an odd derivative of order less than n are the terms in which all the derivatives fall on the same factor. This follows from the simple fact that if the sum of k integers is odd, then at least one of them must be odd. Taking the n'th derivative of (4.19) and setting $\xi = 0$ gives the ODE at $\xi = 0$:

$$t\frac{d}{dt} z_\xi^{(n)} = -n\frac{\partial^n}{\partial \xi^n}\left((-1+Dw)z\right) - \frac{\partial^n}{\partial \xi^n}\left(DWz\right). \qquad (4.23)$$

Since all odd derivatives of order less than n are assumed to vanish at $\xi = 0$, we can apply the observation and the assumptions (4.15), (4.16) that $D = 1$, $w = w_0(t)$ and $z = 0$ at $\xi = 0$, to see that only the n'th order derivative $z_\xi^{(n)}$ survives on the RHS of (4.23). That is, by the induction hypothesis, (4.23) reduces to

$$t\frac{d}{dt}z_\xi^{(n)} = \left[n - (n+1)w_0(t)\right] z_\xi^{(n)}. \tag{4.24}$$

Since under the change of variable $t \to \ln(t)$, (4.24) is a linear first order homogeneous ODE in $z_\xi^{(n)}(t)$ with $z_\xi^{(n)}(t_0) = 0$, it follows by uniqueness of solutions that $z_\xi^{(n)}(t) = 0$ for all $t \geq t_0$. This proves the theorem for the solution component $z(t, \xi)$.

Consider next equation (4.21). Differentiating both sides n times with respect to ξ and setting $\xi = 0$ gives

$$(n+1)A_\xi^{(n)}(t) = -z_\xi^{(n)}(t) = 0, \tag{4.25}$$

thus

$$A_\xi^{(n)}(t) = 0 \tag{4.26}$$

for $t \geq t_0$, which verifyies the theorem for component A.

Consider equation (4.22). Differentiating both sides n times with respect to ξ, setting $\xi = 0$ and applying the observation and the induction hypothesis gives

$$\begin{aligned} nD_\xi^{(n)} &= \frac{\partial^n}{\partial \xi^n}\left(D\frac{1-A}{A}\right) \\ &= D_\xi^{(n)}\left(\frac{1-A}{A}\right) + \sum_{k<n\ odd} c_k D_\xi^{(k)} + D\left(\frac{1-A}{A}\right)_\xi^{(n)} \\ &= 0 \end{aligned} \tag{4.27}$$

for $t \geq t_0$ because $A = 1$ at $\xi = 0$, all lower order odd derivatives are assumed to vanish at $\xi = 0$, and we have already verified the theorem for the component A. This proves

$$D_\xi^{(n)}(t) = 0 \tag{4.28}$$

for $t \geq t_0$, verifying the theorem for component D.

Consider lastly the equation (4.21). Differentiating both sides n times with respect to ξ, setting $\xi = 0$ and applying our observation gives

$$\begin{aligned} t\frac{d}{dt}w_\xi^{(n)} &= -n(-1+w_0(t))w_\xi^{(n)} + w_\xi^{(n)} - \frac{\partial^n}{\partial\xi^n}\left(w^2\right) \\ &= -n(-1+w_0(t))w_\xi^{(n)} + w_\xi^{(n)} - 2ww_\xi^{(n)} \\ &= \left[-n(-1+w_0(t)) + 1 - 2w\right] w_\xi^{(n)} \end{aligned} \tag{4.29}$$

for $t \geq t_0$ because $A = 1$ and $\xi = 0$, all lower order odd derivatives are assume to vanish at $\xi = 0$, and we have established the theorem for the component A. Thus $w_\xi^{(n)}(t)$ solves the first order homogeneous ODE

$$t\frac{d}{dt}w_\xi^{(n)} = \left[-n(-1 + w_0(t)) + 1 - 2w\right] w_\xi^{(n)}, \tag{4.30}$$

starting from zero initial data at $t = t_0$, so again we conclude

$$w_\xi^{(n)}(t) = 0 \tag{4.31}$$

for $t \geq t_0$. This verifies the theorem for the final component w, thereby completing the proof of Theorem 2.□

4.1. **A New Ansatz for Corrections to SM.** In this section we derive the phase portrait which describes any spherical perturbation of the $k = 0$, $p = 0$ Friedman spacetime which is smooth in SSC coordinates. Our condition for smooth solutions is that (z, w, A, B) are smooth functions away from $\xi = 0$, all time derivatives are smooth, and all odd ξ-derivatives vanish at $\xi = 0$. Since solutions are assumed smooth at $\xi = 0$, $t > 0$, Taylor's theorem is valid at $\xi = 0$, so the following ansatz for corrections to SM near $\xi = 0$ is valid in a neighborhood of $\xi = 0$, $t > 0$, with errors bounded by derivatives of the corresponding functions at the corresponding orders.

$$z(t,\xi) = z_{sm}(\xi) + \Delta z(t,\xi) \qquad \Delta z = z_2(t)\xi^2 + z_4(t)\xi^4 \tag{4.32}$$

$$w(t,\xi) = w_{sm}(\xi) + \Delta w(t,\xi) \qquad \Delta w = w_0(t) + w_2(t)\xi^2 \tag{4.33}$$

$$A(t,\xi) = A_{sm}(\xi) + \Delta A(t,\xi) \qquad \Delta A = A_2(t)\xi^2 + A_4(t)\xi^4 \tag{4.34}$$

$$D(t,\xi) = D_{sm}(\xi) + \Delta D(t,\xi) \qquad \Delta D = D_2(t)\xi^2 \tag{4.35}$$

where $z_{sm}, w_{sm}, A_{sm}, D_{sm}$ are the expressions for the unique self-similar representation of the SM when $p = 0$, given by, [22],

$$z_{sm}(\xi) = \tfrac{4}{3}\xi^2 + \tfrac{40}{27}\xi^4 + O(\xi^6), \qquad w_{sm}(\xi) = \tfrac{2}{3} + \tfrac{2}{9}\xi^2 + O(\xi^4), \tag{4.36}$$

$$A_{sm}(\xi) = 1 - \tfrac{4}{9}\xi^2 - \tfrac{8}{27}\xi^4 + O(\xi^6), \quad D_{sm}(\xi) = 1 - \tfrac{1}{9}\xi^2 + O(\xi^4). \tag{4.37}$$

This gives

$$\begin{aligned} z(t,\xi) &= \left(\frac{4}{3} + z_2(t)\right)\xi^2 + \left\{\frac{40}{27} + z_4(t)\right\}\xi^4 + O(\xi^6), \\ w(t,\xi) &= \left(\frac{2}{3} + w_0(t)\right) + \left\{\frac{2}{9} + w_2(t)\right\}\xi^2 + O(\xi^4). \end{aligned}$$

Consistent with Theorem 2, we verify the equations close within this ansatz, at order ξ^4 in z and order ξ^2 in ws with errors $O(\xi^6)$ in z and $O(\xi^4)$ in w. Corrections expressed in this ansatz create a uniform spacetime of density $\rho(t)$, constant at each fixed t, out to errors of order $O(\xi^4)$. That is, since the ansatz,

$$z(\xi,t) = \kappa\rho(t,\xi)r^2 + O(\xi^4) = \left(\frac{4}{3} + z_2(t)\right)\xi^2 + O(\xi^4), \tag{4.38}$$

neglecting the $O(\xi^4)$ error gives $\kappa\rho = (4/3 + z_2(t))/t^2$, a function of time alone. For the SM , $z_2 \equiv 0$ and this gives $\kappa\rho(t) = (4/3)\,t^{-2}$, which is the exact evolution of the density for the SM Friedman spacetime with $p = 0$ in co-moving coordinates, [18]. For the evolution of our specific under-densities in the wave theory, we show $z_2(t) \to -4/3$ as the solution tends to the stable rest point, implying that the instability creates an accelerated drop in the density in a large uniform spacetime expanding outward from the center. (C.f. Section 4.6 below.)

4.2. **Asymptotic equations for Corrections to SM.** Substituting the ansatz (4.32)-(4.35) for the corrections into the Einstein equations $G = \kappa T$, and neglecting terms $O(\xi^4)$ in w and $O(\xi^6)$ in z, we obtain the following closed system of ODE's for the corrections $z_2(\tau))$, $z_4(\tau)$, $w_0(\tau)$, $w_2(\tau)$, where $\tau = \ln t$, $0 < \tau \leq 11$. (Introducing τ renders the equations autonomous, and solves the long time simulation problem.) Letting prime denote $d/d\tau$, the equations for the corrections reduce to the autonomous system

$$z_2' = -3w_0\left(\frac{4}{3} + z_2\right), \tag{4.39}$$

$$w_0' = -\frac{1}{6}z_2 - \frac{1}{3}w_0 - w_0^2, \tag{4.40}$$

$$z_4' = 5\left\{\frac{2}{27}z_2 + \frac{4}{3}w_2 - \frac{1}{18}z_2^2 + z_2w_2\right\} + 5w_0\left\{\frac{4}{3} - \frac{2}{9}z_2 + z_4 - \frac{1}{12}z_2^2\right\}, \tag{4.41}$$

$$w_2' = -\frac{1}{10}z_4 - \frac{4}{9}w_0 + \frac{1}{3}w_2 - \frac{1}{24}z_2^2 + \frac{1}{3}z_2w_0 + \frac{1}{3}w_0^2 - 4w_0w_2 + \frac{1}{4}w_0^2z_2. \tag{4.42}$$

We prove that for the equations to close within the ansatz (4.32)-(4.35), it is necessary and sufficient to assume the initial data satisfies the gauge conditions

$$A_2 = -\frac{1}{3}z_2, \quad A_4 = -\frac{1}{5}z_4, \quad D_2 = -\frac{1}{12}z_2. \tag{4.43}$$

We prove that if these constraints hold initially, then they are maintained by the equations for all time. Conditions (4.43) are not invariant under time transformations, even though the SSC metric form is invariant under arbitrary time transformations, so we can interpret (4.43), and hence the ansatz (4.32)-(4.35), as fixing the time coordinate gauge of our SSC metric. This gauge agrees with FRW co-moving time up to errors of order $O(\xi^2)$.

The autonomous 4×4 system (4.39)-(4.42) contains within it the closed, autonomous, 2×2 sub-system (4.39), (4.40). This sub-system describes the evolution of the corrections (z_2, w_0), which we show in Section 4.3 determines the quadratic correction Qz^2 in (1.1). Thus the sub-system (4.39), (4.40) gives the corrections to SM at the order of the observed anomalous acceleration, accurate within the central region where errors $O(\xi^4)$ in z and orders $O(\xi^3)$ in $v = w/\xi$ can be neglected. The phase portrait for sub-system (4.39), (4.40) exhibits an unstable saddle rest point at $SM = (z_2, w_0) = (0, 0)$ corresponding to the SM, and a stable rest point at $(z_2, w_0) = (-4/3, 1/3)$. These are the rest points referred to in the introduction. From the phase portrait, (see Figure 1), we see that perturbations of SM corresponding to small under-densities will evolve away from the SM near the unstable manifold of $(0, 0)$, and toward the stable rest point M. By (4.36) and (4.37), $A_2 = 4/9, D_2 = 1/9$ at $(z_2, w_0) = (-4/3, 1/3)$, so by (4.37) the metric components A and B are equal to $1 + O(\xi^4)$, implying the metric at the stable rest point $(-4/3, 1/3)$ is Minkowski up to $O(\xi^4)$. Thus during evolution toward the stable rest point, the metric tends to flat Minkowski spacetime with $O(\xi^4)$ errors.

Note that we have only assumed a smooth SSC solution and expanded in finite Taylor series about the center, so our only asymptotic assumption has been that ξ is small, not that the perturbation from the $k = 0$, $p = 0$ Friedman spacetime is small. Thus the phase portrait in Figure 1 is universal in that it describes the evolution of every SSC smooth solution in a neighborhood of $\xi = 0$, $t > 0$. We state this as a theorem:

Theorem 3. *Let (z, w, A, B) be an SSC solution which is smooth in the ambient Euclidean coordinate system x associated with the spherical SSC coordinates, and meeting condition (2.8). Then there exists an SSC time gauge in which the solution satisfies equations (4.39)-(4.42) and (4.43) up to the appropriate orders. Thus the phase portrait of Figure 1 is valid in a neighborhood of $\xi = 0$ with errors $O(1)\xi^6$ in z and $O(1)\xi^4$ in w, where by Taylor's theorem, the $O(1)$ errors are* *bounded by the maximum of the sixth and fourth derivatives of the solution components z and w, respectively.*

4.3. **Redshift vs Luminosity Relations for the Ansatz.** In this section we obtain formulas for Q and C in (1.1) as a function of the corrections z_2, w_0, z_4, w_2 to the SM, we compare this to the values of Q and C as a function of Ω_Λ in DE theory, and we show that remarkably, Q passes through the same range of values in both theories.

Recall that Q and C are the quadratic and cubic corrections to redshift vs luminosity as measured by an observer at the center of the spherically symmetric perturbation of the SM determined by these corrections.[11] The calculation requires taking account of all of the terms that affect the redshift vs luminosity relation when the spacetime is not uniform, and the coordinates are not co-moving.

The redshift vs luminosity relation for the $k = 0$, $p = \sigma\rho$, FRW spacetime, at any time during the evolution, is given by,

$$Hd_\ell = \frac{2}{1+3\sigma}\left\{(1+z) - (1+z)^{\frac{1-3\sigma}{2}}\right\}, \tag{4.44}$$

where only H evolves in time, [9]. For pure radiation $\sigma = 1/3$, which gives $Hd_\ell = z$, and when $p = \sigma = 0$, we get, (c.f. [21]),

$$Hd_\ell = z + \frac{1}{4}z^2 - \frac{1}{8}z^3 + O(z^4). \tag{4.45}$$

The redshift vs luminosity relation in the case of Dark Energy theory, assuming a critical Friedman space-time with the fraction of Dark Energy Ω_Λ, is

$$Hd_\ell = (1+z)\int_0^z \frac{dy}{\sqrt{\mathcal{E}(y)}}, \tag{4.46}$$

where

$$\mathcal{E}(z) = \Omega_\Lambda(1+z)^2 + \Omega_M(1+z)^3, \tag{4.47}$$

and $\Omega_M = 1 - \Omega_\Lambda$, the fraction of the energy density due to matter, (c.f. (11.129), (11.124) of [9]). Taylor expanding gives

$$Hd_\ell = z + \frac{1}{2}\left(-\frac{\Omega_M}{2}+1\right)z^2 + \frac{1}{6}\left(-1-\frac{\Omega_M}{2}+\frac{3\Omega_M^2}{4}\right)z^3 + O(z^4), \tag{4.48}$$

where Ω_M evolves in time, ranging from $\Omega_M = 1$ (valid with small errors at the end of radiation) to $\Omega_M = 0$ (the limit as $t \to \infty$). From (4.48)

[11] The uniformity of the center out to errors $O(\xi^4)$ implies that these should be good approximations for observers somewhat off-center with the coordinate system of symmetry for the waves.

we see that in Dark Energy theory, the quadratic term Q increases exactly through the range

$$.25 \leq Q \leq 5, \tag{4.49}$$

and the cubic term decreases from $-1/8$ to $-1/6$, during the evolution from the end of radiation to $t \to \infty$, thereby verifying the claim in Theorem 1. In the case $\Omega_M = .3$, $\Omega_\Lambda = .7$, representing present time $t = t_{DE}$ in Dark Energy theory, this gives the exact expression,

$$H_0 d_\ell = \mathrm{z} + \frac{17}{40}\,\mathrm{z}^2 - \frac{433}{2400}\,\mathrm{z}^3 + O(\mathrm{z}^4), \tag{4.50}$$

verifying that $Q = .425$ and $C = -.1804$, as recorded in Theorem 1.

In the case of a general non-uniform spacetime in SSC, the formula for redshift vs luminosity as measured by an observer at the center is given by, (see [9]),

$$d_\ell = (1+\mathrm{z})^2 r_e = t_0(1+\mathrm{z})^2 \xi_e\left(\frac{t_e}{t_0}\right), \tag{4.51}$$

where (t_e, r_e) are the SSC coordinates of the emitter, and $(0, t_0)$ are the coordinates of the observer. A calculation based on using the metric corrections to obtain ξ_e and t_e/t_0 as functions of z, and substituting this into (4.51), gives the following formula for the quadratic correction $Q = Q(z_2, w_0)$ and cubic correction $C = C(z_2, w_0, z_4, w_2)$ to redshift vs luminosity in terms of arbitrary corrections w_0, w_2, z_2, z_4 to SM. We record the formulas in the following theorem:

Theorem 4. *Assume a GR spacetime in the form of our ansatz (4.32)-(4.35), with arbitrary given corrections $w_0(t), w_2(t), z_2(t), z_4(t)$ to SM. Then the quadratic and cubic corrections Q and C to redshift vs luminosity in (1.1), as measured by an observer at the center $\xi = r = 0$ at time t, is given explicitly by*

$$Hd_\ell = \mathrm{z}\left\{1 + \left[\frac{1}{4} + E_2\right]\mathrm{z} + \left[-\frac{1}{8} + E_3\right]\mathrm{z}^2\right\} + O(\mathrm{z}^4), \tag{4.52}$$

where

$$H = \left(\frac{2}{3} + w_0(t)\right)\frac{1}{t},$$

so that

$$Q(z_2, w_0) = \frac{1}{4} + E_2, \qquad C(w_0, w_2, z_2, z_4) = -\frac{1}{8} + E_3, \tag{4.53}$$

where $E_2 = E_2(z_2, w_0)$, $E_3 = E_3(z_2, w_0, z_4, w_2)$ are the corrections to the $p = 0$ standard model values in (4.45). The function E_2 is given *explicitly by*

$$E_2 = \frac{24w_0 + 45w_0^2 + 3z_2}{4(2+3w_0)^2}. \tag{4.54}$$

The function E_3 is defined by the following chain of variables:

$$E_3 = 2I_2 + I_3, \tag{4.55}$$

$$I_{2,3} = J_2 + \frac{9w_0}{2(2+3w_0)},\ \ J_3 + 3\left[-1 + \left(\frac{8 - 8J_2 + 3w_0 - 12J_2w_0}{2(2+3w_0)^2}\right)\right],$$

$$J_2 = \frac{1}{4}\left\{1 - \frac{1+9K_2}{\left(1+\frac{3}{2}w_0\right)^2}\right\},$$

$$J_3 = \frac{5}{8}\left\{1 - \frac{1 - \frac{18}{5}K_2 - \frac{81}{5}K_2^2 + \frac{9}{5}w_0 + \frac{27}{5}K_3 + \frac{81}{10}Q_3w_0}{\left(1+\frac{3}{2}w_0\right)^4}\right\},$$

$$K_{2,3} = \frac{2}{3}w_0 + \frac{1}{2}w_0^2 - \frac{1}{12}z_2,\ \ \frac{2}{9}w_0 + w_0^2 + \frac{1}{2}w_0^3 + w_2 - \frac{1}{18}z_2 - \frac{1}{3}z_2w_0.$$

From (4.54) one sees that Q depends only on (z_2, w_0), $Q(0,0) = .25$, (the exact value for the SM), $Q(-4/3, 1/3) = .5$, (the exact value for the stable rest point), and from this it follows that Q increases through precisely the same range (4.49) of DE, from $Q \approx .25$ to $Q = .5$, along the orbit of (4.39), (4.40) that takes the unstable rest point $SM = (z_2, w_0) = (0,0)$ to the stable rest point $(z_2, w_0) = (-4/3, 1/3)$, (c.f. Figure 1).

4.4. **Initial Data from the Radiation Epoch.** In this section we compute the initial data for the $p = 0$ evolution from the restriction of the one parameter family of self-similar a-waves to a constant temperature surface $T = T_*$ at the end of radiation, and convert this to initial data on a constant time surface $t = t_*$, these two surfaces being different when $a \neq 1$. We then define a gauge transformation that converts the resulting initial data to equivalent initial data that meets the gauge conditions (4.43). (Recall that condition (4.43) fixes a time coordinate, or gauge, for the underlying SSC metric associated with our ansatz, and the initial data for the a-waves is given in a different gauge because time since the big bang depends on the parameter a, as well as on the pressure, so it changes when p drops to zero.) The equation of state of pure radiation is derived from the the Stefan-Boltzmann

Law, which relates the initial density ρ_* to the initial temperature T_* in degrees Kelvin by

$$\rho_* = \frac{a_s c}{4} T_*^4, \tag{4.56}$$

where a_s is the Stefan-Boltzmann constant, [14]). According to current theories in cosmology, (see e.g. [14]), the pressure drops precipitously to zero at a temperature $T = T_*$ somewhere between $3000^o K \leq T_* \leq 9000^o K$, corresponding to starting times t_* roughly in the range $10,000yr \leq t_* \leq 30,000yr$ after the Big Bang. We make the assumption that the pressure drops discontinuously to zero at some temperature T_* within this range. That our resulting simulations are numerically independent of starting temperature, (c.f. Section 4.5), justifies the validity of this assumption. Using this assumption, we can take the values of the a-waves on the surface $T = T_*$ as the initial data for the subsequent $p = 0$ evolution. Using the equations we convert this to initial data on a constant time surface $\bar{t} = \bar{t}_*$, where $\bar{t}$ is the time coordinate used in the self-similar expression of the a-waves which assumes $p = \frac{c^2}{3}\rho$. Our first theorem proves that there is a gauge transformation $\bar{t} \to t$ which converts the initial data for a-waves at the end of radiation at $\bar{t} = \bar{t}_*$, to initial data that both meets the assumptions of our ansatz (4.32)-(4.35), as well as the gauge conditions (4.43).

Theorem 5. *Let $\bar{t}$ be the time coordinate for the self-similar waves during the radiation epoch, and define the transformation $\bar{t} \to t$ by*

$$t = \bar{t} + \frac{1}{2}\mu(\bar{t} - \bar{t}_*)^2 - t_B, \tag{4.57}$$

where μ and t_B are given by

$$\mu = \frac{a^2}{2(2 - a^2)}, \tag{4.58}$$

$$t_B = \bar{t}_*(1 - \alpha), \tag{4.59}$$

where

$$\alpha = 4\frac{2 - a^2}{7 - 4a^2}. \tag{4.60}$$

Then upon performing the gauge transformation (4.57), the initial data from the a-waves at the end of radiation $\bar{t} = \bar{t}_*$, meets the conditions for the ansatz (4.32)-(4.35), as well as the gauge conditions (4.43).

Our conclusions are summarized in the following theorem:

Theorem 6. *The initial data for the $p = 0$ evolution determined by the self-similar a-wave on a constant time surface $t = t_*$ with temperature $T = T_*$ at $r = 0$, is given as a function of the acceleration parameter a and the temperature T_*, by*

$$z_2(t_*) = \hat{z}_2, \qquad z_4(t_*) = \hat{z}_4 + 3\hat{w}_0\left(\tfrac{4}{3} + \hat{z}_2\right)\gamma,$$
$$w_0(t_*) = \hat{w}_0, \qquad w_2(t_*) = \hat{w}_2 + \left(\tfrac{1}{6}\hat{z}_2 + \tfrac{1}{3}\hat{w}_0 + \hat{w}_0^2\right)\gamma,$$

where $\hat{z}_2, \hat{z}_4, \hat{w}_0, \hat{w}_2$ and γ are functions of acceleration parameter a given by

$$\hat{z}_2 = \tfrac{3a^2\alpha^2}{4} - \tfrac{4}{3}, \qquad \hat{z}_4 = 2\alpha^3(1-\alpha)\bar{\gamma}Z_2 + \alpha^4 Z_4 - \tfrac{40}{27},$$
$$Z_2 = \tfrac{3a^2}{4}, \; Z_4 = \left[\tfrac{9a^2}{16} + \tfrac{15a^2(1-a^2)}{40}\right],$$
$$\hat{w}_0 = \tfrac{\alpha}{2} - \tfrac{2}{3}, \qquad \hat{w}_2 = \alpha^2(1-\alpha)\bar{\gamma}W_0 + \alpha^3 W_2 - \tfrac{2}{9},$$
$$W_0 = \tfrac{1}{2}, \;\; W_2 = \left[\tfrac{1}{8} + \tfrac{(1-a^2)}{20}\right],$$

where

$$\gamma = \alpha\bar{\gamma} = \alpha\left(\frac{2-a^2}{4}\right), \tag{4.61}$$

and α is given in (4.60).

The time t_* is then given in terms of the initial temperature T_* by

$$t_* = \frac{a\alpha}{2}\sqrt{\frac{3}{\kappa\rho_*}}, \qquad \rho_* = \frac{a_s}{4c}T_*^4. \tag{4.62}$$

Taking the leading order part of the initial data gives a curve parameterized by a in the (z_2, w_0)-plane that cuts through the saddle point SM in system (4.39), (4.40), between the stable and unstable manifold, (the lighter dotted line in Figure 1). This implies that a small under-density corresponding to $a < 1$ will evolve to the stable rest point M, $(z_2, w_0) = (-4/3, 1/3)$, (c.f. Figure 1).

4.5. **The Numerics.** In this section we present the results of our numerical simulations. We simulate solutions of (4.39)-(4.42) for each value of the acceleration parameter $a < 1$ in a small neighborhood of $a = 1$, (corresponding to small under-densities relative to the SM), and for each temperature T_* in the range $3000^oK \leq T_* \leq 9000^oK$. We simulate up to the time t_a, the time depending on the acceleration parameter a at which the Hubble constant is equal to its present measured value $H = H_0 = 100h_0\frac{km}{s\,mpc}$, with $h_0 = .68$. From this we conclude that the dependence on T_* is negligible. We then asked for the value of a that gives $Q(z_2(t_a), w_0(t_a)) = .425$, the value of Q in Dark Energy theory with $\Omega_\Lambda = .7$. This determines the unique value

$a = \underline{a} = 0.999999426$, and the unique time $t_0 = t_{\underline{a}}$. These results are recorded in the following theorem:

Theorem 7. *At present time t_0 along the solution trajectory of (4.39)-(4.42) corresponding to $a = \underline{a}$, our numerical simulations give $H = H_0$, $Q = .425$, together with the following:*

$$z(t_0, \xi) = (-1.142)\xi^2 + (1.385)\xi^4 + O(\xi^6),$$

$$w(t_0, \xi) = 0.247 - (0.348)\xi^2 + O(\xi^4),$$

and

$$A(t_0, \xi) = 1 + (0.381)\xi^2 - (0.277)\xi^4, \tag{4.63}$$

$$D(t_0, \xi) = 1 + (0.095)\xi^2 + O(\xi^4). \tag{4.64}$$

The cubic correction to redshift vs luminosity as predicted by the wave model at $a = \underline{a}$ is

$$C = 0.359. \tag{4.65}$$

Note that (4.63) and (4.64) imply that the spacetime is very close to Minkowski at present time up to errors $O(\xi^4)$, so the trajectory in the (z_2, w_0)-plane is much closer to the stable rest point M than to the SM at present time, c.f. Figure 1. The cubic correction associated with Dark Energy theory with $k = 0$ and $\Omega_\Lambda = .7$ is $C = -0.180$, so (4.65) is a theoretically verifiable prediction which distinguishes the wave theory from Dark Energy theory. A precise value for the actual cubic correction corresponding to C in the relation between redshift vs luminosity for the galaxies appears to be beyond current observational data.

4.6. **The Uniform Spacetime at the Center.** In this section we describe more precisely the central region of accelerated uniform expansion triggered by the instability due to perturbations that meet the ansatz (4.32)-(4.35). By (4.38) we have seen that neglecting terms of order ξ^4 in z, the density $\rho(t)$ depends only on the time. Further neglecting the small errors between (z_2, w_0) and the stable rest point $\left(-\frac{4}{3}, \frac{1}{3}\right)$ at present time t_0 when $a = \underline{a}$, we prove that the spacetime is Minkowski with a density $\rho(t)$ that drops like $O(t^{-3})$, so the instability creates a central region that appears to be a flat version of a uniform Friedman universe with a larger Hubble constant, in which the density drops at a faster rate than the $O(t^{-2})$ rate of the SM.

Specifically, as $t \to \infty$, our orbit converges to $\left(-\frac{4}{3}, \frac{1}{3}\right)$, the stable rest point for the (z_2, w_0) system

$$\begin{pmatrix} z_2 \\ w_0 \end{pmatrix}' = \begin{pmatrix} -3w_0\left(\frac{4}{3} + z_2\right) \\ -\frac{1}{6}z_2 - \frac{1}{3}w_0 - w_0^2 \end{pmatrix}. \tag{4.66}$$

Setting $z_2 = -4/3 + \bar{z}(t)$, $w_0 = 1/3 + \bar{w}(t)$ and discarding higher order terms, we obtain the linearized system at rest point $(-\frac{4}{3}, \frac{1}{3})$,

$$\begin{pmatrix} \bar{z} \\ \bar{w} \end{pmatrix}' = \begin{pmatrix} -1 & 0 \\ -\frac{1}{6} & -1 \end{pmatrix} \begin{pmatrix} \bar{z} \\ \bar{w} \end{pmatrix}. \tag{4.67}$$

The matrix in (4.67) has the single eigenvalue $\lambda = -1$ with single eigenvector $R = (0, 1)$. From this we conclude that all orbits come into the rest point $(-\frac{4}{3}, \frac{1}{3})$ from below along the vertical line $z_2 = -4/3$. This means that $z_2(t)$ and $\rho(t) = z_2(t)/t^2$ can tend to zero at algebraic rates as the orbit enters the rest point, but $w_0(t)$ must come into the rest point exponentially slowly, at rate $O(e^{-t})$. Thus our argument that $\bar{w} = w_0 - 1/3$ is constant on the scale where $\rho(t) = k_0/t^\alpha$ gives the precise decay rate,

$$\rho(t) = \frac{k_0}{t^{3(1+\bar{w})}}. \tag{4.68}$$

That is, $\bar{w} \equiv \bar{w}(t) \to 0$ and $k_0 \equiv k_0(t)$ are changing exponentially slowly, but the density is dropping at an inverse cube rate, $O(1/t^{3(1+\bar{w})})$, which is *faster* than the $O(1/t^2)$ rate of the standard model.

Therefore, neglecting terms of order ξ^4 together with the small errors between the metric at present time t_0 and the stable rest point, the spacetime is Minkowski with a density $\rho(t)$ that drops like $O(t^{-3})$, a faster rate than the $O(t^{-2})$ of the SM. Furthermore, we show that neglecting relativistic corrections to the velocity of the fluid near the center where the velocity is zero, evolution toward the stable rest point creates a flat, center independent spacetime which evolves outward from the origin, and whose size is proportional to the Hubble radius.

We conclude that the effect of the instability triggered by a perturbation of the SM consistent with ansatz (4.32)-(4.35) near the stable rest point $\left(-\frac{4}{3}, \frac{1}{3}\right)$, is to create an anomalous acceleration consistent with the anomalous acceleration of the galaxies in a large, flat, uniform, center-independent spacetime, expanding outward from the center of the perturbation.

5. Conclusion

Our purpose is to identify a mechanism that could account for the anomalous acceleration of the galaxies within Einstein's original theory, without the cosmological constant. We find such a mechanism by deriving a universal phase portrait for spherical perturbations of the Friedman space-time of the Standard Model of Cosmology. It is universal in the sense that it describes the evolution near the center of symmetry of *any* smooth radial perturbation of the $k = 0$, $p = 0$ Friedman spacetime that is locally inertial at the center. The phase portrait places SM at an *unstable* saddle rest point, and the resolution of this instability creates the same anomalous accelerations as the cosmological constant, without assuming it. The phase portrait of the instability shows that only under-dense and over dense perturbations of SM are observable, (not SM itself), and the under-dense case would imply that we live within a large region of approximate uniform density that is expanding outward from us at an accelerated rate relative to the SM. The idea that the Milky Way lies near the center of a large region of under-density has already been proposed and studied in the physics literature. (See [4] and the Appendix below.)

The central region created by the instability is different from, but looks a lot like, a speeded up Friedman universe tending more rapidly to flat Minkowski space than the SM. The phase portrait for the perturbations provides a verifiable mathematical explanation for the anomalous acceleration of the galaxies that does not invoke Dark Energy.

At this stage we have made no assumptions regarding the space-time far from the center of the perturbations that trigger the instabilities in the SM. The purpose of our paper is not to solve all the problems of Cosmology in one grand solution. Our purpose is to introduce and demonstrate a new instability in the Friedman space-time of the Standard Model of Cosmology, to identify mechanisms that trigger it, to show how it naturally could account for the anomalous acceleration within Einstein's original theory without Dark Energy, and then to derive new predictions from it. Given that SM is unstable, the paper raises the fundamental question as to whether it is reasonable to expect to observe an unperturbed Friedman space-time, with or without Dark Energy, on the scale of the supernova data. The consistency of this model with other observations in astrophysics would require additional assumptions.[12]

[12]Note that when $p = 0$, the only sound speed is the fluid velocity, so one should expect that solutions constructed asymptotically near $\xi = 0$ could be extended by initial data whose evolution maintains a positive fluid velocity up to present time.

6. Appendix: The Problem of Regularity at the Center in LTB Coordinates

In LTB coordinates, the metric is spherically symmetric and diagonal, but particles are assumed co-moving so particle paths are geodesics, and this involves a change of radial coordinate $r \to \hat{r}$ and time coordinate $t \to \hat{t}$ relative to SSC. It is not difficult to show by construction of an integrating factor [27, 18] that (under obvious assumptions), metrics in SSC are coordinate equivalent to metrics in LTB form. But although the condition for smoothness at $r = 0$ is easily expressed in SSC in terms of r-derivatives at $r = 0$, (odd order derivatives of the metric components and scalar functions should vanish at $r = 0$), identifying such a condition for smoothness at $\hat{r} = 0$ in LTB appears to be a subtle issue not adequately addressed in the literature. Lacking clear criteria for smoothness has led theorists to admit solutions with LTB coordinate singularities at $\hat{r} = 0$, and aside from the problem of determining which solutions are physical, and what mechanisms might create "singularities", lacking such criteria allows for solutions with derivatives at $\hat{r} = 0$ either undefined, or unconstrained by the equations, [29, 38, 37]. Thus unlike SSC, expanding LTB solutions in finite Taylor series about the origin is problematic. We finish by recording formulas for the first three derivatives at $\hat{r} = 0$ of density functions which are smooth in SSC at $r = 0$ to show that no such simple criteria in LTB identifies smoothness at $\hat{r} = 0$ in terms of $\hat{r}$-derivatives at $\hat{r} = 0$.

Consider then a coordinate transformation that takes a $p = 0$ gravitational metric from LTB coordinates $(\hat{t}, \hat{r})$ over to SSC coordinates (t, r) given by

$$t = t(\hat{t}, \hat{r}), \quad r = r(\hat{t}, \hat{r}).$$

Now LTB and SSC are diagonal metrics such that the coordinates meet the conditions that the fluid is co-moving with respect to $\hat{r}$, $\hat{r} = const$ are geodesics, $\hat{t}$ is geodesic time along $\hat{r} = const$, and r measures arclength distance at fixed t in the radial direction, [9]. In the following lemma we record formulas for derivatives of the scalar density function $\rho(t, r)$ in terms of its SSC derivatives and the mapping from SSC to LTB, under the assumption that the function is smooth in SSC in the sense that all its even r-derivatives vanish at $r = 0$.

Lemma 2. *Assume that $\rho(t, r)$ is a scalar density function which extends to a smooth function $\rho(t, |\mathbf{x}|)$ in SSC coordinates, so that it is*

Note also there are large scale aspherical anomalies observed in the microwave background radiation, [3].

given near $r = 0$ by

$$\rho(t,r) = f_0(t) + f_2(t)r^2 + \cdots , \tag{6.69}$$

where the dots indicate that the expansion includes only even powers of r. Assume that the mapping $(t,r) \to (\hat{t},\hat{r})$ from SSC to LTB coordinates is smooth and invertible on $r \geq 0$, and meets the minimal regularity condition that all derivatives of $\frac{\partial t}{\partial \hat{r}}(\hat{t},\hat{r})$ up to order three have continuous one-sided limits at $\hat{r} = 0$, together with

$$\lim_{\hat{r}\to 0} r(\hat{t},\hat{r}) = r(\hat{t},0) = 0. \tag{6.70}$$

Finally, let

$$\hat{\rho}(\hat{t},\hat{r}) = \rho(t(\hat{t},\hat{r}), r(\hat{t},\hat{r})) \tag{6.71}$$

denote the representation of the function $\rho(t,r)$ in LTB coordinates. Then among odd order derivatives, the first three partial derivatives of $\hat{\rho}$ with respect to $\hat{r}$ are given by

$$\frac{\partial\hat{\rho}}{\partial\hat{r}}(\hat{t},0) = \frac{\partial\rho}{\partial t}\frac{\partial t}{\partial\hat{r}}, \tag{6.72}$$

$$\frac{\partial^2\hat{\rho}}{\partial\hat{r}^2}(\hat{t},0) = \frac{\partial^2\rho}{\partial t^2}\left(\frac{\partial t}{\partial\hat{r}}\right)^2 + \frac{\partial\hat{\rho}}{\partial t}\frac{\partial^2 t}{\partial\hat{r}^2} + \frac{\partial^2\rho}{\partial r^2}\left(\frac{\partial r}{\partial\hat{r}}\right)^2, \tag{6.73}$$

$$\begin{aligned}\frac{\partial^3\hat{\rho}}{\partial\hat{r}^3}(\hat{t},0) &= \frac{\partial^3\rho}{\partial t^3}\left(\frac{\partial t}{\partial\hat{r}}\right)^3 + 3\frac{\partial^2\rho}{\partial t^2}\frac{\partial t}{\partial\hat{r}}\frac{\partial^2 t}{\partial\hat{r}^2} \\ &\quad + \frac{\partial\rho}{\partial t}\frac{\partial^3 t}{\partial\hat{r}^3} + 3\frac{\partial^3\rho}{\partial r^2\partial t}\left(\frac{\partial r}{\partial\hat{r}}\right)^2\frac{\partial t}{\partial\hat{r}} + 3\frac{\partial^2\rho}{\partial r^2}\frac{\partial r}{\partial\hat{r}}\frac{\partial^2 r}{\partial\hat{r}^2}.\end{aligned} \tag{6.74}$$

Proof of Lemma 2: Taking the first partial derivative of $\hat{\rho}$ with respect to $\hat{r}$ using (6.71) gives

$$\frac{\partial}{\partial\hat{r}}\hat{\rho}(\hat{t},\hat{r}) = \frac{\partial\rho}{\partial t}\frac{\partial t}{\partial\hat{r}} + \frac{\partial\rho}{\partial r}\frac{\partial r}{\partial\hat{r}}, \tag{6.75}$$

so assuming $\frac{\partial\rho}{\partial r} = 0$ at $(t,0)$ by (6.69) gives (6.72). Then

$$\begin{aligned}\frac{\partial^2}{\partial\hat{r}^2}\hat{\rho}(\hat{t},\hat{r}) &= \frac{\partial^2\rho}{\partial t^2}\left(\frac{\partial t}{\partial\hat{r}}\right)^2 + \frac{\partial\rho}{\partial t}\frac{\partial^2 t}{\partial\hat{r}^2} \\ &\quad + 2\frac{\partial^2\rho}{\partial r\partial t}\frac{\partial t}{\partial\hat{r}}\frac{\partial r}{\partial\hat{r}} + \frac{\partial^2\rho}{\partial r^2}\left(\frac{\partial r}{\partial\hat{r}}\right)^2 + \frac{\partial\rho}{\partial r}\frac{\partial^2 r}{\partial\hat{r}^2},\end{aligned} \tag{6.76}$$

and again using $\frac{\partial\rho}{\partial r} = 0$ at $(t,0)$ by (6.69) gives (6.73). For the third derivative, differentiate (6.73) and use that all partial derivatives of $\rho(t,r)$ that are odd order in r vanish at $r = 0$. $\Box$

We conclude from (6.72)-(6.74) that it would be difficult to identify the density functions satisfying the SSC smoothness condition that all odd SSC r-derivatives vanish at $r = 0$ by a condition on derivates at $\hat{r} = 0$ in LTB coordinates, and such a condition would depend on the solution dependent mapping from SSC to LTB. Lacking this, the smoothness of LTB metrics and scalars at $\hat{r} = 0$ is unresolved.

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