The Mathematics of Self-Similar Waves
Blake Temple, October, 2011

We have presented our new self-similar waves that perturb the standard model of cosmology [80], [82] as a potential physical mechanism that could explain the anomalous acceleration of the galaxies. Our point was that waves created from local fluctuations during the radiation phase of the expansion could create a wave that expanded relative to a background Friedmann spacetime, and thereby introduce a quadratic correction to redshift vs luminosity observed at a later time. In some sense, this has downplayed the original mathematics contained within those papers. As a contribution to PDE's and conservation laws, this work stands as a surprising and original change of direction for the field of conservation laws as well.

Narrowly viewed from the PDE context, independent of the physical connection to cosmology, the mathematics in [80], [82] provides the first ever construction of a family of self-similar time asymptotic waves for the relativistic p-system with general relativistic sources. The family contains and perturbs a unique expanding wave solution (Friedmann) whose expansion is uniform in space. This is the first ever such family of self-similar solutions ever discovered for the relativistic p-system with GR sources. To give you an idea how remarkable this is, recall that the p-system without sources, in one space dimension, admits two characteristic families of self-similar solutions consisting of shock waves and rarefaction waves, and all solutions of compact support decay time-asymptotically to such waves. Our GR self-similar solutions are analogous to the rarefaction waves...they are self-similar, and they expand outward from a center. Who could imagine that the p-system with General Relativistic corrections coupling the spacetime to the fluid, would also admit a family of self-similar "rarefaction waves"?!

The construction of these GR self-similar solutions is based on novel methods that come from ideas in general relativity, and relies on a subtle and serendipitous mathematical discovery. Namely, the Einstein equations admit a miraculous new symmetry when the pressure is proportional to the density ($p = \frac{1}{3}\rho c^2$ for pure radiation), and we discovered it. This symmetry pretty much could never be discovered from the equations themselves, (Jeff Groah and I worked a long time on these equations), because, first, the equations are so complicated when you try to close them up in the fluid and metric components (Bianchi identities are required!), that you would never find them if you didn't expect them to be there ahead of time. And second, the equations are only manageable in a coordinate system (Standard Schwarzschild Coordinates SSC) in which the solutions by themselves would be extremely difficult to interpret physically. Anyone who reads through our paper and sees how all the terms not depending on r/t cancel out of the equations, appears to be witnessing a miracle.

We found this symmetry by a line of reasoning that surely was never followed before. We believed there should be time-asymptotic expansion waves during the radiation phase because we knew that the strong nonlinearities required to get decay of complicated fluctuations to simple wave forms was in place when $p = \frac{1}{3}\rho c^2$, and we set out to numerically simulate such a wave. To start the simulation, we transformed the Standard Model Friedmann universe over to SSC coordinates, and serendipitously found a representation of the components of the gravitational metric of the standard model of cosmology in these coordinates that reduced to functions of the single self-similar variable r/t. (SSC representations of a metric are not unique, but depend on an integrating factor.) We concluded that in these coordinates, the standard model must satisfy an ODE, leading to our conjecture that the Einstein equations themselves must reduce to ODE's in that same self-similar variable. That is what motivated us to look for them. We found the equations (not so easy!!) and the analysis of them gives rise to the new self-similar solutions.

The coordinate mapping for the standard model from co-moving coordinates over to SSC, gave us the key to interpret them. That is, it told us how to transform the new solutions back to coordinates where they made physical sense. In particular, in SSC, the solutions have a cusp singularity at the origin, and without the standard model to give us the key to interpret this singularity, we very likely would have dismissed these cusps as non-physical. The mathematical structure of these solutions is remarkable. Writing the equations as a first order system with independent variable $\xi = r/t$, a change of variables makes the equations non-autonomous in all non-differentiated terms. This creates a singularity at the center of the solution whose structure is just perfect so that it transforms to a rest point of an ODE under another change of variables. Finally, rest point analysis applies at the center and picks out the stable manifold which identifies the one parameter family of spacetimes extending the standard model as the stable manifold of solutions regular at the center.

There are many interesting mathematical issues arising from this new mathematical reasoning.

For example, we could explain it this way: We found new self-similar solutions to the relativistic p-system with general relativistic corrections. That alone is a great result in conservation laws. The standard model of cosmology during the most important epoch is one of them. And all of them look just like the standard model at the center, with the first divergence coming in the redshift vs luminosity relation. The analysis of redshift vs luminosity could be viewed as a new method of characterizing these new solutions, and of distinguishing the different solutions in the family. The Reciprocity Theorem of general relativity was required in the analysis, and one could interpret this as introducing new techniques from GR to address the regularity of time asymptotic waves for conservation laws.

The mathematics tells us that nonlinear dissipation driving decay to simple wave forms should be maximal during the radiation phase. Indeed, the pressure closes off in the density as $p = \frac{c^2}{3}\rho$, so the fluid part of the Einstein equations for radial waves closes into a 2×2 genuinely nonlinear system in the velocity and density uncoupled from the temperature or entropy. That is, the linearly degenerate field, present in the compressible Euler equations when the equation of state depends on the entropy, is closed off, and this is the only characteristic field that fights nonlinear decay. Thus, the Einstein equations for pure radiation reduce to the relativistic p-system with GR sources. At the center of a locally inertial coordinate frame, they are *exactly* the relativistic p-system, precisely the system for which the mechanism of decay via nonlinearities was studied by Glimm and Lax and all who followed them. The sound speed is over half the speed of light, the modulus of genuine nonlinearity ($\nabla\lambda_i \cdot R_i >> 0$) is highly nonzero in both characteristic fields, and the mathematics tells us that nonlinearities are optimal for decay to self-similar waves. Therefore we have found, for the first time, a family of self-similar waves for a system of conservation laws with GR sources, that perturb and contain a uniform solution (the Friedmann spacetime), and play the role of time asymptotic wave forms. This introduces a plethora of new and interesting PDE questions.