Tensors In General ⇔ Summation Convention

- \( x : M \to \mathbb{R}^n \) a coordinate system on an \( n \)-dimensional manifold \( M \)
- \( T_p M \) = tangent space of \( M \) at \( p \). All vectors can be represented in a coordinate system
- \( \frac{\partial}{\partial x_i} |_p \), \( i = 1, \ldots, n \) the \( x \)-coordinate basis for \( T_p M \)
- \( X_p = a^i \frac{\partial}{\partial x^i} \) the \( x \)-coordinate name for a vector \( X_p \in T_p M \)
- \( a^i \) the \( x \)-components of \( X_p \)
- Under change of coordinates \( x_1 \mapsto y \), the components \( b^i \) basis transform by \( (B = B^i_a = \frac{\partial x^i}{\partial y^a}) \)

\[
\frac{\partial x_i}{\partial y^a} = \frac{\partial x_i}{\partial y^a} \frac{\partial y^a}{\partial x_i}, \quad \bar{a}^i = \frac{\partial y^a}{\partial x_i} a^i
\]
\( T_P^*M \equiv \) cotangent space of \( M \) at \( P \). All covectors can be represented in a coordinate system.

- \( dx^i \bigg|_P, \ i = 1, \ldots, n \) the \( x \)-coordinate basis for \( T_P^*M \)
- \( \omega_p = a_i \, dx^i \) the \( x \)-coordinate name for a covector \( \omega_p \in T_P^*M \)
- \( a_i \) the \( x \)-components of \( \omega_p \)
- \( \omega_p (X_p) = a_i \, dx^i \left( b^j \frac{\partial}{\partial x^j} \right) = a_i \, b^i = \mathbf{a} \cdot \mathbf{b} \)
- Under change of coordinates \( x \mapsto y \), the components \& basis transform by

\[
\bar{a}_\alpha = \frac{\partial x^i}{\partial y^\alpha} a_i, \quad dy^\alpha = \frac{\partial y^\alpha}{\partial x^i} dx^i
\]

- Covectors preserve the "coordinate dot product": 
  \( a_i \, b^i = \bar{a}_\alpha \bar{b}^\alpha \) & thereby keep track of hyperplanes -co-dimension 1 planes in \( T_P^*M \).
A coordinate dependent matrix $g_{ij}(p)$ tells how to compute the inner product on $T_pM$:

$$\langle X_p, Y_p \rangle = \text{"inner product between vectors up on } M\text{"}$$

$$= g_{ij} a^i b^j = \|X_p\| \|Y_p\| \cos \theta$$

$$\hat{X} = \frac{\partial a}{\partial x} \quad \hat{Y} = \frac{\partial b}{\partial x}$$

In a different coordinate system $\hat{Y}$:

$$\langle \hat{X}_p, \hat{Y}_p \rangle = \hat{g}_{\hat{a} \hat{b}} \hat{a}^\hat{a} \hat{b}^\hat{b}$$

$$\hat{X}_p = \frac{\partial \hat{a}}{\partial x^\hat{a}} \quad \hat{Y}_p = \frac{\partial \hat{b}}{\partial x^\hat{b}}$$

where

$$\hat{g}_{\hat{a} \hat{b}} = g_{ij} \frac{\partial x^i}{\partial y^\hat{a}} \frac{\partial x^j}{\partial y^\hat{b}} \Rightarrow \quad \hat{g} = B^T g B$$
A matrix $A^j_i$ keeps track of linear transformations of $T_p M$:

$$T: T_p M \rightarrow T_p M, \quad \overline{X}_p \rightarrow \overline{Y}_p = T \overline{X}_p$$

$$\overline{X}_p = a^i \frac{\partial}{\partial x^i}, \quad \overline{Y}_p = b^j \frac{\partial}{\partial x^j}$$

$$\downarrow$$

$$b^j = A^j_i a^i$$

- $A^j_i$ is the $x$-coordinate representation of $A$
- Under change of coordinates, $A^j_i$ transforms to $\overline{A}^a_b$, where

$$b^a = \overline{A}^a_b \overline{a}_b$$

* $\overline{a}_b$ component of $\overline{X}_p$ * $\overline{A}^a_b$ component of $\overline{X}_p$

$$\overline{A}^a_b = A^i_j \frac{\partial x^i}{\partial y^a} \frac{\partial y^b}{\partial x^j}$$

$$\overline{A} = B^{-1} A B$$
Tensors:

- $\delta_{ij}$ is a $(0)$-tensor
- $A^i_j$ is a $(1)$-tensor
- $R^i_{jke}$ is a $(1)$-tensor

Riemann Curvature

Example:

$$\overline{R}^a_{\beta\delta\sigma} = R^i_{jke} \frac{\partial x^k}{\partial y^\beta} \frac{\partial x^e}{\partial y^\delta} \frac{\partial y^i}{\partial x^j}$$

Extends to $(m)$-tensors

Principle: Up index transforms contravariantly

$$\overline{A}^\alpha = A^\alpha \frac{\partial y^\alpha}{\partial x^\alpha}$$

down index transforms covariantly

$$\overline{A}_{\beta} = A_{\gamma} \frac{\partial x^\beta}{\partial y^\gamma}$$
- We can use the metric to "raise & lower" indices -

\[ A_{ij} \equiv g_{ik} A^k_j \]

\( A_{ij} \) transforms like a \((0,2)\) tensor.

Eg: \( \bar{A}_\beta^a = g_{\alpha \beta} \bar{A}^\sigma_\sigma \) etc.

- Raising indices: let \( g^{ij} \rightarrow \) row, \( g_{ij} \rightarrow \) col.

Then \( g^{ik} A_{kij} = A^i_j \)

Same way -

- In general: \( A_{i\ldots k} = g_{im\ldots k} A_{m\ldots k} \)

\( A_{i\ldots k} = g^{ih} A_{h\ldots k} \)
Theorem: raising & lowering converts contravariant indices to covariant and visa versa.

Theorem: contracting an up index with a down index is coordinate independent operation:

Eg. $A^i_j$, a (1,1)-tensor implies

\[ A^2_1 = \overline{A}^{\alpha}_{\beta} \]

\[ \text{"sum up-down index from 1-N"} \]

\[ \text{\( \overline{A}^{\alpha}_{\beta} \)} \]

 HW
HW: Prove that the contraction of 2 indices is independent of coordinates. I.e., show that if \( A^i_j \) is a 1-1 tensor, then

\[
A^i_i = \bar{A}^\alpha_\alpha
\]

Sum \( i = 1, \ldots, n \).

This is the trace of \( A \):

\[
\text{Trace of } A = \text{sum of diagonal entries in } x\text{-coordinates}
\]

Trace of \( A \) in y-coords.
Finally. If $A^i_j$ are the "component" what are the "coordinate basis elements"?

Basis: $dx^i \otimes \frac{\partial}{\partial x^i}$ basis

$$A = A^i_j \ dx^i \otimes \frac{\partial}{\partial x^i}$$

component basis

$d x^i \otimes \frac{\partial}{\partial x^i}$ operate on pairs $(x^p, w_p) \in T_p M \times T^*_p M$ by

$$d x^i \otimes \frac{\partial}{\partial x^i} (a^r \frac{\partial}{\partial x^r} b dx^i) = a^r b_i$$

Turns out: $dy^x \otimes \frac{\partial}{\partial y^x} = \partial y^x (dx^i \otimes \frac{\partial}{\partial x^i} \frac{\partial x^i}{\partial y^x}) \frac{\partial x^i}{\partial y^x}$

*This gives tensors an invariant interpretation as a linear operator on $T_p M \times T^*_p M$ etc. (i.e., by summation convention.)