Surfaces $M^2 \subseteq \mathbb{R}^3$ (Ch 4 Book) VI (0

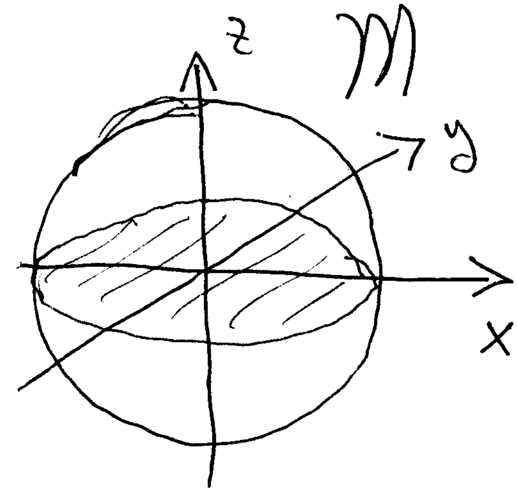
Defn: a 2-d surface (manifold) in $\mathbb{R}^3$ is a surface that can be covered by a collection of coordinate charts

$$\underset{\sim}{x} : \mathcal{U} \longrightarrow M \subseteq \mathbb{R}^3,$$

M covered by the union of $\underset{\sim}{x}(\mathcal{U}) \subseteq \mathbb{R}^3$, $\mathcal{U}_{open} \subseteq \mathbb{R}^2$, $\underset{\sim}{x}(u^1,u^2) = (x(u^1,u^2), y(u^1,u^2), z(u^1,u^2))$ st all 3 functions are smooth, $\underset{\sim}{x}^{-1}$ exists, and nondegenerate ($\frac{\partial \underset{\sim}{x}}{\partial u^1} \times \frac{\partial \underset{\sim}{x}}{\partial u^2} \neq 0$ so pos area $\overset{\underset{\sim}{x}}{\longleftrightarrow}$ positive area). "Smooth" means as many cont. deriv's as you need ~ C^2 ok" such that $\underset{\sim}{x}^{-1} \circ \underset{\sim}{y} : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth & invertible on the overlaps.

<u>Example</u>: Sphere of radius ρ in $\mathbb{R}^3$

$$x^2+y^2+z^2=\rho^2$$



- Solve for z: $z=\sqrt{\rho^2-x^2-y^2}$

$(x, y, \sqrt{\rho^2-x^2-y^2}) \in M$

$u^1 = x \quad u^2 = y$

$\underset{\sim}{x}(u^1,u^2) = (u^1, u^2, \sqrt{\rho^2-(u^1)^2-(u^2)^2})$

covers top half of sphere (not regular on ∂).

To cover: $\underset{\sim}{y}(v^1,v^2) = (v^1, \sqrt{\rho^2-(v^1)^2-(v^2)^2}, v^2)$

$v^1 = x, \; v^2 = z$

$\underset{\sim}{z}(w^1,w^2) = (\sqrt{\rho^2-(w^1)^2-(w^2)^2}, w^1, w^2)$

Still ∂-pts where not regular (eg $(-1,0,0)$)

Throw in $\underset{\sim}{x}^-(u^1,u^2) = (u^1, u^2, -\sqrt{\rho^2-(u^1)^2-(u^2)^2})$ etc

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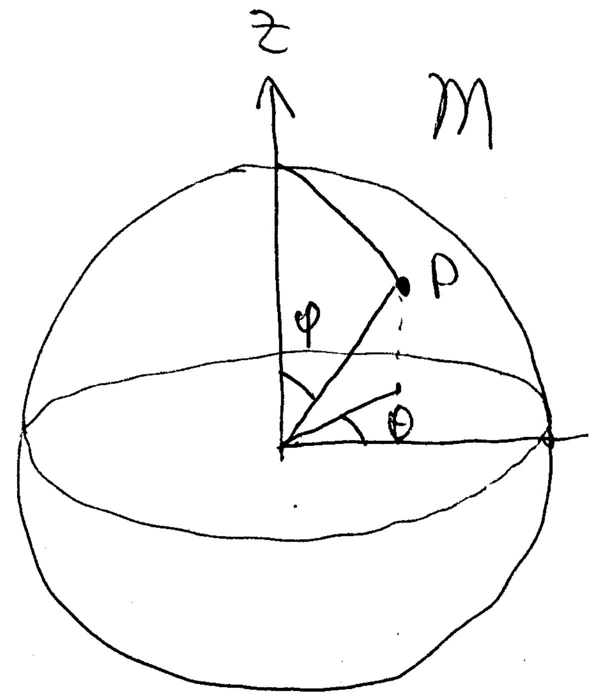
- In general: $z = f(x,y)$, Monge Patch is coord system of form

$$\underset{\sim}{x}(u^1,u^2) = (u^1, u^2, f(u^1,u^2))$$



- Spherical coords

$P = (r\cos\theta, r\sin\theta, z)$

$z = \rho\cos\varphi$

$r = \rho\sin\varphi$

$P = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$

φ = latitude, θ = longitude

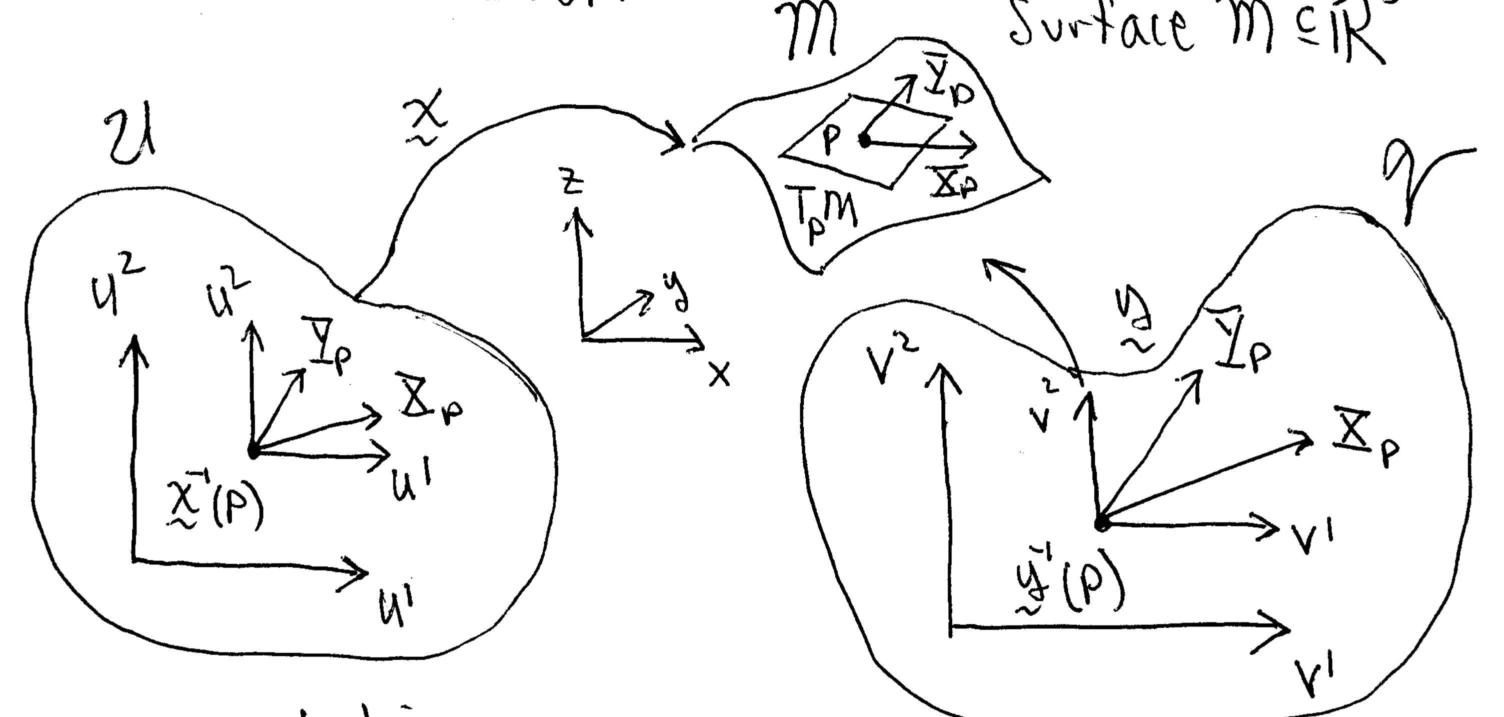
$$\underset{\sim}{x}(\theta,\varphi) = \rho(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$u^1 = \theta$, $u^2 = \varphi$

$$\underset{\sim}{x}(u^1,u^2) = \rho(\sin u^2\cos u^1, \sin u^2\sin u^1, \cos u^2)$$

Theory of Surfaces – Ch 4

Notation ≈ Book



- New notation (Book):

$\underset{\sim}{x} : \mathcal{U} \longrightarrow \mathcal{M}$ (not other way around)

$\underset{\sim}{x}(a,b) = P$, $\underset{\sim}{x}^{-1}(P) = (a,b) \in \mathbb{R}^2 \Leftrightarrow \underset{\sim}{x}^{-1}(P) = (u^1(P), v^1(P))$

- Each tangent vector to $\mathcal{M}$ at P $(\bar{X}_P \in T_P\mathcal{M})$ has a representation in each coord system @ P

$$a^i \frac{\partial}{\partial u^i} = b^\alpha \frac{\partial}{\partial v^\alpha}$$

use (u^1,u^2), (v^1,v^2) to save (x,y,z) for $\mathbb{R}^3 \supseteq \mathcal{M}$

- $\underset{\sim}{x}^{-1}(P) = (u^1(P), u^2(P))$, $\underset{\sim}{y}^{-1}(P) = (v^1(P), v^2(P))$

(2)

Curves $\gamma(t)$ in $\mathcal{M}$: $\gamma : \mathbb{R} \to \mathcal{M} \subseteq \mathbb{R}^3$

$$t \mapsto \gamma(t) \in \mathcal{M}$$

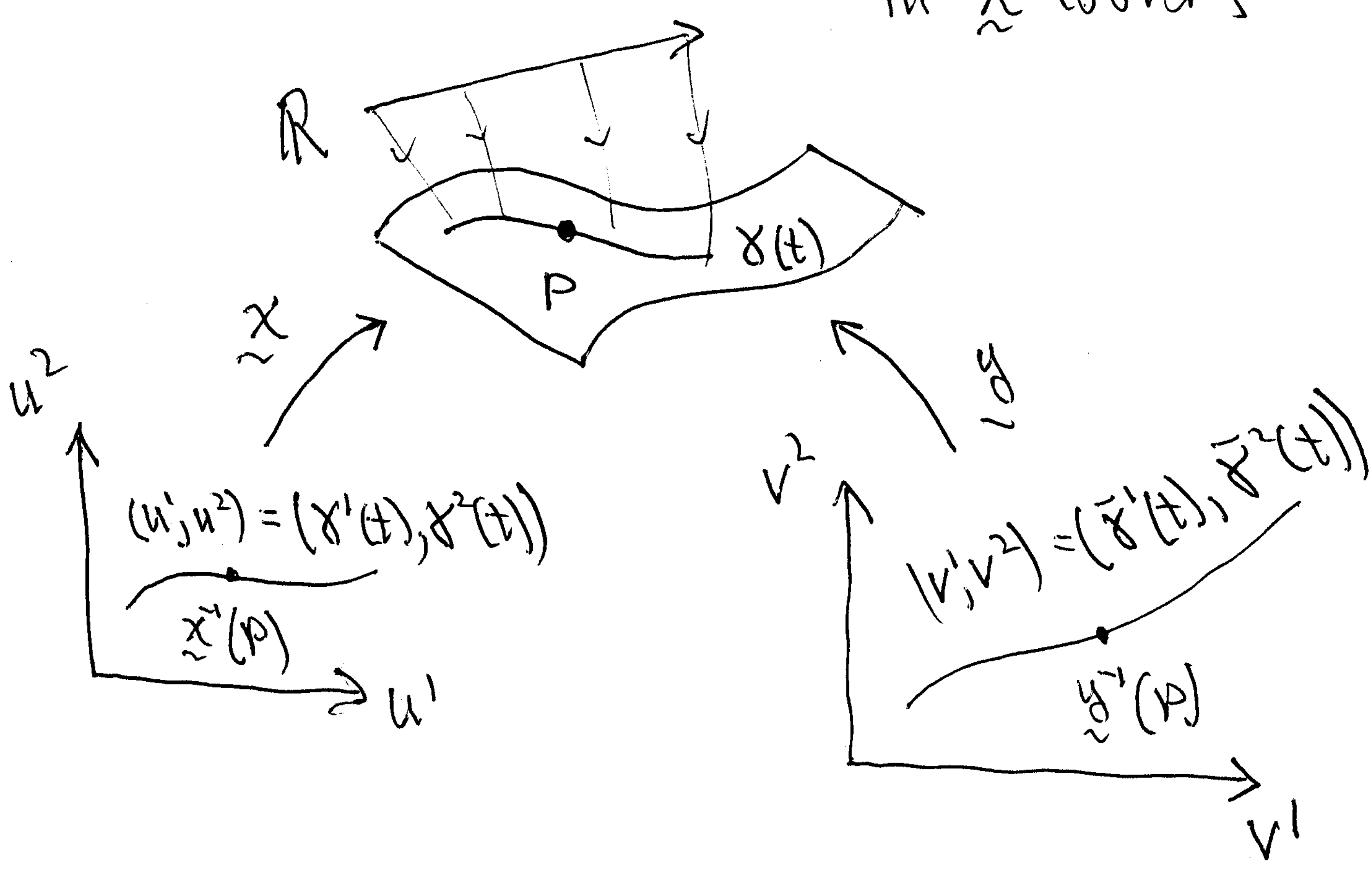
- $\gamma(t)$ has a representation in each coord syst.

$$\gamma(t) = \underset{\sim}{x} \circ (\gamma^1(t), \gamma^2(t))$$

$$\gamma(t) = \underset{\sim}{y} \circ (\bar{\gamma}^1(t), \bar{\gamma}^2(t))$$

I.e., $(\gamma^1(t), \gamma^2(t)) = \underset{\sim}{x}^{-1} \circ \gamma(t)$, etc

I.e., $(u^1, u^2) = (\gamma^1(t), \gamma^2(t))$ gives the curve in $\underset{\sim}{x}$-coord's

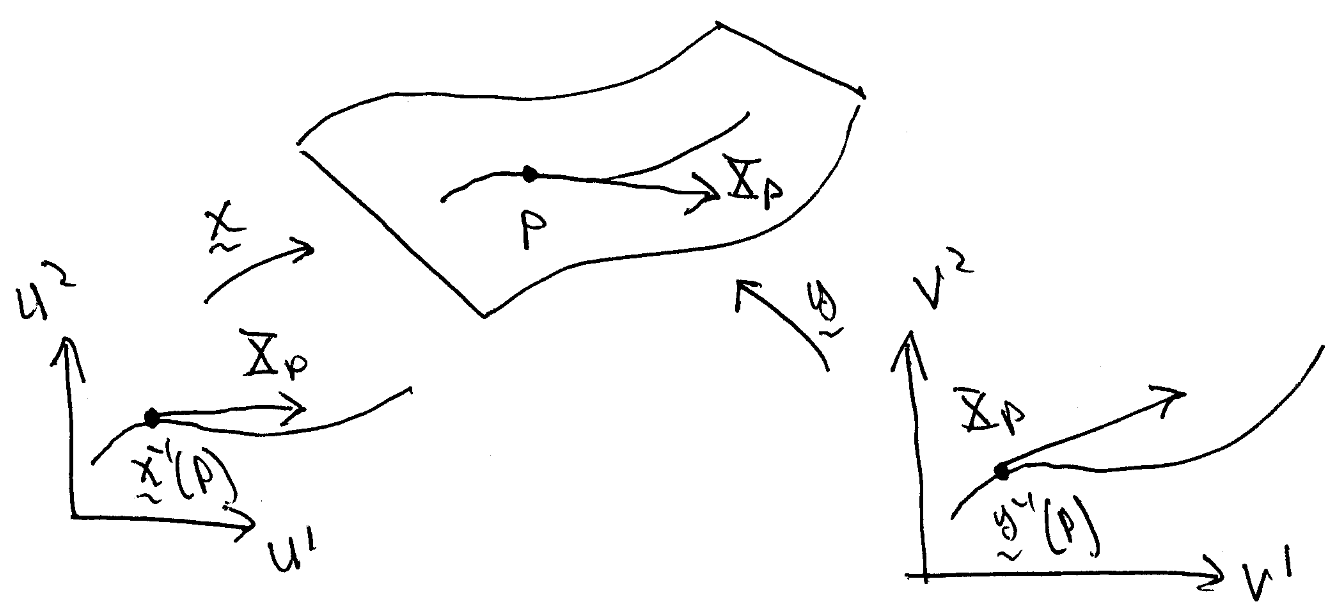


• A smooth curve $\gamma(t)$ passing thru p has a tangent vector at p

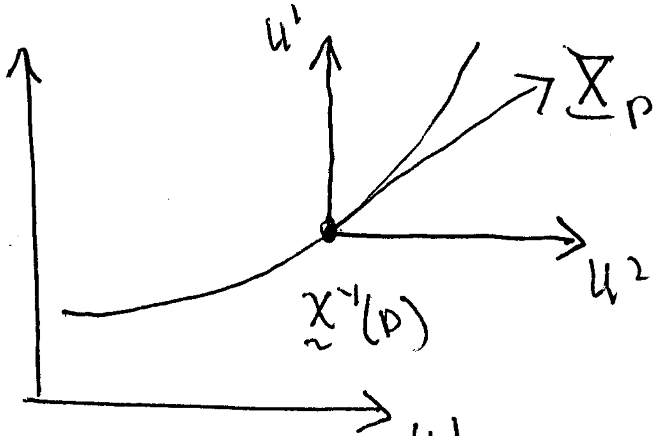
$$\frac{d}{dt}\gamma(t_0) \equiv \dot{\gamma}(t_0) = X_p \in T_p M$$

$$\gamma(t_0) = p$$

This tangent vector has a representation in each coordinate system $\underset{\sim}{x}$, $\underset{\sim}{y}$;



- In $\underset{\sim}{x}$-coordinates:



$$\underset{\sim}{x}^{-1}\circ\gamma(t) = (\gamma^1(t), \gamma^2(t))$$

$$X_p = \text{"}(\dot\gamma^1(t_0), \dot\gamma^2(t_0))\text{"} = \dot\gamma^i \frac{\partial}{\partial u^i} \text{ in } \underset{\sim}{x}\text{-coords}$$

components of X_p in $\underset{\sim}{x}$-coords

unit coord directions

$$\underset{\sim}{y}^{-1}\circ\gamma(t) = (\bar\gamma^1(t), \bar\gamma^2(t))$$

$$X_p = \text{"}(\dot{\bar\gamma}^1(t_0), \dot{\bar\gamma}^2(t_0))\text{"} = \dot{\bar\gamma}^\alpha \frac{\partial}{\partial v^\alpha}$$

$\underset{\sim}{y}$-components

Lets check that they transform like vectors:

① $$\boxed{\dot{\bar\gamma}^\alpha = B^\alpha_i\, \dot\gamma^i} \qquad B^\alpha_i = \frac{\partial v^\alpha}{\partial u^i}$$

② $$\boxed{\frac{\partial}{\partial v^\alpha} = B^i_\alpha \frac{\partial}{\partial u^i}} \qquad B^i_\alpha = \frac{\partial u^i}{\partial v^\alpha}$$

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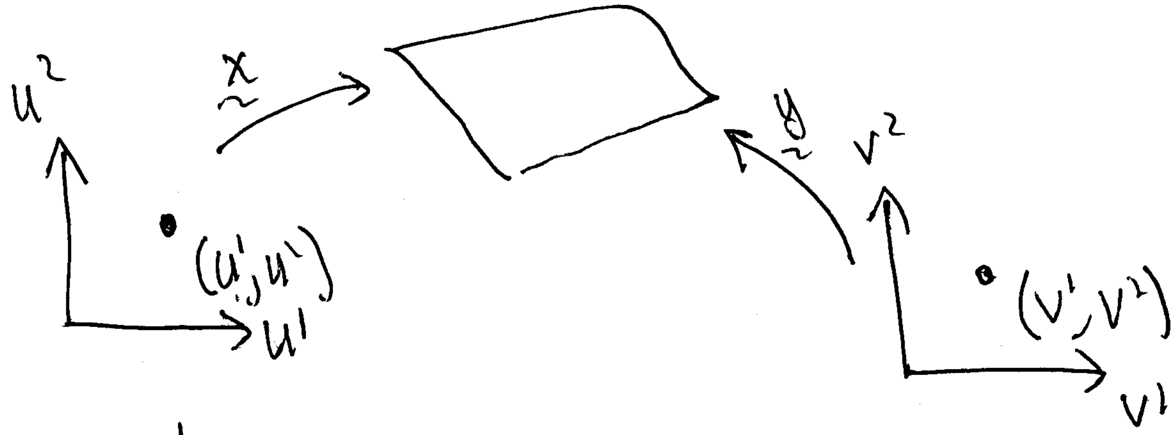


$$(v^1, v^2) = \underset{\sim}{y}^{-1} \circ \underset{\sim}{x}\,(u^1, u^2) = (v^1(u^1, u^2), v^2(u^1, u^2))$$

So

$$(\bar{\gamma}^1(t), \bar{\gamma}^2(t)) = (v^1(\gamma^1(t), \gamma^2(t)), v^2(\gamma^1(t), \gamma^2(t)))$$

curve in $\underset{\sim}{y}$-coords

differentiate :

$$\dot{\bar{\gamma}}^\alpha(t) = \frac{d}{dt}\bar{\gamma}^\alpha(t) = \frac{d}{dt} v^\alpha(\gamma^1(t), \gamma^2(t))$$

$$\overset{\text{chn rule}}{=} \frac{\partial v^\alpha}{\partial u^1}\dot{\gamma}^1(t) + \frac{\partial v^\alpha}{\partial u^2}\dot{\gamma}^2(t)$$

$$= \frac{\partial v^\alpha}{\partial u^i}\dot{\gamma}^i(t)$$

$$\boxed{\dot{\bar{\gamma}}^\alpha(t) = \frac{\partial v^\alpha}{\partial u^i}\dot{\gamma}^i(t)} \qquad B^\alpha_i = \frac{\partial v^\alpha}{\partial u^i}$$

② Using ①:

$$\dot{\gamma}^i \frac{\partial}{\partial u^i} = \dot{\gamma}^\alpha \frac{\partial}{\partial v^\alpha} = \dot{\gamma}^i \underline{\frac{\partial v^\alpha}{\partial u^i} \frac{\partial}{\partial v^\alpha}}$$

2 names for X_p

must be: $$\boxed{\frac{\partial}{\partial u^i} = \frac{\partial v^\alpha}{\partial u^i} \frac{\partial}{\partial v^\alpha}}$$

$$\boxed{\frac{\partial}{\partial v^\alpha} = \frac{\partial u^i}{\partial v^\alpha} \frac{\partial}{\partial u^i}} \qquad B^i_\alpha = \frac{\partial u^i}{\partial v^\alpha}$$

By reversing roles of u, v it follows that

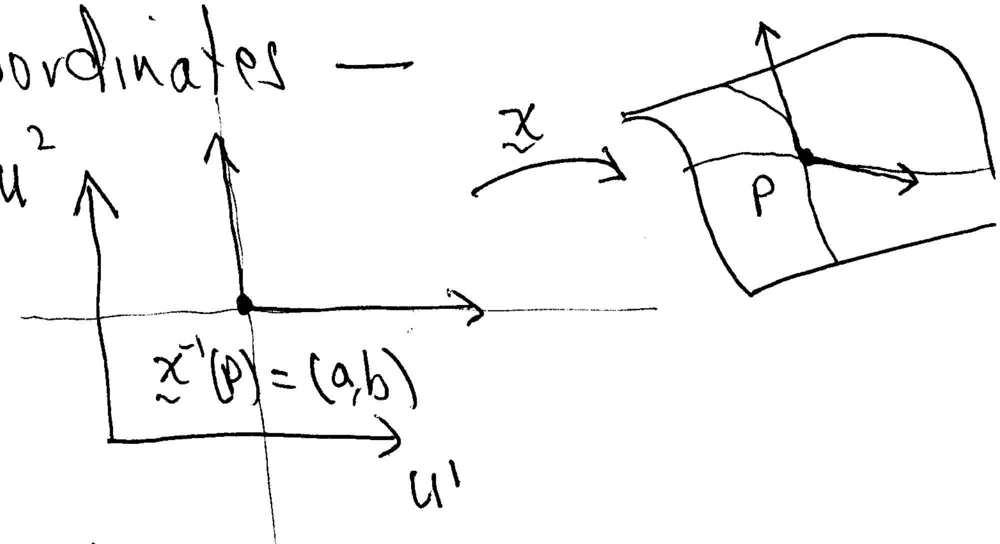
$$B^\alpha_i = \left(\frac{\partial u^i}{\partial v^\alpha}\right)^{-1} = \frac{\partial v^\alpha}{\partial u^i} \quad \checkmark$$

__Defn__: $T_p M$ is the collection of all tangent vectors to regular curves in M passing thru p.

__Lemma__: $T_p M$ is a vector space of dim 2. (i.e., closed under $+$ & $\cdot$) see book.

• $\underset{\sim}{x}$-Coord. vector fields $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}$:

Consider $\underset{\sim}{x}$-coordinates —



Consider curves defined by coord axes passing thru $(a,b) = \underset{\sim}{x}^{-1}(p)$ in $\underset{\sim}{x}$-coords:

$$\alpha(t) = \underset{\sim}{x}(u^1, b) \quad u^1 \equiv t$$

$$\beta(t) = \underset{\sim}{x}(a, u^2) \quad u^2 \equiv t$$

$$\dot{\alpha}(t) = \frac{\partial \underset{\sim}{x}}{\partial u^1}(u^1, b) \equiv \underset{\sim}{x}_1$$

evaluate at

$$\dot{\beta}(t) = \frac{\partial \underset{\sim}{x}}{\partial u^2}(a, u^2) \equiv \underset{\sim}{x}_2$$

so: $\underset{\sim}{x}_1$ = vector in $T_p(M)$ corresponding to $\frac{\partial}{\partial u^1}$

$\underset{\sim}{x}_2$ = vector in $T_p(M)$ corresponding to $\frac{\partial}{\partial u^2}$

<u>Check</u>: $\alpha(t) = \underset{\sim}{x}(u^1, b) \quad t = u^1$

$$= \underset{\sim}{x}(\alpha^1(t), \alpha^2(t))$$

$$\alpha^1(t) = \alpha^1(u^1) = u^1 \Rightarrow \dot{\alpha}^1(t) = 1$$

$$\alpha^2(t) = b \Rightarrow \dot{\alpha}^2(t) = 0$$

So $\dot{\alpha}(t) = \dot{\alpha}^1 \frac{\partial}{\partial u^1} + \dot{\alpha}^2 \frac{\partial}{\partial u^2} = \frac{\partial}{\partial u^1}$ ✓

- Example: $\mathcal{M}$ = sphere $x^2+y^2+z^2=\rho^2$

Curve: $\gamma(t) = (x(t), y(t), z(t))$, $x(t)^2+y(t)^2+z(t)^2=\rho$

$= \underset{\sim}{x}(\gamma^1(t), \gamma^2(t))$ $u^1=\gamma^1(t), u^2=\gamma^2(t)$

$\gamma(t) = \underset{\sim}{x}\circ(\gamma^1(t), \gamma^2(t)) \equiv \rho\,(\sin\gamma^2(t)\cos\gamma^1(t), \sin\gamma^2(t)\sin\gamma^1(t), \cos\gamma^2$

$\dot{\gamma}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$ vector tan to $\mathcal{M}$

$$= \frac{d}{dt}\underset{\sim}{x}\circ(\gamma^1(t), \gamma^2(t))$$

$$= \frac{\partial \underset{\sim}{x}}{\partial u^1}\dot{\gamma}^1(t) + \frac{\partial \underset{\sim}{x}}{\partial u^2}\dot{\gamma}^2(t)$$

$\underset{\sim}{x}_1 = \frac{\partial}{\partial u^1}$, $\underset{\sim}{x}_2 = \frac{\partial}{\partial u^2}$

identify a vector with its representation in each coord system

$$= \dot{\gamma}^i\frac{\partial}{\partial u^i}$$

$$\underset{\sim}{x}_1 \equiv \frac{\partial}{\partial u^1} = \frac{\partial}{\partial u^1}\rho\,(\sin u^2\cos u^1, \sin u^2\sin u^1, \cos u^2)$$
$$= \rho\,(-\sin u^2\cos u^1, \sin u^2\cos u^1, 0)$$

$$\underset{\sim}{x}_2 \equiv \frac{\partial}{\partial u^2} = \frac{\partial}{\partial u^2}\rho(\quad) = \rho\,(\cos u^2\cos u^1, \cos u^2\sin u^1, -\sin u^2)$$

Functions: Assume we have a function defined on surface — eg, say $f(x,y,z) =$ temperature at $(x,y,z) \in \mathbb{R}^3 \supseteq M$.

Q1: if $\underline{X}_p \in T_pM$, what is "the rate at which f is increasing in direction $\underline{X}_p$"? Ans $\nabla f \cdot \underline{X}_p$

Answer from vector calc — $D_{\underline{X}_p} f = \nabla f \cdot \underline{X}_p$

Q2: how do we calculate this in $\underset{\sim}{x}$-coordinates?

Ans: Say $\underline{X}_p = \dot{\gamma}(t_0)$ for some curve $\gamma(t)$, $\gamma(t_0) = p$. Then $f(\gamma(t))$ is the value of f along the curve γ.

Differentiating: $\dfrac{df}{dt} = \dfrac{d}{dt} f(\gamma(t)) = \nabla f \cdot \dot{\gamma}(t)$

Chn Rule

$= \nabla f \cdot \underline{X}_p$

makes sense in $\mathbb{R}^3$

In $\underset{\sim}{x}$-coordinates: $\gamma(t) = \underset{\sim}{x} \circ (\gamma^1(t), \gamma^2(t))$

$$\frac{df}{dt} = \frac{d}{dt} f\left(\underset{\sim}{x} \circ (\gamma^1(t), \gamma^2(t))\right)$$

"$\underset{\sim}{x}$-components of γ"

Now $\underset{\sim}{x} : (u^1, u^2) \longmapsto P \in M$

$\underset{\sim}{x}(u^1, u^2) = P \in M$

$(f \circ \underset{\sim}{x})(u^1, u^2)$ gives (temperature) f at each $(u^1, u^2) \in \mathcal{U}$

Call $f(u^1, u^2) \equiv f \circ \underset{\sim}{x}(u^1, u^2) \equiv$ "f as a fn of (u^1, u^2)"

Then $$\frac{df}{dt} = \frac{d}{dt} f(\gamma^1(t), \gamma^2(t))$$

f as a fn of (u^1, u^2)

$$= \frac{\partial f}{\partial u^1} \dot{\gamma}^1(t) + \frac{\partial f}{\partial u^2} \dot{\gamma}^2(t)$$

$$= \left(\dot{\gamma}^1 \frac{\partial}{\partial u^1} + \dot{\gamma}^2 \frac{\partial}{\partial u^2}\right) f \Big|_p = X_p(f)$$

- <u>Defn</u>: Let $X_p = a^i \frac{\partial}{\partial u^i}$ be the $\underset{\sim}{x}$-word representation of X_p. Then

$$X_p(f) = \left(a^i \frac{\partial}{\partial u^i}\right) f = a^1 \frac{\partial f}{\partial u^1} + a^2 \frac{\partial f}{\partial u^2} = \nabla f \cdot (a^1, a^2)$$

is the rate at which f increases in direction X_p

- <u>Note</u>: We think of $X_p = a^i \frac{\partial}{\partial u^i}$ as "acting on" or "operating on" function f to give the rate of change

- <u>Note</u>: $\frac{\partial}{\partial u^1}$ & $\frac{\partial}{\partial u^2}$ act on the $\underset{\sim}{x}$-coordinate representation of f, namely

$$f(u^1, u^2) \equiv f \circ \underset{\sim}{x}(u^1, u^2) = f(\underset{\sim}{x}(u^1, u^2))$$

original f given as a function of $P \in \mathbb{R}^3$!

Example: M = Sphere. Say temperature

$T = f(x,y,z) = xz$

$x = \rho \sin\varphi \cos\theta \quad \theta = u^1$

$y = \rho \sin\varphi \sin\theta \quad \varphi = u^2$

$z = \rho \cos\varphi$

$\underset{\sim}{x}$-coord rep. of f is:

$$f(u^1,u^2) = \rho^2 \sin\varphi \cos\theta \cos\varphi = \rho^2 \sin u^2 \cos u^2 \cos u^1$$

- Say $\gamma(t)$ is the $\frac{1}{2}$ great circle $\theta = \text{const}$,

$$\gamma(t) = \underset{\sim}{x} \circ (\gamma^1(t), \gamma^2(t)) = \underset{\sim}{x} \circ (\theta, t), \quad \gamma^2(t) = t = u^2$$

$$= \rho(\sin t \cos\theta, \sin t \sin\theta, \cos t)$$

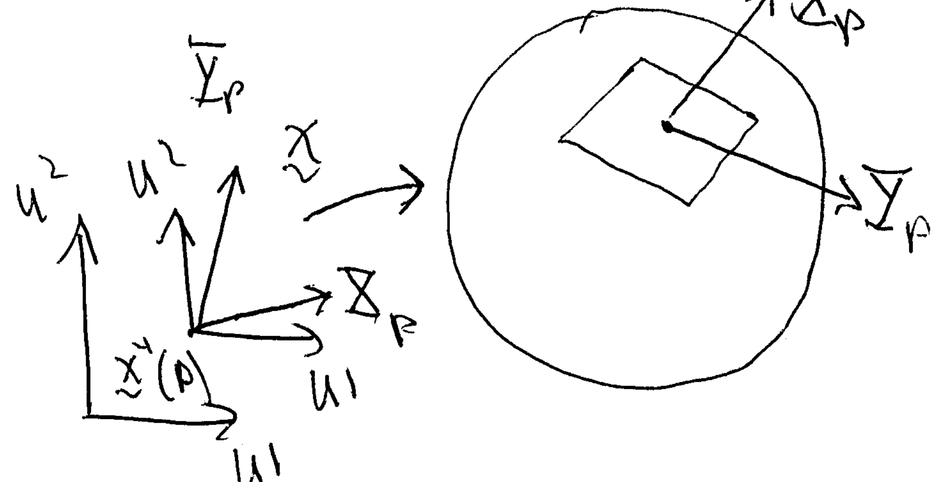
$$\dot{\gamma}(t) = \frac{\partial}{dt} \underset{\sim}{x} \circ (\theta, t) = \underset{\sim}{x}_2 = X$$

- The rate at which T increases along $\gamma(t)$:

$$\frac{d}{dt} f(\gamma(t)) = \frac{d}{dt} \rho^2 \sin t \cos t \cos\theta = \rho^2 \cos\theta (\cos^2 t - \sin^2 t)$$

also $= \frac{\partial f}{\partial u^1} \underbrace{\dot{\gamma}^1(t)}_{0} + \frac{\partial f}{\partial u^2} \underbrace{\dot{\gamma}^2(t)}_{1} = \frac{\partial}{\partial u^2} f = X(f)$ ← same

The metric:



- X_p, Y_p are vectors in $\mathbb{R}^3$ with representati in x-coords — $X_p = a^i \frac{\partial}{\partial u^i}$, $Y_p = b^j \frac{\partial}{\partial u^j}$

$$\langle X_p, Y_p \rangle = \underset{\substack{\uparrow \\ \text{dot product in } \mathbb{R}^3}}{X_p \cdot Y_p} = \left\langle a^i \frac{\partial}{\partial u^i}, b^j \frac{\partial}{\partial u^j} \right\rangle$$

$$\underset{\substack{\uparrow \\ \text{linearity}}}{=} a^i b^j \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle$$

Define: $g_{ij} = \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle = \langle x_i, x_j \rangle$

- We already know how g transforms:

$$\bar{g}_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \quad B^i_\alpha = \frac{\partial x^i}{\partial y^\beta}$$

$$\boxed{\bar{g}_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}}$$

$\bar{g}_{\alpha\beta} = \langle \underset{\sim}{y}_i, \underset{\sim}{y}_j \rangle$ from $\underset{\sim}{y}$-coordinates

- Since $\underset{\sim}{x}_i = \frac{\partial}{\partial y^i}$ we have in each coord syst:

$$\langle X_p, Y_p \rangle = g_{ij} a^i b^j$$

$$\underline{X}_p = a^i \frac{\partial}{\partial u^i} \quad \underline{Y}_p = b^j \frac{\partial}{\partial u^j}$$

<u>Defn</u>: $g \equiv \langle \, , \, \rangle$ is called the metric or <u>First Fundamental Form</u> of the surface.

$g \equiv g_{ij}(P)$ a matrix that changes from pt to pt.

Ex: $M \equiv$ sphere $\quad x^2+y^2+z^2=\rho^2$

$$\underset{\sim}{x}(u^1,u^2) = \rho\left(\sin u^2 \cos u^1,\ \sin u^2 \sin u^1,\ \cos u^2\right)$$

$$u^1 \equiv \theta\ ,\ u^2 \equiv \phi$$

$$\underset{\sim}{x}_1 = \rho\left(-\sin u^2 \sin u^1,\ \sin u^2 \cos u^1,\ 0\right)$$

$$\underset{\sim}{x}_2 = \rho\left(\cos u^2 \cos u^1,\ \cos u^2 \sin u^1,\ -\sin u^2\right)$$

$$g_{11} = \underset{\sim}{x}_1 \cdot \underset{\sim}{x}_1 = \rho^2\left(\sin^2 u^2 \sin^2 u^1 + \sin^2 u^2 \cos^2 u^1\right)$$

$$= \rho^2 \sin^2 u^2$$

$$g_{22} = \rho^2\left(\cos^2 u^2 \cos^2 u^1 + \cos^2 u^2 \sin^2 u^1 + \sin^2 u^2\right)$$

$$= \rho^2\left(\cos^2 u^2 + \sin^2 u^2\right) = \rho^2$$

$$g_{12} = \rho^2\left(-\sin u^2 \sin u^1 \cos u^2 \cos u^1 + \sin u^2 \cos u^1 \cos u^2 \sin u^1\right) = 0$$

$$g_{ij} = \begin{bmatrix} \rho^2 \sin^2 u^2 & 0 \\ 0 & \rho^2 \end{bmatrix}$$

- The metric keeps track of length, angles and area up on $\mathcal{M}$:

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \underset{\substack{\uparrow \\ \text{dot prod} \\ \text{up on } \mathcal{M}}}{\mathbf{X} \cdot \mathbf{Y}} = \|\mathbf{X}\| \|\mathbf{Y}\| \cos\theta = g_{ij}\, a^i b^j$$

$\underline{x}$-comp's of $\mathbf{X}, \mathbf{Y}$

$$\mathbf{X} = a^i \frac{\partial}{\partial u^i} = a^i \underset{\sim}{x}_i \,, \quad \mathbf{Y} = b^j \frac{\partial}{\partial u^j} = b^j \underset{\sim}{x}_j$$

in $\underline{x}$-coords

up on $\mathcal{M} \subseteq \mathbb{R}^3$

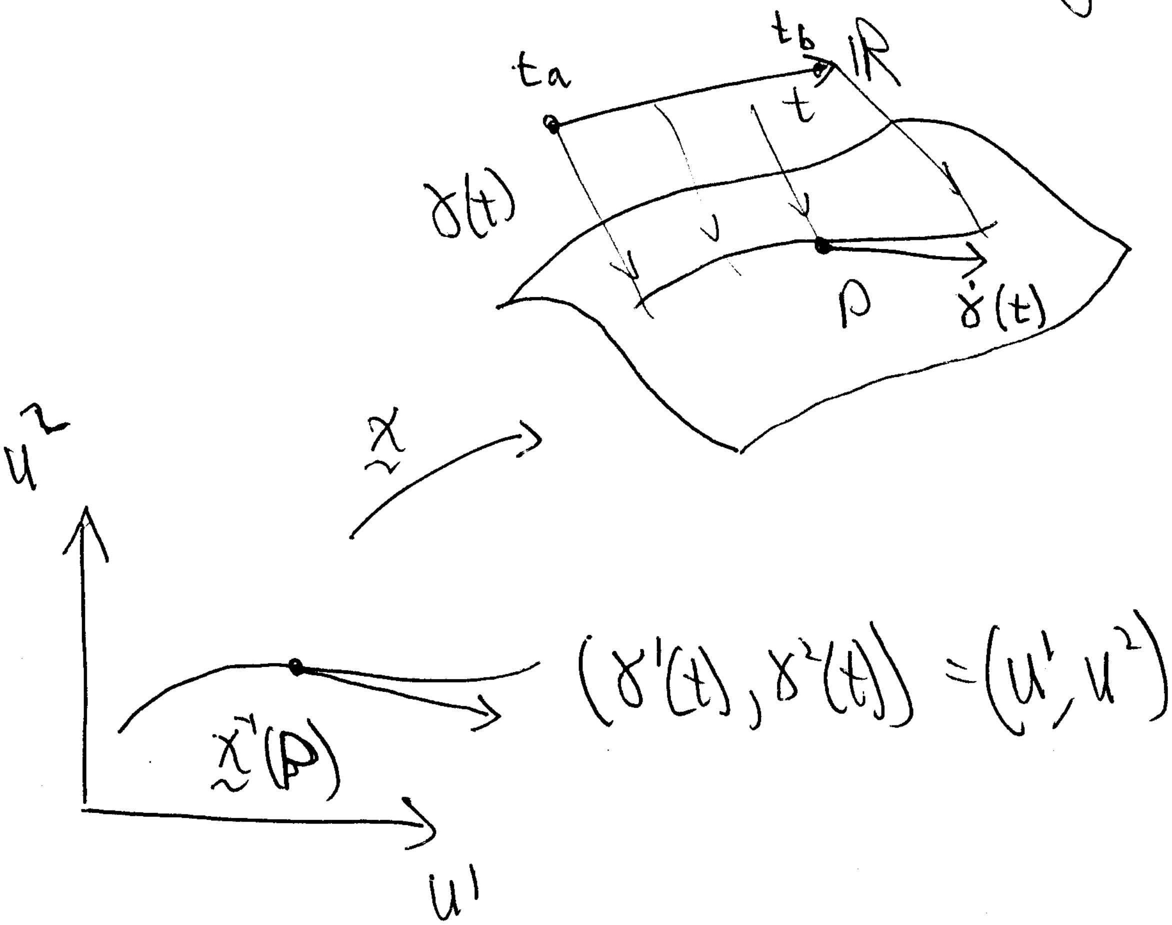
This gives lengths & $\angle$'s directly.

- Turns out: $\det g_{ij} \equiv |g_{ij}| = g > 0$

$g^{1/2} \equiv$ amplification factor for area

$$\int_{\mathcal{U}} g^{1/2}\, du^1 du^2 = \int_{\mathcal{M}} dS \equiv \text{surface area on } \mathcal{M}$$

Length of a curve in $\mathcal{M}$: (arclength)



How to compute length of $\gamma(t)$ up in $\mathcal{M}$ by computing in coord system $\underset{\sim}{x}$.

Ans: $$L_a^b = \int_{t_a}^{t_b} \|\dot{\gamma}(t)\| \, dt$$

You have to compute $\dot{\gamma}(t)$ using the metric at $\underset{\sim}{x}^{-1}(\gamma(t))$

__Why__: Consider a curve in $\mathbb{R}^3$ $\gamma(t)$

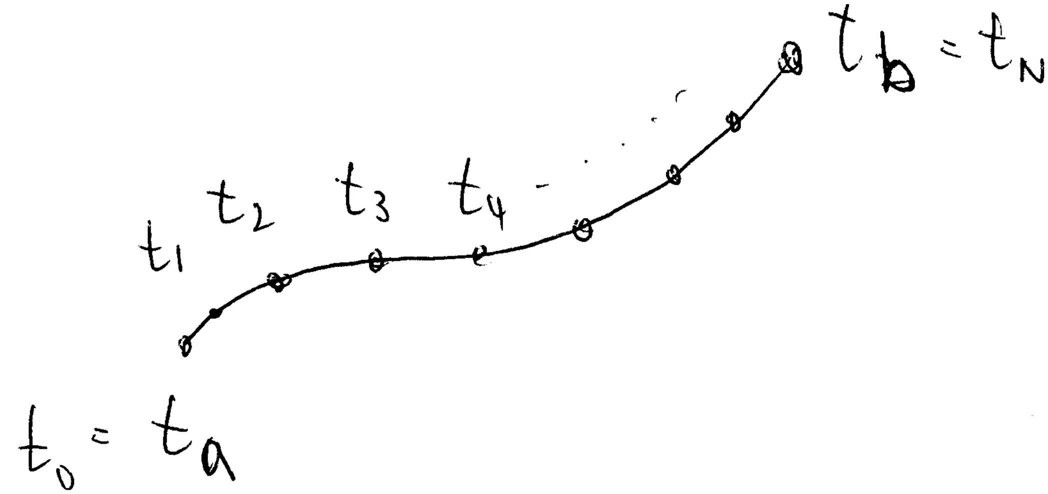


Fig (*)

$$L_a^b \approx \sum_{i=1}^{N} \|\gamma(t_i) - \gamma(t_{i-1})\| = \sum_{i=1}^{N} \left\| \frac{\gamma(t_i) - \gamma(t_{i-1})}{t_i - t_{i-1}} \right\| |t_i - t_{i-1}|$$

$$\approx \sum_{i=1}^{N} \|\dot{\gamma}(t_i)\| \Delta t_i \quad \text{with errors} \approx |\Delta t|^2$$

as $\Delta t = \operatorname{Max} \Delta t_i \to 0$

$$\Rightarrow \boxed{L_a^b = \int_{t_a}^{t_b} \|\dot{\gamma}(t)\| dt} \qquad (*)$$

Since we can calculate $\|\dot{\gamma}(t)\|$ down in $\underline{x}$-coords as $\left[g_{ij} \dot{\gamma}^i(t) \dot{\gamma}^j(t) \right]^{1/2}$, (*) gives a way to compute arclength in coordinate systems

- The deeper point here is that all errors of order $|\Delta t|^2$ are squeezed to zero in the the approximation of the integral, and as a result, we can get lengths on $\mathcal{M}$ by just keeping track of lengths of tangent vectors in each $T_p\mathcal{M}$. That is why giving g_{ij} at each P (a tensor!) suffices to record all lengths up on $\mathcal{M}$, and that is why the whole framework of differential geometry works.
- To make this point more clear, assume in Figure (*) that each $\Delta t_i = \Delta t = \frac{t_b - t_a}{N}$ and note by Taylors theorem

$$\gamma(t_i) = \gamma(t_{i-1}) + \dot{\gamma}(t_{i-1})\Delta t + E_i$$

where $|E_i| \leq C|\Delta t|^2$, $C \equiv$ const indept of i.

Also, the arclength Δs_i along γ between t_{i-1} and t_i is approximated by

$$\Delta s_i = \|\gamma(t_i) - \gamma(t_{i-1})\| + \bar{E}_i$$

where $\bar{E}_i \le \bar{C}\,|\Delta t|^2$. Thus, being more precise in doing the approximations after Figure (*) yields

$$L_a^b = \sum_{i=1}^{N} \Delta s_i = \sum_{i=1}^{N} \left\{ \|\gamma(t_i) - \gamma(t_{i-1})\| + \bar{E}_i \right\}$$

$$= \sum_{i=1}^{N} \left\{ \|\dot{\gamma}(t_{i-1})\|\Delta t + \bar{E}_i + E_i \right\}$$

Now in the limit $\Delta t \to 0$, the 1st term tends to $\int_{t_a}^{t_b} \|\dot{\gamma}(t)\|\,dt$, & the sum of all errors is

$$\sum_{i=1}^{N} \left(\bar{C}\Delta t^2 + C\Delta t^2\right) \underset{\uparrow}{=} \frac{t_b - t_a}{\cancel{\Delta t}} (\bar{C} + C)\,\Delta t^{\cancel{2}} = (t_b - t_a)(\bar{C} + C)\Delta t$$

$$N = \frac{t_b - t_a}{\Delta t} \qquad \to 0 \text{ as } \Delta t \to 0!$$

- Conclude: In order to get lengths of curves up on M, it suffices to only keep track of the lengths of tangent vectors — and tangent vectors can be viewed as giving a length and direction up on M "neglecting second order errors".
- We now show carefully that tangent vectors give lengths and directions "to second order up on M. To make this precise, let $\gamma(s)$ be a regular curve parameterized wrt arclenth starting at $P = \gamma(0)$, & let $X_0 = \dot{\gamma}(0)$, so then $\|X_0\| = 1$. Let $X_s = s X_0$ so $\|X_s\| =$

Theorem: $X_s = \gamma(s) - \gamma(0) + E$ where $|E| \leq C \|X_s\|^2$.

The theorem makes precise the statement that $\mathbf{X} \equiv \mathbf{X}_s$ gives a length and a direction up on $\mathcal{M}$ "to within errors that are 2nd order in $\|\mathbf{X}\|$".

Pf. By Taylors Theorem,

$$\gamma(s) = \gamma(0) + \dot{\gamma}(0)s + E \quad \text{where } |E| \leq C s^2$$

Since $\mathbf{X}_s = \dot{\gamma}(0)s$, & $\|\mathbf{X}_s\| = s$, were done. $\square$

<u>Conclude</u>: The whole framework of differential geometry is based on the fact that to get everything about distances up on $\mathcal{M}$, it suffices to define a gadget g_{ij} at each p, where $g_{ij}(p)$ only records lengths "to leading order @ p" i.e., neglecting 2nd order errors

• Philosophical Question: "Why is calculus so fundamentally important"?

Answer: The laws of science come to us stated in terms of derivatives — Eg $\vec{F} = m\vec{a}$ — and it suffices by calculus to give them in terms of derivatives because integrations does not require 2nd order error to work.

More generally: The laws of physics & science come to us stated in terms of tensors! Eg "$G = 8\pi T$", Einstein's Eqn, is stated in terms of tensors.

Differential

- Notation for the metric g:

$\underset{\sim}{x}$-coords: $\quad ds^2 = g_{ij}\, du^i du^j \quad$ (summation conv.)

Or: $\quad ds^2 = g_{11}(du^1)^2 + 2g_{12}\, du^1 du^2 + g_{22}(du^2)^2$

ds = "the increment in arclength corresponding to increments du^1 & du^2 in $\underset{\sim}{x}$-coords"

- Of course: components $g_{ij}(p)$ change from pt to pt so this really means the increment in s along tangent plane - close enough so that you get the correct change when you integrate.

- Now we can integrate the differential ds along a curve to get the change in arclength:

$\gamma(t) = \underset{\sim}{x} \circ (\gamma^1(t), \gamma^2(t)) \quad u^1 = \gamma^1(t),\ u^2 = \gamma^2(t)$

$$L_a^b = \int_{t_a}^{t_b} ds = \int_{t_a}^{t_b} \sqrt{g_{ij}\, du^i du^j} = \int_{t_a}^{t_b} \sqrt{g_{ij} \frac{du^i}{dt} \frac{du^j}{dt}}\, dt = \int_{t_a}^{t_b} \|\gamma(t)\|\, dt$$

We can think of ds as acting on tangent vectors "like" a 1-form, but ds is <u>not linear</u>, being the square root of something bilinear:

I.e., recall the 1-form du^i acts linearly on tangent vectors by picking out i-th component:

$$X = a^j \frac{\partial}{\partial u^j} \Rightarrow du^i(X) = a^i = \text{"i-th comp of } X\text{"}$$

Said differently: $du^i(X)$ is the increment in coordinate u^i corresponding to motion X in $T_p(M)$.

- Similarly, we can define

$$ds(X) = \sqrt{g_{ij}\, du^i(X)\, du^j(X)} = \sqrt{g_{ij} a^i a^j} = \|X\|$$

which is the "increment in arclength" correspondin to motion X in $T_p(M)$

- Then along a curve $\gamma(t)$ with tangent

$$X = \dot{\gamma}^i(t) \frac{\partial}{\partial u^i}$$

we define

$$ds = ds(X)\, dt$$

Then

$$L_a^b = \int_\gamma ds = \int_{t_a}^{t_b} ds(X)\, dt = \int_{t_a}^{t_b} \|X\|\, dt$$

as before.

Ex: metric on a sphere: $ds^2 = \rho^2 \underbrace{(\sin^2\varphi\, d\theta^2 + d\varphi^2)}$
(Sherical coordinates)

line element on unit sphere denoted $d\Omega^2$

$u^1 = \theta$, $u^2 = \varphi$

$\underset{\sim}{x}(u^1, u^2) = \rho^2 (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi)$

• Ex: What about $M \equiv$ all of $\mathbb{R}^3$, and view $\underset{\sim}{x}(\theta, \varphi, \rho) = \rho(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ as a coordinate system for M.

Q: what is g_{ij} in $\underset{\sim}{x}$-coords $u^1 = \theta, u^2 = \varphi, u^3 = \rho$

Ans: $g_{ij} = \langle \underset{\sim}{x}_i, \underset{\sim}{x}_j \rangle$

$$\underset{\sim}{x}_1 = \frac{\partial \underset{\sim}{x}}{\partial u^1}, \quad \underset{\sim}{x}_2 = \frac{\partial \underset{\sim}{x}}{\partial u^2}, \quad \underset{\sim}{x}_3 = \frac{\partial \underset{\sim}{x}}{\partial u^3}$$

HW. $g_{ij} = \begin{bmatrix} \rho^2\sin^2\varphi & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\underbrace{ds^2 = d\rho^2 + \rho^2\sin^2\varphi\, d\theta^2 + \rho^2 d\varphi^2}_{} = d\rho^2 + \rho^2 d\Omega^2$$

"The line element for $\mathbb{R}^3$ in spherical coordinates"

$\boxed{d\Omega^2 = \sin^2\varphi\, d\theta^2 + d\varphi^2}$

$d\Omega^2 \equiv$ line element on unit sphere S^2 (standard notation!)