Curvature of Riemannian manifolds

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In mathematics, specifically differential geometry, the infinitesimal geometry of Riemannian manifolds with dimension at least 3 is too complicated to be described by a single number at a given point. Riemann introduced an abstract and rigorous way to define it, now known as the curvature tensor. Similar notions have found applications everywhere in differential geometry.

For a more elementary discussion see the article on curvature which discusses the curvature of curves and surfaces in 2 and 3 dimensions.

The curvature of a pseudo-Riemannian manifold can be expressed in the same way with only slight modifications.

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Ways to express the curvature of a Riemannian manifold

The Riemann curvature tensor

Main article: Riemann curvature tensor

The curvature of Riemannian manifold can be described in various ways; the most standard one is the curvature tensor, given in terms of a Levi-Civita connection (or covariant differentiation) ∇ and Lie bracket $[\cdot,\cdot]$ by the following formula:

$$R(u,v)w = \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w.$$

Here $R(u,v)$ is a linear transformation of the tangent space of the manifold; it is linear in each argument. If $u = \partial/\partial x_i$ and

$v = \partial/\partial x_j$ are coordinate vector fields then $[u,v] = 0$ and therefore the formula simplifies to

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w$$

i.e. the curvature tensor measures *noncommutativity of the covariant derivative*.

The linear transformation $w \mapsto R(u,v)w$ is also called the **curvature transformation** or **endomorphism**.

NB. There are a few books where the curvature tensor is defined with opposite sign.

Symmetries and identities

The curvature tensor has the following symmetries:

$$R(u,v) = -R(v,u)$$

$$\langle R(u,v)w, z \rangle = -\langle R(u,v)z, w \rangle$$

$$R(u,v)w + R(v,w)u + R(w,u)v = 0$$

The last identity was discovered by Ricci, but is often called the *first Bianchi identity*, just because it looks similar to the Bianchi identity below. These three identities form a complete list of symmetries of the curvature tensor, i.e. given any tensor which satisfies the identities above, one could find a Riemannian manifold with such a curvature tensor at some point. Simple calculations show that such a tensor has $n^2(n^2 - 1) / 12$ independent components. Yet another useful identity follows from these three:

$$\langle R(u,v)w,z\rangle = \langle R(w,z)u,v\rangle$$

The **Bianchi identity** (often the **second Bianchi identity**) involves the covariant derivatives:

$$\nabla_u R(v,w) + \nabla_v R(w,u) + \nabla_w R(u,v) = 0$$

Sectional curvature

Main article: Sectional curvature

Sectional curvature is a further, equivalent but more geometrical, description of the curvature of Riemannian manifolds. It is a function $K(\sigma)$ which depends on a *section* σ (i.e. a 2-plane in the tangent spaces). It is the Gauss curvature of the σ-*section* at *p*; here σ-*section* is a locally-defined piece of surface which has the plane σ as a tangent plane at *p*, obtained from geodesics which start at *p* in the directions of the image of σ under the exponential map at *p*.

If v, u are two linearly independent vectors in σ then

$$K(\sigma) = K(u,v)/|u\wedge v|^2 \text{ where } K(u,v) = \langle R(u,v)v,u\rangle$$

The following formula indicates that sectional curvature describes the curvature tensor completely:

$$6\langle R(u,v)w,z\rangle =$$
$$[K(u+z,v+w) - K(u+z,v) - K(u+z,w) - K(u,v+w) - K(z,v+w) - K(v+z,u) + K(u,w) + K(v,z)] -$$

$$[K(u+w,v+z)-K(u+w,v)-K(u+w,z)-K(u,v+z)-K(w,v+z)-K(u+w,v)+K(v,w)+K(u,z)].$$

Curvature form

Main article: Curvature form

The connection form gives an alternative way to describe curvature. It is used more for general vector bundles, and for principal bundles, but it works just as well for the tangent bundle with the Levi-Civita connection. The curvature of n-dimensional Riemannian manifold is given by an antisymmetric $n\times n$ matrix $\Omega = \Omega^i{}_j$ of 2-forms (or equivalently a 2-form with values in $so(n)$, the Lie algebra of the orthogonal group $O(n)$, which is the structure group of the tangent bundle of a Riemannian manifold).

Let e_i be a local section of orthonormal bases. Then one can define the connection form, an antisymmetric matrix of 1-forms $\omega = \omega^i{}_j$ which satisfy from the following identity

$$\omega^k{}_j(e_i) = \langle \nabla_{e_i} e_j, e_k \rangle$$

Then the curvature form $\Omega = \Omega^i{}_j$ is defined by

$$\Omega = d\omega + \omega \wedge \omega$$

The following describes relation between curvature form and curvature tensor:

$$R(u,v)w = \Omega(u \wedge v)w.$$

This approach builds in all symmetries of curvature tensor except the *first Bianchi identity*, which takes form

$$\Omega \wedge \theta = 0$$

where $\theta = \theta^i$ is an *n*-vector of 1-forms defined by $\theta^i(v) = \langle e_i, v \rangle$. The *second Bianchi identity* takes form

$$D\Omega = 0$$

D denotes the exterior covariant derivative

The curvature operator

It is sometimes convenient to think about curvature as an operator Q on tangent bivectors (elements of $\Lambda^2(T)$), which is uniquely defined by the following identity:

$$\langle Q(u \wedge v), w \wedge z \rangle = \langle R(u, v)z, w \rangle.$$

It is possible to do this precisely because of the symmetries of the curvature tensor (namely antisymmetry in the first and last pairs of indices, and block-symmetry of those pairs).

Further curvature tensors

In general the following tensors and functions do not describe the curvature tensor completely, however they play an important role.

Scalar curvature

Main article: Scalar curvature

Scalar curvature is a function on any Riemannian manifold, usually denoted by *Sc*. It is the full trace of the curvature tensor; given an orthonormal basis $\{e_i\}$ in the tangent space at p we have

$$Sc = \sum_{i,j} \langle R(e_i, e_j)e_j, e_i \rangle = \sum_i \langle \mathrm{Ric}(e_i), e_i \rangle,$$

where *Ric* denotes Ricci tensor. The result does not depend on the choice of orthonormal basis. Starting with dimension 3, scalar curvature does not describe the curvature tensor completely.

Ricci curvature

Main article: Ricci curvature

Ricci curvature is a linear operator on tangent space at a point, usually denoted by *Ric*. Given an orthonormal basis $\{e_i\}$ in the tangent space at p we have

$$Ric(u) = \sum_i R(u, e_i)e_i.$$

The result does not depend on the choice of orthonormal basis. Starting with dimension 4, Ricci curvature does not describe the curvature tensor completely.

Explicit expressions for the Ricci tensor in terms of the Levi-Civita connection is given in the article on Christoffel symbols.

Weyl curvature tensor

Main article: Weyl tensor

The **Weyl curvature tensor** has the same symmetries as the curvature tensor, plus one extra: its Ricci curvature must vanish. In dimensions 2 and 3 Weyl curvature vanishes, but if the dimension $n > 3$ then the second part can be non-zero.

- The curvature tensor can be decomposed into the part which depends on the Ricci curvature, and the Weyl tensor.
- If $g' = fg$ for some positive scalar function f — a conformal change of metric — then $W' = W$.
- For a manifold of constant curvature, the Weyl tensor is zero.
  - Moreover, $W=0$ if and only if the metric is locally conformal to the standard Euclidean metric (equal to fg, where g is the standard metric in some coordinate frame and f is some scalar function).

Ricci decomposition

Main article: Ricci decomposition

Although individually, the Weyl tensor and Ricci tensor do not in general determine the full curvature tensor, the Riemann curvature tensor can be decomposed into a Weyl part and a Ricci part. This decomposition is known as the Ricci decomposition, and plays an important role in the conformal geometry of Riemannian manifolds. In particular, it can be used to show that if the metric is rescaled by a conformal factor of e^{2f}, then the Riemann curvature tensor changes to (seen as a $(0,4)$-tensor):

$$e^{2f}(R + (\mathrm{Hess}(f) - df \otimes df + \frac{1}{2}\|\mathrm{grad}(f)\|^2 g) \circ g)$$

where $\circ$ denotes the Kulkarni–Nomizu product and Hess is the Hessian.

Calculation of curvature

For calculation of curvature

- of hypersurfaces and submanifolds see second fundamental form,
- in coordinates see the list of formulas in Riemannian geometry or covariant derivative,
- by moving frames see Cartan connection and curvature form.
- the Jacobi equation can help if one knows something about the behavior of geodesics.

References

- Kobayashi, Shoshichi and Nomizu, Katsumi (1996 (New edition)). *Foundations of Differential Geometry, Vol. 1*. Wiley-Interscience. ISBN 0471157333.

Notes

Categories: Riemannian geometry | Curvature (mathematics)