MIDTERM EXAM
Math 116
Temple-Spring 2010

−Print your name, section number and put your signature on the upper right-hand corner of this exam. Write only on the exam.

−Show all of your work, and justify your answers for full credit.

<u>SCORES</u>

#1

#2

#3

#4

#5

TOTAL:

1. (20 pts) Let $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})$ be the **x**-coordinate basis for the tangent space $T_P\mathcal{M}$ of an 2-dimensional manifold $\mathcal{M}$ at a point $P \in \mathcal{M}$, and let $\mathbf{X}_P = a^i \frac{\partial}{\partial x^i}$ be a vector at P, g_{ij} the the **x**-components of the metric at P (the components of a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$-tensor), and A^i_j the **x**-components of a linear transformation at P (the components of a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$-tensor). Let $\bar{a}^\alpha \frac{\partial}{\partial y^\alpha}$, $\bar{g}_{\alpha\beta}$ and $\bar{A}^\alpha_\beta$ be the corresponding components relative to a basis $(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2})$ given by $\frac{\partial}{\partial y^\alpha} = B^i_\alpha \frac{\partial}{\partial x^i}$ (sum repeated up-down indices from 1 to 2). Let B denote the 2×2 matrix B^i_α, and let B^α_i denote the matrix B^{-1}.

   (a) *Using the summation convention*, write the transformations that give the **y**-components $\bar{a}^\alpha$, $\frac{\partial}{\partial y^\alpha}$, $\bar{g}_{\alpha\beta}$ and $\bar{A}^\alpha_\beta$ in terms of the unbarred **x**-components .

   (b) Using B, B^{-1}, g and A as usual, with $\mathbf{a} = \begin{pmatrix} a^1 \\ a^2 \end{pmatrix}$ and $\frac{\partial}{\partial \mathbf{x}} = \left(\frac{\partial}{\partial x^1}, \ \frac{\partial}{\partial x^2} \right)$, etc., write the corresponding transformations in matrix notation.

2. (20 pts) Recall that the components of a 1-form $a_i\,dx^i$ transform by the rule $\bar{a}_\alpha = a_i B^i_\alpha$. Show that if a^i are the components of a vector, and in each coordinate system we define

$$a_i = g_{ij} a^j,$$

then a_i transform like the components of a 1-form. (This is called the lowering of the index by the metric.)

3. (20 pts) Let $\gamma(s) = (x(s), y(s), z(s))$ denote a regular curve in R^3 parameterized by arclength s, and let $\mathbf{T}(s) = \gamma'(s)$, $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$ and $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$ define the curvature $\kappa(s)$, unit normal $\mathbf{N}(s)$, and binormal $\mathbf{B}(s)$.

(a) Writing $\frac{dx}{ds} = x'(s)$, $\frac{dy}{ds} = y'(s)$, $\frac{dz}{ds} = z'(s)$, use the differential identities

$$ds = \sqrt{dx^2 + dy^2 + dz^2},$$

and

$$dx = x'(s)ds, \quad dy = y'(s)ds, \quad dz = z'(s)ds$$

to prove $\|\mathbf{T}(s)\| = 1$.

(b) Prove: If $\kappa(s) \neq 0$, then $\mathbf{N}(s) \perp \mathbf{T}(s)$. (Hint: differentiate $\mathbf{T}(s) \cdot \mathbf{T}(s) = 1$.)

(c) Prove $\|\mathbf{B}(s)\| = 1$.

(d) Justify defining $\tau(s)$ via $\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s)$ by proving $\mathbf{B}'(s)$ is parallel to $\mathbf{N}(s)$.

(e) Derive the *Frennet-Seret* equations:

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

4. Let $\gamma(t) = (r \cos t, r \sin t, bt)$.

(a) (5 pts) Find $\gamma(s)$, the arclength parameterization when s goes from $t = 0$ to t.

(b) (15 pts) Find the curvature κ and torsion τ as a function of t.

4. (cont)

5. (20 pts) Find a 2×2 matrix B such that

$$B^T g B = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$$

where $\delta_i = \pm 1$ and

$$g = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

(Hint: Use the matrix methods used in proof of Sylvester's Theorem.)

5. (cont)