# MIDTERM SOLUTIONS



#1

(a) $\bar{a}^\alpha = a^i B^\alpha_i \ , \ \frac{\partial}{\partial y^\alpha} = B^i_\alpha \frac{\partial}{\partial x^i}$

$\bar{g}_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta \ , \ \bar{A}^\alpha_\beta = A^i_j B^j_\beta B^\alpha_i$

(b) $\bar{g} = B^T g B \ , \ \bar{A} = B^{-1} A B$

$\bar{a} = B^{-1} a \ , \ \frac{\partial}{\partial x} B = \frac{\partial}{\partial y}$

#2

$$\bar{a}_\alpha = \bar{g}_{\alpha\beta}\, \bar{a}^\beta$$

$$= g_{ij} B^i_\alpha \underbrace{B^j_\beta B^\beta_k}_{\delta_{jk} \Rightarrow j=k} a^k$$

$$= g_{ij} B^i_\alpha a^j = \underbrace{g_{ij} a^j}_{a_i} B^i_\alpha$$

$$= a_i B^i_\alpha$$

as claimed

#3 $B^T g B = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$ $g = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

- Idea: $g$ symmetric, so find ON basis of e-vects, diagonalize, then mult by $\begin{bmatrix} 1/\sqrt{|\lambda_1|} & \\ & 1/\sqrt{|\lambda_2|} \end{bmatrix}$ left & right.
- evals: $\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 9 = \lambda^2 - 2\lambda - 8$

$$= (\lambda-4)(\lambda+2) \Rightarrow \lambda = -2, 4$$

- e-vects: $\begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix} \begin{bmatrix} 1 \\ r \end{bmatrix} = 0 \Rightarrow \begin{matrix} 1-\lambda+3r=0 \\ 3+r-\lambda r=0 \end{matrix} \Rightarrow r = \frac{\lambda-1}{3}$

$\lambda=-2, r=-1, \lambda=4, r=+1 \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \tilde{B} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & +1 \end{bmatrix} \Rightarrow$ colms are O.N. $\Rightarrow \tilde{B}^T = \tilde{B}^{-1}$

(#3) cont $\quad \tilde{B}^T g \tilde{B} = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/2 \end{bmatrix} \Rightarrow Q^T \tilde{B}^T g \tilde{B} Q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \tilde{B} Q = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/2 \end{bmatrix} = \frac{\sqrt{2}}{4} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{\sqrt{2}}{4} \begin{bmatrix} \sqrt{2} & 1 \\ -\sqrt{2} & 1 \end{bmatrix}$$

<u>Check</u>: $\frac{\sqrt{2}}{4} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1 \\ -\sqrt{2} & 1 \end{bmatrix} \frac{\sqrt{2}}{4}$

$$= \frac{1}{8} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2\sqrt{2} & 4 \\ 2\sqrt{2} & 4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 2 \\ \sqrt{2} & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

#4

(a) $\|\gamma'(s)\| = \left\|\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)\right\|$

$$= \left(\frac{dx^2+dy^2+dz^2}{ds^2}\right)^{1/2} = 1$$

(b) $0 = \frac{d}{dt}\left\{T(s)\cdot T(s)\right\} = 2T'(s)\cdot T(s) = 2K(s)N\cdot T$

(c) $\|B(s)\| = \|T(s)\times N(s)\| = \|T(s)\|\|N(s)\|\sin\theta = 1$

(d) $B'(s) \perp B(s) \quad (\|B(s)\| = 1)$

$B'(s) \perp T(s)$ since $0 = \frac{d}{ds}(B\cdot T) = B'\cdot T + \cancel{B\cdot T'}$

$\quad T' = KN$

$\therefore B'(s) \parallel N(s)$ ✓

(e) Only need: $N' = -KT + \tau B$

$0 = N\cdot T \Rightarrow 0 = N'\cdot T + NT' = N'\cdot T + K \Rightarrow N'\cdot T = -K$

$0 = N\cdot B \Rightarrow 0 = N'B + NB' \Rightarrow N'\cdot B - \tau \Rightarrow N'\cdot B = \tau$

$0 = N'\cdot N$

Thus: $N' = (N'\cdot T)T + (N'\cdot N)N + (N'\cdot B)B = -KT + \tau B$ ✓

(#5) (a) $s = \int_0^t |\gamma'(t)|\,dt = \int_0^t \sqrt{r^2\sin^2 t + r^2\cos^2 t + b}\,dt$ 

$$= \int_0^t \sqrt{r^2+b^2}\,dt = \sqrt{r^2+b^2}\,t$$

(b) $\gamma(s) = \left(r\cos\frac{s}{\sqrt{r^2+b^2}},\ r\sin\frac{s}{\sqrt{r^2+b^2}},\ b\,\frac{s}{\sqrt{r^2+b^2}}\right)$

$$\gamma'(s) = \frac{1}{\sqrt{r^2+b^2}}\left(-r\sin\frac{s}{\sqrt{r^2+b^2}},\ r\cos\frac{s}{\sqrt{r^2+b^2}},\ b\right) = T(s)$$

$$T'(s) = \frac{1}{r^2+b^2}\left(-r\cos\frac{s}{\sqrt{r^2+b^2}},\ -r\sin\frac{s}{\sqrt{r^2+b^2}},\ 0\right)$$

$$= \frac{r}{r^2+b^2}(-\cos t, -\sin t, 0) \Rightarrow N(s) = -(\cos t, \sin t, 0)$$

$$B(s) = T(s)\times N(s) = \frac{1}{\sqrt{r^2+b^2}}\begin{vmatrix} -r\sin t & r\cos t & b \\ -\cos t & -\sin t & 0 \\ \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \end{vmatrix}$$

$$= (+b\sin t\,\underset{\sim}{i} - b\cos t\,\underset{\sim}{j} + r)\frac{1}{\sqrt{r^2+b^2}}$$

$$B'(s) = \frac{1}{(\sqrt{r^2+b^2})^2}(+b\cos t, +b\sin t, 0) = -\frac{b}{r^2+b^2}N$$

$$\Rightarrow \boxed{K(s) = -\frac{b}{r^2+b^2}}$$