Dimensional Analysis

- The values of physical quantities must be measured in terms of a chosen unit of
  time (s), length (m), mass (kg).

\[ F = ma \]

The magnitude of \( F \), \( m \), \( a \), change = "rescale" with change of units. To determine how they scale, we find their dimensions -

\[ [F] = "\text{dimensions of } F" \]

\[ [F] = [ma] = [m][a] = [m][\frac{dx}{dt^2}] \]

\[ [m] = M \quad [\frac{dx}{dt}] = [x] \quad [y] \]

\[ [a] = \frac{1}{T^2} \quad M, L, T \text{ are fundamental dimensions} \]
Conclude: \[ F = \frac{ML}{T^2} \text{ "dimensions of mass times accel"} \]

Ex: If the mass doubles, the length triples, the time five times, then it will rescale \( F \) by \( \frac{2 \times 3}{5^2} \)

"Dimensions: keep track of rescalings under change of units"

Principle: In every physical equation, the dimensions of each term are the same... \( \Rightarrow \) not all equations can be physical equations.

Otherwise, changing dimensions changes the equations of motion.
Ex: Newton's Law of Gravity

\[ \vec{F} = \frac{G M_p M_s}{r^2} \]

Gravity is an Inverse square force law proportional to \( M_E M_s \)

"You have to have a dimensional constant here to make both sides have same dimension"

Q: What are the dimensions of the gravitational constant \( G \)?

\[
\begin{align*}
[F] &= [M_p \dot{\vec{a}}] = \frac{ML}{T^2} \\
\left[ G \frac{M_p M_s}{\| \vec{r} \|^2} \right] &= \left[ G \frac{M_p M_s}{\| \vec{r} \|^2} \right] = \left[ G \right] \frac{M^2}{L^2}
\end{align*}
\]
Conclude:

\[ \frac{ML^2}{T^2} = [G] \frac{M^2}{L^2} \implies [G] = \frac{L^3}{M T^2} \]

"The theory implies a universal constant \( G \), \([G] = \frac{L^3}{M T^2}\) to be measured."

*Note:* \( M_p \hat{a} = -G \frac{M_p M_s}{\| \mathbf{r} \|^2} \frac{\mathbf{r}}{\| \mathbf{r} \|^2} \)

"The acceleration of the body is independent of the mass of the planet."

"Every object, feather & earth, will describe the same path thru grav. field" \( \implies \) Equivalence Principle

\( \implies \) Led Einstein to suspect that gravity was about the paths, not about forces \( \implies \) GR
Thus write: (The equation for a planet)

\[ \vec{a} = - \frac{G M_s}{r^3} \vec{r} \]

\[ [G M_s] = [G] [M_s] = \frac{L^3}{T^2} \]

ODE for planetary motion (Newton)

\[ \frac{d^2 \vec{r}(t)}{dt^2} = - \frac{G M_s}{\| \vec{r}(t) \|^3} \frac{\vec{r}(t)}{\| \vec{r}(t) \|} \]

ODE for position \( \vec{r}(t) = (x(t), y(t), z(t)) \)
Kepler's Laws - All derivable from (*)

1. Planets move in ellipses about sun, with sun at focus of ellipse

2. Planets sweep out equal area in equal time

3. The mean distance from sun = \( L_p \)
   
   Period = \( T_p \) (depends on planet)

satisfies

\[
\frac{L_p^3}{T_p^2} = \text{same \ A \ planet}
\]

Note: "Some thing independent of planet exists (\( \frac{L_p^3}{T_p^2} \)) with same dimensions as the gravitational constant (GMs)"
• Dimensional Analysis implies deep connections with solutions that enable you to guess answers ahead of time.

E.g.: since equations contain a universal constant \([G\text{M} \text{S}^2] = \frac{L^3}{T^2}\), you might guess there is something independent of planet’s mass that same dimensions. 

⇒ get ideas before doing any real work in solving the equations.