The phase portrait in 2-D

Autonomous Systems in the plane

- Notation: \( \mathbf{x} = (x, y) \in \mathbb{R}^2 \)

Autonomous ODE

\[
\dot{\mathbf{x}} = f(\mathbf{x})
\]

\[
f(\mathbf{x}) = \left( \begin{array}{c} f_1(x, y) \\ f_2(x, y) \end{array} \right)
\]

No explicit dependence on \( t \)

- Look for solutions \( \mathbf{x}(t) = (x(t), y(t)) \) of the ivp

\[
\dot{\mathbf{x}} = f(\mathbf{x})
\]

\[
\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^2
\]
- **Visualization**: \( f(x) \) is a vector field on the plane - look for \( \dot{x}(t) \) tangent to \( f(x) \) at each point.

\[
\begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}
\]

\( x(t) \) solves \( \dot{x} = f(x) \) if:

1. \( \dot{x}(t) = f(x(t)) \) is tangent to \( f(x(t)) \) at each \( x(t) \)

2. The speed \( \|\dot{x}\| = \frac{ds}{dt} = \|f(x(t))\| \) at each point.
**Example:** Force = \(-kx\)

\[
\begin{align*}
\text{Spring constant} \\
\text{Equation: } ma &= \text{force} = -kx \\
m\ddot{x} &= -kx \\
\dddot{x} + \frac{k}{m}x &= 0 \\
\dddot{x} + \omega^2 x &= 0, \quad \omega = \sqrt{\frac{k}{m}} 
\end{align*}
\]

Harmonic oscillator has periodic solutions

\[x(t) = A \cos \omega t + B \sin \omega t\]

constants \(A, B\).
Write as a first order system:

\[
\begin{align*}
    x &= x' \
    y &= x = v' \
    y' &= x = -\omega^2 x
\end{align*}
\]

\[
\begin{align*}
    \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} 
\end{align*}
\]

Because it is linear, we can write it in matrix form:

\[
\begin{align*}
    \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} 
\end{align*}
\]

The constant coefficient matrix is \( A \).
Solution to the initial value problem:
\[ \dot{x} = f(x) \]
\[ \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]
\[ x(0) = x_0, \quad y(0) = y_0 \]

Since we know:
\[ x(t) = A \cos \omega t + B \sin \omega t \]
\[ y(t) = \dot{x}(t) = -A \omega \sin \omega t + B \omega \cos \omega t \]

We can choose \( A, B \) to meet any initial conditions:
\[ x_0 = A \cos \theta + B \sin \theta = A \]
\[ y_0 = -A \omega \sin \theta + B \omega \cos \theta = B \omega \]

Solve i.v.p.
\[ A = x_0, \quad B = \theta / \omega \]

\[ x(t+2\pi) = x(t) \Rightarrow \text{periodic orbits} \]
Phase portrait:

Vector field: \( f(x) = \begin{pmatrix} y \\ -w^2x \end{pmatrix} \)

\( f(0) = 0 \) rest point

Not hard to show are ellipses -
They all circle around the rest point \((0,0)\).
Big Picture: general $2 \times 2$ autonomous system / nonlinear

$\dot{x} = f(x)$

nonlinear vector field

Assume $f$ Lipschitz continuous in $x$

$\exists K > 0$ such that

$$||f(x_2) - f(x_1)|| \leq K ||x_2 - x_1||$$

(\star)

for all $x_1$ & $x_2$ in the region where you look for solutions

$\Rightarrow$ The ivp has a unique solution

$\forall x_0, y$ defined until it goes out of region where (\star) holds
Conclude, solution orbits/trjectories cannot cross:

\[ x_0 \]

two solutions with the same initial condition?

\[ f(\bar{x}) = 0 \]

\[ \Rightarrow \text{The qualitative feature of solutions is determined by the structure of solutions around the rest points when} \]

\[ f(\bar{x}) = 0 \]
\[ \begin{align*}
\dot{x} &= -2xy - x^3 y \\
\dot{y} &= -2xy - y^3 x
\end{align*} \]

non-linear

stable/unstable rest points
or nodes (like harmonic oscillator)
(marginally stable)
Program: Linearize the equation around the rest points where \( f(\bar{x}) = 0 \).

Get 2x2 constant coefficient system

\[
\begin{pmatrix}
  x \\
  y \\
\end{pmatrix} = \begin{pmatrix}
  a & b \\
  c & d \\
\end{pmatrix} \begin{pmatrix}
  x \\
  y \\
\end{pmatrix}
\]

* Determine the structure of solutions by eigenvalue methods at each rest point.

* Connect the orbits so "no trajectories cross."