Feb 13 2012 MAT119A

2-D Phase Plane Ch 5

2×2 Nonlinear Autonomous System –

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(\underset{\sim}{x}) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}.$$

- Rest Point: $f(\bar{\underset{\sim}{x}}) = 0 \Rightarrow \underset{\sim}{x}(t) = \bar{\underset{\sim}{x}}$ is a constant soln.

- Assume: f is Lipshitz Cont in $\underset{\sim}{x}$ in Domain $\mathcal{U} \subseteq \mathbb{R}^2$ where soln's take values –

$\exists K$ st

$$\|f(\underset{\sim}{x}_2) - f(\underset{\sim}{x}_1)\| \leq K \|\underset{\sim}{x}_2 - \underset{\sim}{x}_1\|$$

$\exists!$ Thm: There exists a unique soln of ivp $\forall \underset{\sim}{x}_0 \in \mathcal{U}$: I.e. $\exists!$ Soln of

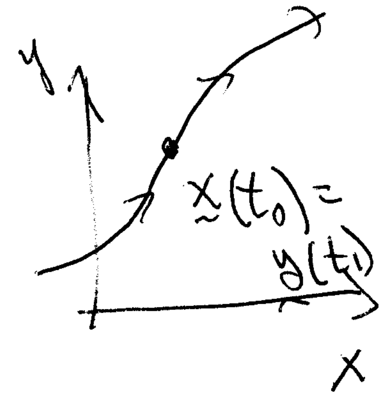
$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$$
$$\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0.$$

Soln exists for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$

- <u>Theorem</u>: If $\underset{\sim}{x}(t)$ is a solution, so is $\underset{\sim}{x}(t+c)$ for <u>any</u> constant c.

Pf. Assume $\dot{\underset{\sim}{x}}(t) = f(\underset{\sim}{x}(t))$. Then set

$$\underset{\sim}{y}(t) = \underset{\sim}{x}(t+c)$$



(Same values, different time –)

Then

$$\dot{\underset{\sim}{y}}(t) = \frac{d}{dt}\underset{\sim}{x}(t+c) = \dot{\underset{\sim}{x}}(t+c)$$

$$= f(\underset{\sim}{x}(t+c))$$

$$= f(\underset{\sim}{y}(t)) \checkmark$$

<u>Note</u>: This requires <u>autonomous</u>: Eg if

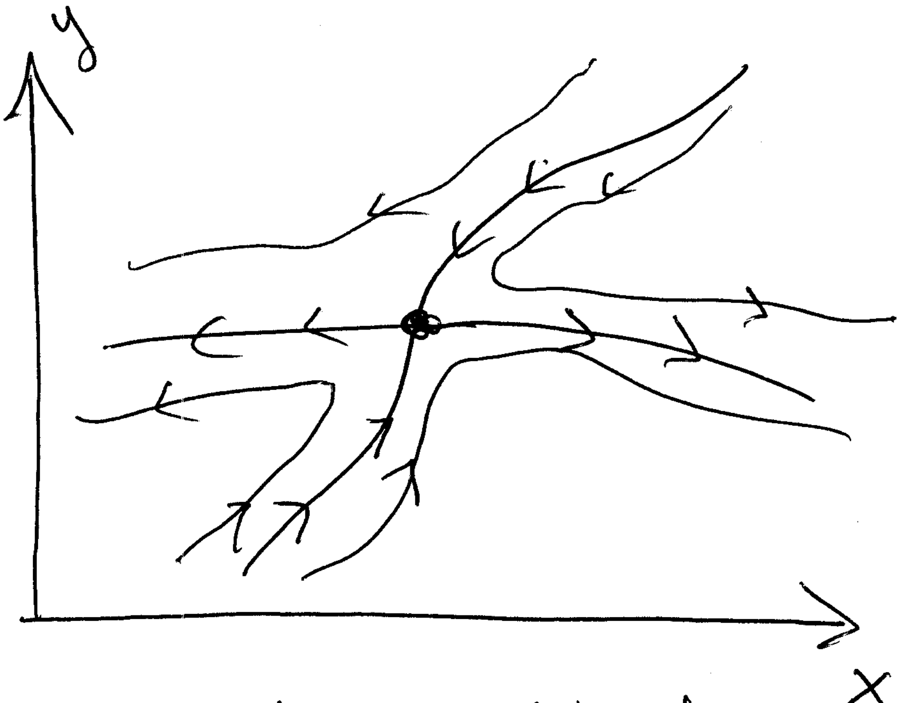
$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, t), \quad \underset{\sim}{y}(t) = \underset{\sim}{x}(t+c)$$

$$\dot{\underset{\sim}{y}} = \dot{\underset{\sim}{x}}(t+c) = f(\underset{\sim}{x}(t+c), t+c) = f(\underset{\sim}{y}(t), t+c)$$

$$\neq f(\underset{\sim}{y}(t), t) \,!$$

- **Defn**: The phase portrait for soln's of $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ is the graph of solution trajectories in the xy-plane.

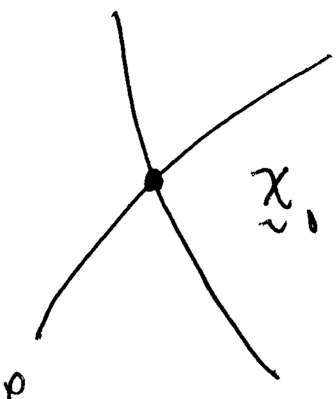
Defn: The trajectory of a solution $\underset{\sim}{x}(t)$ is the image curve in the xy-plane



"time dependence suppressed - time direction indicated by arrows..."

Cor: Two different solution trajectories cannot cross in the xy-plane:

Pf Assume two solutions $\underset{\sim}{x}(t)$ & $\underset{\sim}{y}(t)$ cross at point $(x_0, y_0) = \underset{\sim}{x}_0$ in the xy-plane.



Then $\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0 = \underset{\sim}{y}(t_1)$ at some t_0, t_1.

Now let $c = t_1 - t_0$. Then

$$\underset{\sim}{z}(t) = \underset{\sim}{y}(t + c) \Rightarrow \underset{\sim}{z}(t_0) = \underset{\sim}{y}(\cancel{t_0} + t_1 - \cancel{t_0}) = y(t_1) = \underset{\sim}{x}_0$$

Thus: both $\underset{\sim}{x}(t)$ & $\underset{\sim}{z}(t)$ solve

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$$
$$\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0$$

$\Rightarrow \exists!$ Thm $\Rightarrow$ they must be <u>same</u> soln ✓

$\Rightarrow \underset{\sim}{x}(\cdot)$ & $\underset{\sim}{y}(\cdot)$ had <u>same</u> <u>trajectory</u>

• <u>Defn</u>: The phase portrait is the graph of the <u>solution trajectories</u> in the xy-plane.

<u>Defn</u>: Soln <u>Trajectory</u> is the curve in xy-plane given by image of $\underset{\sim}{x}(t)$

<u>Cor</u>: trajectories can only "intersect" at rest pts (where <u>solutions</u> never touch.)

<u>Conclude</u>: Rest points are fundamental,

☐ Simplest Autonomous Systems: Linear, Constant Coeff, Homogeneous

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \qquad (*)$$

Rest Point @ $\bar{\underset{\sim}{x}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

- Note: Solutions of (*) form a vector space i.e., solution space is closed under addition & scalar multiplication of solutions:

Eg: If $\underset{\sim}{x}(t)$, $\underset{\sim}{y}(t)$ solve $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, $\dot{y} = A\underline{y}$

Then set $\underset{\sim}{z}(t) = c_1 \underset{\sim}{x} + c_2 \underset{\sim}{y}$

$$\dot{\underset{\sim}{z}}(t) = c_1 \dot{\underset{\sim}{x}}(t) + c_2 \dot{\underline{y}}(t)$$

$$= c_1 A \underset{\sim}{x}(t) + c_2 A \underline{y}(t)$$

$$= A\left(c_1 \underset{\sim}{x}(t) + c_2 \underline{y}(t)\right)$$

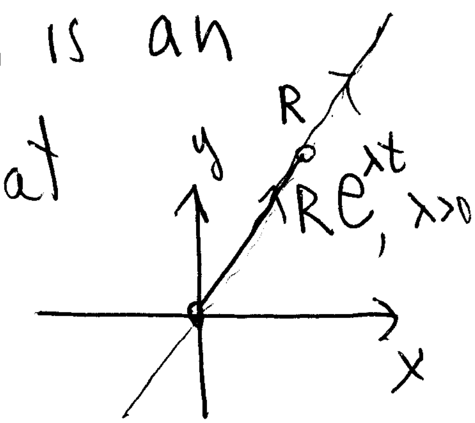
$$= A \underset{\sim}{z}(t) \checkmark$$

In particular: if $\dot{\underline{x}} = A\underset{\sim}{x}$, the $-\dot{\underset{\sim}{x}} = A(-\underset{\sim}{x})$ is also a solution.

Classification of Rest Points:

Main Theorem: If (λ, R) is an eigenpair of matrix A so that

$$AR = \lambda R,$$



Then $\underset{\sim}{x}(t) = Re^{\lambda t}$ solves $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$.

Proof: Let $\underset{\sim}{x}(t) = Re^{\lambda t}$. Then

$$\dot{\underset{\sim}{x}}(t) = R\lambda e^{\lambda t}$$

$$A\underset{\sim}{x}(t) = ARe^{\lambda t} = R\lambda e^{\lambda t}$$

- Note: The eigensolutions tend to rest point $\bar{\underset{\sim}{x}} = (0,0)$ in forward time when $\lambda < 0$, and in backward time when $\lambda > 0$.

<u>Conclude</u>: To solve $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, find two eigensolutions

$$\underset{\sim}{x}_1 = R_1 e^{\lambda_1 t}$$
$$\underset{\sim}{x}_2 = R_2 e^{\lambda_2 t}$$

& general solution is:

$$\underset{\sim}{x}(t) = C_1 R_1 e^{\lambda_1 t} + C_2 R_2 e^{\lambda_2 t}$$

Q: How do we know this is <u>all</u> soln's?

Ans: $\exists !$ soln to $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$

$$\underset{\sim}{x}(0) = \underset{\sim}{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

So $\underset{\sim}{x}(t) = C_1 R_1 e^{\lambda_1 t} + C_2 R_2 e^{\lambda_2 t}$

$$\underset{\sim}{x}(0) = C_1 R_1 + C_2 R_2 = \begin{bmatrix} | & | \\ R_1 & R_2 \\ | & | \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Can alway solve if R_1, R_2 form a basis.

We have: Theorem: If (λ_1, R_1) (λ_2, R_2) are eigenpairs of matrix A, and R_1 & R_2 are linearly independent so that

$$\det \begin{bmatrix} | & | \\ R_1 & R_2 \\ | & | \end{bmatrix} \neq 0,$$

then the general solution of

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$$

is

$$\underset{\sim}{x}(t) = c_1 R_1 e^{\lambda_1 t} + c_2 R_2 e^{\lambda_2 t}, \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} | & | \\ R_1 & R_2 \\ | & | \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Proof: The unique soln of the initial value problem is $\dot{\underline{x}} = A\underline{x}$, $\underline{x}(0) = \underline{x}_0$ is $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} | & | \\ R_1 & R_2 \\ | & | \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$\Rightarrow$ we can solve evey ivp $\Rightarrow$ we have evey solution!

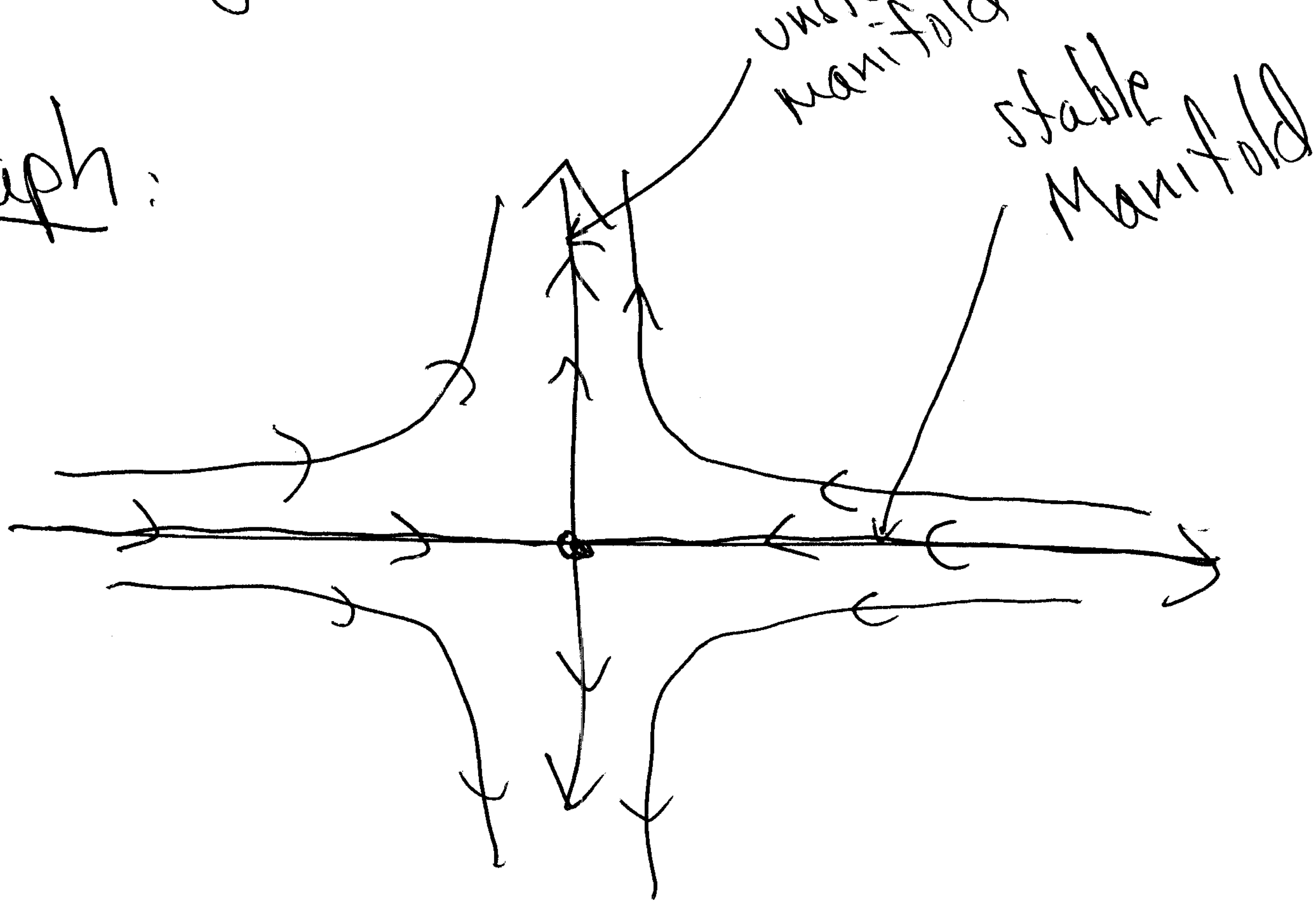
3) **Ex:** $\dot{\underset{\sim}{x}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\left(-1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ & $\left(1, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ are eigenpairs –

$$\underset{\sim}{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$$

is the general soln –

Graph:



☒ Canonical Cases —

(1) $A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ distinct eigenvalues, same sign
$\lambda, \mu > 0$, $\lambda, \mu < 0$

(2) $A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ distinct eigenvalues, opp sign
$\lambda < 0 < \mu$

(3) $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ complex evals $\lambda = \alpha \pm i\beta$

(4) $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ coincident evalues — basis of e-vectors
($\underline{x}(t) = \vec{v} e^{\lambda t}$ is soln)

(5) $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ degenerate case — <u>one</u> e-value, <u>one</u> e-vector

<u>Note</u>: $\lambda = 0 \Rightarrow$ 2nd order isn't leading term?!
$\Rightarrow$ not so interesting ~ degenerate

$$\lambda^2 \mp \underbrace{\mathrm{tr}A}_{\tau}\,\lambda + \underbrace{\det A}_{\Delta}$$

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - \Delta}}{2}$$

$\tau^2 - \Delta = 0$ where evals go complex