2x2 Nonlinear Autonomous System -
\[
\dot{x} = (x, y) = f(x) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}
\]

- **Rest Point**: \( f(\bar{x}) = 0 \) \( \Rightarrow \bar{x}(t) = \bar{x} \) is a constant soln.

- **Assume**: \( f \) is Lipschitz cont in \( x \) in Domain \( U \subseteq \mathbb{R}^2 \) where soln's take value\

\[\exists K > 0 \forall x, y \in U : \| f(x) - f(y) \| \leq K \| x - y \|\]

**Thm**: There exists a unique soln of ivp

\( \forall x_0 \in U \): i.e. \( \exists! \) Soln of
\[
\begin{align*}
\dot{x} &= f(x) \\
x(t_0) &= x_0.
\end{align*}
\]

Soln exists for \( t \in (t_0 - \varepsilon, t_0 + \varepsilon) \)
• **Theorem**: If \( \dot{x}(t) \) is a solution, so is \( \dot{x}(t+c) \) for any constant \( c \).

**Proof**: Assume \( \dot{x}(t) = f(x(t)) \). Then set

\[
y(t) = x(t+c)
\]

(Same values, different time -)

Then

\[
\dot{y}(t) = \frac{d}{dt} x(t+c) = \dot{x}(t+c)
\]

\[
= f(x(t+c))
\]

\[
= f(y(t)) \checkmark
\]

**Note**: This requires **autonomous**. E.g. if

\[
\dot{x} = f(x, t), \quad \dot{y}(t) = x(t+c)
\]

\[
\dot{y} = \dot{x}(t+c) = f(x(t+c), t+c) = f(y(t), t+c)
\]

\[
\neq f(x(t), t)
\]
Defn: The phase portrait for soln's of \( x = f(x) \) is the graph of solution trajectories in the xy-plane.

Defn: The trajectory of a solution \( x(t) \) is the image curve in the xy-plane.

"time dependency suppressed - time direction indicated by arrows..."
Cor: Two solution trajectories cannot cross in the xy-plane.

Proof: Assume two solutions \( x(t) \) and \( y(t) \) cross at point \( (x_0, y_0) = x_0 \) in the xy-plane.

Then \( x(t_0) = x_0 = y(t_1) \) at some \( t_0, t_1 \).

Now let \( c = t_1 - t_0 \). Then

\[
\begin{align*}
    z(t) &= y(t + c) \Rightarrow z(t_0) &= y(t_0 + t_1 - x_0) = y(t_1) = x_0
\end{align*}
\]

Thus: both \( x(t) \) and \( y(t) \) solve

\[
\dot{x} = f(x)
\]

\[
x(t_0) = x_0
\]

\[
\Rightarrow \exists! \text{Thm} \Rightarrow \text{they must be same soln} \Rightarrow x(t) \text{ and } y(t) \text{ had same trajectory}
\]
Defn: The phase portrait is the graph of the solution trajectories in the $xy$-plane.

Con: trajectories can only "intersect" at rest pts (where solutions never touch.)

Conclude: Rest points are fundamental.

- Simplest Autonomous Systems: Linear, Constant Coeff, Homogeneous

\[ \dot{x} = A \dot{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

Rest Point @ \( \bar{x} = (0) \).
• Note: Solutions of (⋆) form a vector space, i.e., solution space is closed under addition and scalar multiplication of solutions.

Eg: If \( x(t), y(t) \) solve \( \dot{x} = Ax, \quad \dot{y} = Ay \)
Then set \( z(t) = c_1 x + c_2 y \)

\[ \dot{z}(t) = c_1 \dot{x}(t) + c_2 \dot{y}(t) \]
\[ = c_1 A x(t) + c_2 A y(t) \]
\[ = A (c_1 x(t) + c_2 y(t)) \]
\[ = A z(t) \checkmark \]

In particular: if \( \dot{x} = Ax \), then \( -\dot{x} = A(\dot{-x}) \) is also a solution.
Classification of Rest Points:

Main Theorem: If \((\lambda, R)\) is an eigenpair of matrix \(A\) so that \(AR = \lambda R\),

Then \(x(t) = Re^{\lambda t}\) solves \(\dot{x} = Ax\).

Proof: Let \(x(t) = Re^{\lambda t}\). Then

\[
\dot{x}(t) = R\lambda e^{\lambda t}
\]

\[
Ax(t) = AR e^{\lambda t} = R\lambda e^{\lambda t}
\]

Note: The eigen solutions tend to rest point \(\bar{x} = (0,0)\) in forward time when \(\lambda < 0\), and in backward time when \(\lambda > 0\).
Conclude: To solve $\dot{x} = Ax$, find two eigensolutions

$$x_1 = R_1 e^{\lambda_1 t}$$
$$x_2 = R_2 e^{\lambda_2 t}$$

The general solution is:

$$x(t) = C_1 R_1 e^{\lambda_1 t} + C_2 R_2 e^{\lambda_2 t}$$

Q: How do we know this is all solutions?

Ans: Every solution to $\dot{x} = Ax$

$$x(0) = x_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

So

$$x(t) = C_1 R_1 e^{\lambda_1 t} + C_2 R_2 e^{\lambda_2 t}$$

$$x(0) = C_1 R_1 + C_2 R_2 = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Can always solve if $R_1, R_2$ form a basis.
We have: Theorem: If \((\lambda_1, R_1), (\lambda_2, R_2)\) are eigenvectors of matrix \(A\), and \(R_1, 8\) \(R_2\) are linearly independent so that
\[
\det \begin{bmatrix} R_1 & R_2 \end{bmatrix} \neq 0,
\]
then the general solution of
\[
\dot{x} = Ax
\]
is
\[
x(t) = c_1 R_1 e^{\lambda_1 t} + c_2 R_2 e^{\lambda_2 t}, \quad [c_1] = [R_1 R_2]^{-1} [x_0].
\]

Proof: The unique solution of the initial value problem is \(\dot{x} = Ax\) is
\[
[x(t)] = [R_1 R_2]^{-1} [x_0],
\]
\[
x(0) = x_0
\]
\(\Rightarrow\) we can solve every \(y_0\) \(\Rightarrow\) we have every solution.
Ex: \[ X = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \]

\((-1, 0)\) & \((1, i)\) are eigenpairs -

\[ X(t) = c_1(0)e^{-t} + c_2(0)i e^t \]

is the general soln -

Graph:

- Unstable manifold
- Stable manifold
 Canonical Cases -

1. \( A = \begin{bmatrix} \lambda & 0 \\ 0 & M \end{bmatrix} \) distinct eigenvalues, same sign
   \( \lambda, M > 0, \lambda, M < 0 \)

2. \( A = \begin{bmatrix} \lambda & 0 \\ 0 & M \end{bmatrix} \) distinct eigenvalues, opp sign
   \( \lambda < 0 < M \)

3. \( \begin{bmatrix} a & b \\ -b & d \end{bmatrix} \) complex evals \( \lambda = \alpha \pm i \beta \)

4. \( \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \) coinciding evals - basis
   \( X(t) = \text{vec} \) of e-vectors
   \( \Rightarrow \text{dim is soln} \)

5. \( \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \) degenerate case - one e-value, one e-vector

Note: \( \lambda = 0 \Rightarrow 2nd \text{order isn't leading term} \Rightarrow \text{not so interesting - degenerate} \)
\[ \lambda^2 = \text{tr} A \lambda + \det A \]

\[ \lambda = \frac{c \pm \sqrt{c^2 - \Delta}}{2} \]

\[ c^2 - \Delta = 0 \text{ where evals go complex} \]

Diagram:
- Saddle
- Unstable node
- Stable node
- Unstable spiral
- Stable spiral
- Center
- Degenerate nodes