Conservative systems —

<u>Theorem</u>: Every system of form $\ddot{\underset{\sim}{x}} = -\nabla U(\underset{\sim}{x})$ has an <u>energy</u> $E = \frac{1}{2}|\dot{\underset{\sim}{x}}|^2 + U(\underset{\sim}{x})$ meaning:

$E(\underset{\sim}{x}(t))$ is constant along solutions.

"Solutions can only move along level curves of E"

Holds for $\underset{\sim}{x}(t) = (x_1(t) \dots x_n(t)) \in \mathbb{R}^n$

<u>Proof</u>: Let $\underset{\sim}{x}(t)$ solve $\dot{\underset{\sim}{x}} = -\nabla U(\underset{\sim}{x})$. Then

$$\frac{d}{dt}E(t) = \frac{d}{dt}\left(\frac{1}{2}(\dot{x}_1^2 + \dots + \dot{x}_n^2)\right) - \frac{d}{dt}U(\underset{\sim}{x}(t))$$

$$= \dot{x}_1\ddot{x}_1 + \dots + \dot{x}_n\ddot{x}_n - \nabla U \cdot \dot{\underset{\sim}{x}}$$

$$= \dot{\underset{\sim}{x}} \cdot \ddot{\underset{\sim}{x}} - \nabla U \cdot \dot{\underset{\sim}{x}}$$

$$= \dot{\underset{\sim}{x}} \cdot \underbrace{(\ddot{\underset{\sim}{x}} - \nabla U)}_{0} = 0 \quad \checkmark$$

- To find the energy – "mult by the velocity & create a total derivative"

$$\ddot{\underset{\sim}{x}} + \nabla u = 0$$

$$\dot{\underset{\sim}{x}}(\ddot{\underset{\sim}{x}} + \nabla u) = 0$$

$$\underbrace{\dot{\underset{\sim}{x}}\ddot{\underset{\sim}{x}}}_{\frac{d}{dt}\left(\frac{1}{2}\dot{\underset{\sim}{x}}^2\right)} + \underbrace{\nabla u \cdot \dot{\underset{\sim}{x}}}_{\frac{d}{dt} u(\underset{\sim}{x}(t))} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\underset{\sim}{x}}^2\right) + \frac{d}{dt} u(\underset{\sim}{x}(t)) = 0$$

$$\frac{d}{dt}\underbrace{\left\{\frac{1}{2}\dot{\underset{\sim}{x}}^2 + u(\underset{\sim}{x}(t))\right\}}_{E(t)} = 0$$

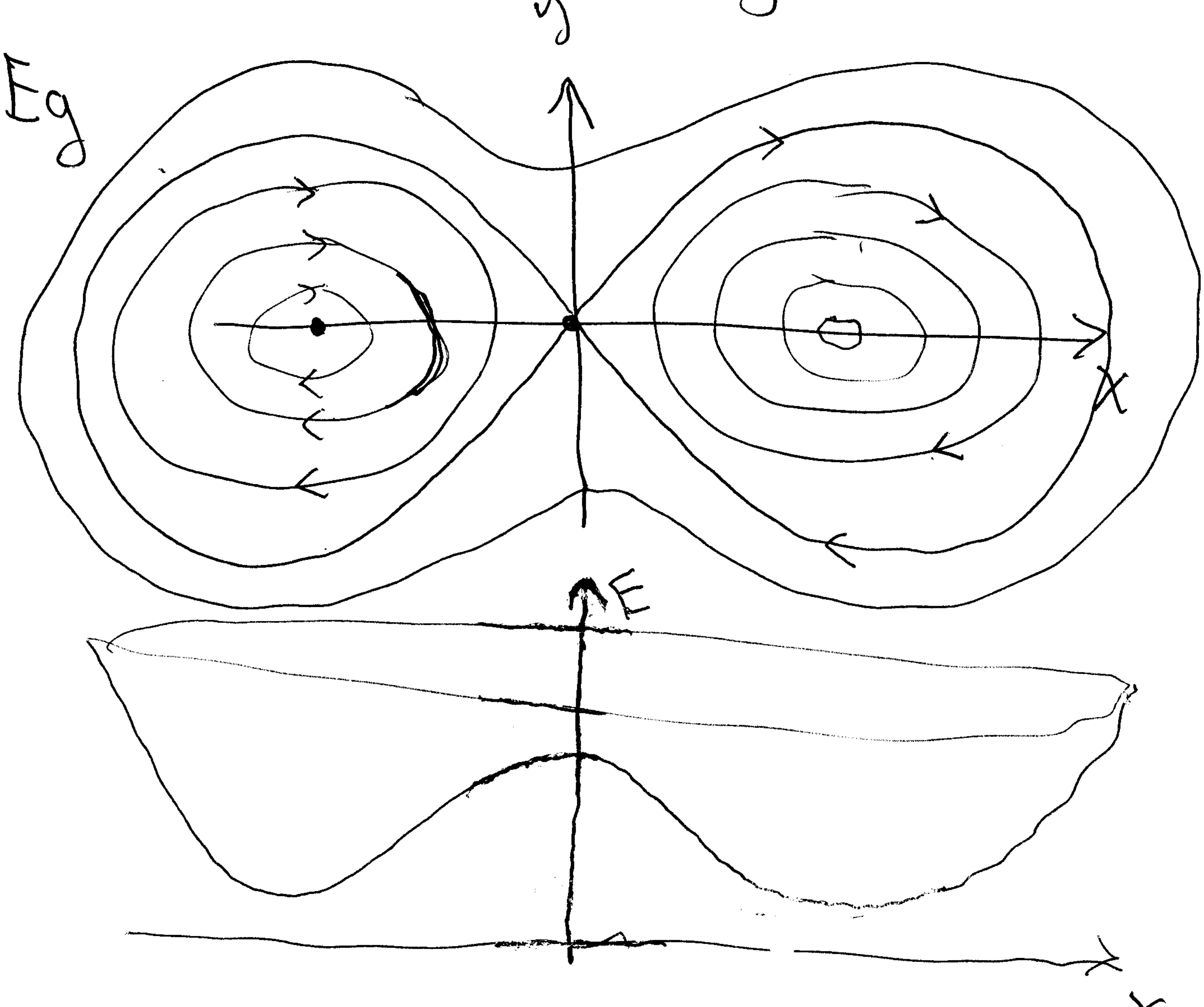
- If x is a scalar — $\ddot{x} = -\nabla U = -U'(x)$

$\Rightarrow$ 1st order system: $x = x$, $y = \dot{x}$ $\quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -U'(x) \end{pmatrix}$

Energy: $E = \frac{1}{2}\dot{x}^2 + U(x) = \frac{1}{2}y^2 + U(x)$

- Solutions must move along level curves of E

Eg



Double potential Eg: $E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$

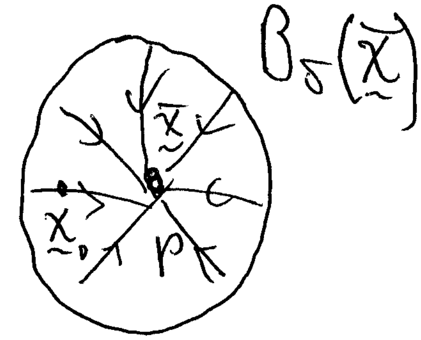
$\ddot{x} = x - x^3$

More general:

might not come from $\ddot{x} = -\nabla u$

Defn: Given $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$, a conserved quantity or energy is a function $E(\underset{\sim}{x})$ that is constant on solutions, and non-constant in any open ball. A system with conserved quantity is called a conservative system.

Theorem: A conservative system cannot have an attracting fixed pt or a repelling fixed pt.



Pf. All trajectories $\underset{\sim}{x}(t)$ starting from $\underset{\sim}{x}_0 \in B_\delta(\bar{\underset{\sim}{x}}) = \{\underset{\sim}{x} : |\underset{\sim}{x} - \bar{\underset{\sim}{x}}| < \delta\}$ tend toward $\bar{\underset{\sim}{x}}$. Thus $E(\underset{\sim}{x}_0) = E(\underset{\sim}{x}(t)) = E(\bar{\underset{\sim}{x}})$

$\Rightarrow$ E is constant on $B_\delta(\bar{\underset{\sim}{x}})$ $\times$

- Recall: Solutions of nonlinear systems can blow up in finite time –

Eg $\dot{x} = x^2 \qquad x(t) = \dfrac{1}{\frac{1}{x_0} - t} \xrightarrow{t \to \frac{1}{x_0}} \infty$

Could happen in 2x2 nonlinear system –

Eg: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} \dfrac{1}{\frac{1}{x_0} - t} \\ \dfrac{1}{\frac{1}{y_0} - t} \end{pmatrix}$

<u>Ex</u>: Show that if $\dot{\underset{\sim}{x}} = f(\underline{x})$ is conservative, then the solution of ivp

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$$

$$\underset{\sim}{x}(0) = \underset{\sim}{x}_0$$

does not blow up so long as $S = \{\underset{\sim}{x} : E(\underset{\sim}{x}) = E(\underset{\sim}{x}_0)\}$ is bounded.

Solution: $E(\underset{\sim}{x}(t)) = E(\underset{\sim}{x}_0)$ all along the solution, so $\underset{\sim}{x}(t)$ stays in the set S and so cannot go off to infinity at any value of t.

Note: if S is closed & bounded $\Leftrightarrow$ compact & $f(\underset{\sim}{x})$ is continuous then:

Thm – a continuous function is bounded on a compact set.

$\Rightarrow |\dot{\underset{\sim}{x}}| = |f(\underset{\sim}{x})| \leq M$ on S $\Rightarrow$ the derivative is bounded $\forall t$ $\Rightarrow$ solution $\underset{\sim}{x}(t)$ exists for all time –

True in any number of dimensions

<u>Ex</u>: Consider 2x2 conservative system

$$\ddot{x} = -U'(x) \qquad \dot{x} = y$$

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -U'(x) \end{pmatrix} = f(\underset{\sim}{x})$$

Let $\bar{x}$ be an isolated, regular local min of U: $U'(\bar{x})=0$, $U''(\bar{x})>0$

$\bar{\underset{\sim}{x}} = (\bar{x}, 0)$ is a rest point.

(1) Evals of linearized equations

$$\dot{\underset{\sim}{x}} = \bar{\underset{\sim}{x}} + Df(\bar{\underset{\sim}{x}})(\underset{\sim}{x} - \bar{\underset{\sim}{x}})$$

are purely imaginary $\Rightarrow$ center

(2) Nonlinear trajectories are closed orbits in a nbhd of $\bar{\underset{\sim}{x}}$.

$\rightsquigarrow$ "Energy makes the closed orbits of the linearized eqn's perturb to nonlinear eqn's"

Soln (1) $$f(\underset{\sim}{x}) = f(\underset{\sim}{\bar{x}}) + \underbrace{\begin{pmatrix} \nabla f_1(\underset{\sim}{\bar{x}}) \\ \nabla f_2(\underset{\sim}{\bar{x}}) \end{pmatrix}}_{\begin{pmatrix} 0 & 1 \\ -U''(\bar{x}) & 0 \end{pmatrix}} (\underset{\sim}{x} - \underset{\sim}{\bar{x}}) + \text{hot}$$

evals: $\lambda^2 - \tau + \Delta = 0$

$\tau = 0 \quad \Delta = U''(\bar{x})$

$$\boxed{\lambda = \pm i\sqrt{U''(\bar{x})}}$$

~~Soln (2) For solution trajectories suff. close to $\underset{\sim}{\bar{x}} = (\bar{x}, 0)$, $E(\underset{\sim}{x}(t)) = E(\underset{\sim}{\bar{x}})$~~

~~$E(\underset{\sim}{x}(t)) = \frac{1}{2}y^2 + U(x) =$~~

Soln(2): Nonlinear solutions move along level curves of $E = \frac{1}{2}y^2 + U(x)$

But: $\nabla E(x,y) = \begin{pmatrix} U'(x) \\ y \end{pmatrix} \Rightarrow \nabla E(\bar{x},0) = 0$

$$H(x,y) = \begin{bmatrix} -\nabla E_x- \\ -\nabla E_y- \end{bmatrix}_{\underset{\sim}{\bar{x}}} = \begin{bmatrix} U''(\bar{x}) & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ evals $\lambda = U''(\bar{x}), 1 > 0 \Rightarrow \underset{\sim}{\bar{x}} = (\bar{x}, 0)$ is isolated rel min $\Rightarrow$

"Level curves of E are closed curves with no rest pts near $\underset{\sim}{\bar{x}}$" ✓ (Not a complete proof).