Lagrangian Principle

Equations of Physics —

Lagrangian Principle of Least Action

- Recall bead on rotating hoop with friction —

- We derived equations from Newton's Laws —

\[ m\ddot{a} = F \]

Derivation more complicated because of the restoring force that holds mass to hoop —

"Balance of forces" — \( F \approx \) gravity + centripetal force + force that holds mass to hoop —
Equations we derived:

\[ mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr^2\omega^2\sin\phi \cos\phi \]
\[ m\ddot{\alpha} = F_t + F_g + F_n \]

\( F_t = \text{Friction Force} \)
\( F_g = \text{Grav. Force} \)
\( F_n = \text{centripetal force} + \text{restoring force on hoop} \)

Q: Is there a **systematic** way to derive the equations of motion with constraints (like hoop)?

Ans: Yes when energy is conserved (holonomic constraints). Not so easy otherwise (non-holonomic constraint).

For us we need \( b = 0 \) (no friction) to get cons of energy.

Calculus of Variations:

Problem: Given a function $L(x, \dot{x})$ is "Lagrangian" find the equations for the curve $x(t)$ taking $x(a) = x_1$, $x(b) = x_2$ that "minimizes the action":

$$A[x(t)] = \int_{a}^{b} L(x(t), \dot{x}(t)) \, dt$$
Idea: whatever curve minimizes the action in $\mathcal{X} = (x, y, z)$ coordinates will minimize the action in any other co-ords: I.e., say $\tilde{\mathcal{X}} = \Phi (\tilde{q}) = \begin{pmatrix} \phi_1 (\tilde{q}) \\ \phi_2 (\tilde{q}) \\ \phi_3 (\tilde{q}) \end{pmatrix}$

$\Phi : (x, y, z) \mapsto (q_1, q_2, q_3)$

(Eg., $q$ could be spherical co-ords: $q_1 = \rho$ $q_2 = \phi$ $q_3 = \Theta$)

Chain rule for Curve - $\tilde{\mathcal{X}} = \mathcal{X}(t)$

$\mathcal{X}(t) = \Phi (\hat{q}(t))$

$\dot{\mathcal{X}}(t) = D\Phi \cdot \dot{\hat{q}}(t)$ $D\Phi = \begin{bmatrix} -\nabla \phi_1 \\ -\nabla \phi_2 \\ -\nabla \phi_3 \end{bmatrix}$
Conclude:

\[ A [\mathbf{x}(t)] = \int_a^b L (\mathbf{x}(t), \dot{\mathbf{x}}(t)) \, dt = \int_a^b L (\mathbf{\bar{x}}(t), D\mathbf{\bar{x}}(t)) \, dt \]

\[ = \int_a^b \mathcal{L} (\mathbf{q}(t), \dot{\mathbf{q}}(t)) \, dt = A [\mathbf{q}(t)] \]

"The action is independent of coordinates, so the curve that minimizes the action is independent of coordinates."
Theorem: (Calculus of Variations) The curve $x(t)$ minimizes the action [really is a critical point of the action] iff

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \quad x = (x^1, x^2, x^3)$$

I.e.: (\textit{4}) is the ODE for the curve $x(t)$ that minimizes the action.

Define: $\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0$ are the Euler–Lagrange equations.

Con: $x(t) = \Phi q(t)$ solves (\textit{4}) iff $q(t)$ satisfies

$$\frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{q}^i} - \frac{\partial \tilde{L}}{\partial q^i} = 0$$

Pf: $q(t)$ minimizes the action for $I$. \checkmark
**Theorem:** (Calculus of Variations) The curve $x(t)$ minimizes the action \[ \text{[really is a critical point of action J iff \( x(t) \) satisfies]} \]

\[
\frac{d}{dt} \frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial x^i} = 0, \quad x = (x^1, x^2, x^3) \quad (EL)
\]

More precisely: (EL) is the ODE for the curve $x(t)$ that is a critical point of the action in the sense that

\[
0 = \frac{d}{d\varepsilon} \left. \frac{1}{2} \int A \left[ x(t) + \varepsilon \gamma(t) \right]^2 \right|_{\varepsilon = 0} \quad A \gamma(t)
\]

\( \gamma(t) \) any smooth curve $\gamma : [a, b] \to \mathbb{R}^3$ so that $\gamma(a) = \gamma(b) = 0$

$y(t) = x(t) + \varepsilon \gamma(t)$ is any other curve taking

$y(a) = x_1$

$y(b) = x_2$
Pf. Not so hard: See Modern Geometry-Methods & Applications Vol I Dubrovin Fomenko, Novikov

Defn.: \[ \frac{d}{dt} \partial \mathcal{L} - \partial \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = 0 \] are called the Euler-Lagrange Equations.

Cor.: \( \dot{x}(t) = \mathbf{\Gamma}_{\dot{x}}(t) \) solves (E-L) iff \( g(t) \) satisfies
\[ \frac{d}{dt} \partial \mathcal{L} - \partial \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = 0 \]

Pf. "\( g(t) \) minimizes the action for \( \mathcal{L} \)")

"The action \( A[\dot{x}(t)] = \overline{A}[g(t)] \) is indep. of what coordinates you express the curve in"
**Conclusion**: The (EL) equations are coordinate independent equations. 

I.e. Given coordinates $\xi$, "find the Lagrangian $L(\xi, \dot{\xi})"$ and the equations are

\[
\frac{\partial L}{\partial \dot{\xi}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\xi}} = 0
\]

**Idea to make Newton's Laws coordinate independent** — Find the Lagrangian.

Newton: $m \ddot{\mathbf{a}} = \mathbf{F}$ (too general)

Assume **Conservative**:

\[
m \ddot{\mathbf{a}} = -\nabla U \implies \dot{x} = -\frac{1}{m} \nabla U(x)
\]
\textbf{Theorem:} This works with

\[ L(x, \dot{x}) = KE - PE = \frac{1}{2} m (\dot{x})^2 - U(x) \]

\textbf{Proof:}

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \]

\[ \frac{d}{dt} \frac{\partial}{\partial \dot{x}^i} \left\{ \frac{1}{2} m (\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2 \right\} - U(x^1, x^2, x^3) \]

\[ - \frac{\partial}{\partial x^i} \left\{ \frac{1}{2} m (\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2 \right\} - U(x^1, x^2, x^3) \]

\[ = \frac{d}{dt} (m \dot{x}^i) - \frac{\partial U}{\partial x^i} = m \ddot{x}^i - \frac{\partial U}{\partial x^i} \]

\[ \Rightarrow \quad m \ddot{x} - \nabla U(x) = 0 \]
Conclude: Newton's Laws in a conservative force field are equivalent to
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0
\]

↑

"generalized momentum" "generalized force"

Conclude: To get the equations of motion in a different coordinate system \( x = \Phi q \), just find the Lagrangian
\[
L(q, \dot{q}) = \frac{1}{2} m \left| \dot{\Phi} \dot{q} \right|^2 - U(\Phi(q))
\]
\[= \text{KE} - \text{PE} \text{ in } q \text{-coordinates} \]

\[\Rightarrow \text{Equations: } \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \]
Problem: What happens when there are constraints?

Defn: We call the constraint **holonomic** if the Lagrangian \( L(q, \dot{q}) \) is the constrained KE-PE & the equations are

\[
\frac{d}{dt} \mathbf{\dot{q}} = \mathbf{\ddot{q}} = \frac{\mathbf{\mathcal{F}}}{\mathbf{m}}
\]

Thm: "Whenever masses are constrained to move along surfaces with infinite restoring force in an otherwise conservative force field, the constraints are holonomic."

Ex: Pendulum, Bead on Rotating Hoop, ... ramps, (without friction &) [c.f. Arnold]
Ex: Derive equations for bead on rotating hoop w/o friction

- ph moves in conservative gravitational force field constrained to rotating hoop \( \Rightarrow \) holonomic.

- \( KE = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m |\dot{x}|^2 \)

- \( PE = U(x) = mgz \)

  (i.e. \( -\nabla U = (0,0,-mg) = -mg \hat{k} \))

- Coordinates: \( (r, \theta, \phi) = (r', \phi^2, \phi^3) = \phi \)
  (Spherical Coordinates)
\[ x = R \sin \theta = r \sin \varphi \sin \theta \]
\[ y = R \sin \theta = r \sin \varphi \sin \theta \]
\[ z = -r \cos \varphi \quad (z \text{ measured from center of hoop!}) \]

\[ x^2 + y^2 + z^2 = \frac{1}{2} mr^2 (\dot{\varphi}^2 + \omega^2 \sin^2 \varphi) \]

\[ U(\varphi) = mgz = -mg r \cos \varphi \]

\[ KE - PE = \frac{1}{2} mr^2 (\dot{\varphi}^2 + \omega^2 \sin^2 \varphi) + mg r \cos \varphi = L(\varphi, \dot{\varphi}) \]

Here constrained variable is \( q = \varphi \)

**Equations:**
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0 \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \varphi} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} = mr^2 \ddot{\varphi} = mr^2 \dot{\varphi} \]

\[ \frac{\partial L}{\partial \varphi} = mr^2 \omega^2 \sin \varphi \cos \varphi - mg r \sin \varphi \]
Equations:

$$mr^2 \ddot{\phi} = -mg \sin \phi + mr^2 \omega^2 \sin \phi \cos \phi$$

"Never had to balance the forces."
The Lagrangian Formalism:

Newton (Conservative Force Field)

\[ m \dddot{x} + \nabla U = 0 \quad \vec{F} = -\nabla U \quad (N) \]
\[ m \dddot{x}^i + \frac{\partial}{\partial x^i} U = 0 \quad F_i = -\frac{\partial}{\partial x_i} U \]

\[ \frac{d}{dt} \frac{\partial}{\partial \dot{x}^i} \left( \frac{1}{2} m \dot{x}^2 \right) + \frac{\partial}{\partial x^i} U = 0 \]

\[ \frac{d}{dt} \frac{\partial}{\partial \dot{x}^i} \left( \frac{1}{2} mL^2 \right) - \frac{\partial}{\partial x^i} \left\{ \frac{1}{2} mL^2 - U \right\} = 0 \]

\[ \frac{d}{dt} \frac{\partial}{\partial \dot{x}^i} L - \frac{\partial}{\partial x^i} L = 0 \]

- generalized momentum
- generalized force

Vector Form: \[ \frac{d}{dt} \frac{\partial}{\partial \dot{x}^i} L - \frac{\partial}{\partial x^i} L = 0 \]
What about the energy?

For (N), \( E = \frac{1}{2} m \dot{x}^2 + U(x) \Rightarrow \text{write in terms of} \ L \)

\[ = m \dot{x}^2 - \left( \frac{1}{2} m \ddot{x}^2 - U(x) \right) \]

\[ = m \dot{x}^2 - L \]

\[ m \dot{x}^2 = m \left( \dot{x}^1 \right)^2 + \left( \dot{x}^2 \right)^2 + \left( \dot{x}^3 \right)^2 \]

\[ \dot{x} \cdot m \dot{x} = \dot{x} \cdot \frac{\partial L}{\partial \dot{x}} \]

Good Guess: Generalized Energy

\[ E = \dot{x} \cdot \frac{\partial L}{\partial \dot{x}} - L = \left( \sum_{i=1}^{3} \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} \right) - L \]
\textbf{Theorem}: \( E = \dot{x} \cdot \frac{\partial L}{\partial \dot{x}} - L \) is constant along solution \( x(t) \) of \( (EL) \) \( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \).

\textbf{Proof}: Set \( E(t) = \dot{x} \cdot \frac{\partial L}{\partial \dot{x}} - L(x) \) on solution \( x(t) \)

\[
\frac{dE}{dt} = \dot{x} \cdot \frac{\partial L}{\partial \dot{x}} + \dot{x} \cdot \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial x} L(x) - \frac{\partial L}{\partial x} \dot{x} \cdot x
\]

\[
= \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \right) \cdot \dot{x} = 0 \checkmark
\]
A nice way to write Lagrange's Equations as a 1st order system:

1. \[ E = \dot{x} \frac{\partial L}{\partial \dot{x}} - L, \quad p = \frac{\partial L}{\partial \dot{x}} \]

"Solve" \[ p = \frac{\partial L}{\partial \dot{x}} \] for \[ \dot{x} = f(x, p) \]

Then \[ E = f(x, p) \dot{p} - L(x, f(x, p)) = H(x, p) \]

\[ \frac{\partial H}{\partial x} = \frac{\partial f}{\partial x} \dot{p} - \frac{\partial L}{\partial x} - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \dot{p} \right) = -\frac{\partial L}{\partial x} \]

But \[ \frac{d}{dt} \frac{\partial}{\partial x} \] is not defined. Hence, \[ \frac{\partial H}{\partial p} = \frac{\partial f}{\partial p} \dot{p} + f(x, p) - \frac{\partial L}{\partial p} \frac{\partial f}{\partial p} = \dot{x} \]
Conclude as 1st order system:

\[
\begin{align*}
\dot{x} &= H_p \\
\dot{p} &= -H_x
\end{align*}
\]

\((*)\) Hamilton's Eqns

Note: Just write the energy \( E \) as a fn of \((x, p) \) & \((*)\) is the 1st order system

\[
E = \frac{p^2}{2m} + U(x)
\]

\[
p = \frac{\partial E}{\partial x} = mx
\]

\[
\dot{x} = \frac{p}{m}
\]

\[
\dot{p} = -\frac{\partial}{\partial x} (\frac{p^2}{2m} + U(x)) = -U'(x)
\]
Newton's Laws in conservative force field has an energy that is conserved:

\[ m\ddot{x} = -\nabla U(x) \Rightarrow E = \frac{1}{2}m\dot{x}^2 + U(x) \]

\[ \frac{dE}{dt}(x(t)) = 0 \text{ on solns of } m\ddot{x} = -\nabla U(x) \]

Lagrange's method shows that many other types of ODE's have an energy—

**Example:** Set \( L(x, \dot{x}) = x\dot{x}^2 - x^3 \)

\[ \text{(EL)} \quad \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = \frac{d}{dt} (2x\dot{x}) - \dot{x}^2 + 3x^3 \]

\[ = 2x\ddot{x} + 2\dot{x}^2 - \dot{x}^2 + 3x^3 \]

\[ = 2x\ddot{x} + \dot{x}^2 + 3x^3 = 0 \]
Solutions of (1) have the energy
\[ E = \dot{x} \frac{\partial L}{\partial \dot{x}} - L \cdot \frac{\partial L}{\partial x} = 2x \dot{x} \]
\[ = \dot{x} (2x \dot{x}) - (x \dot{x}^2 - x^3) \]
\[ = 2x \dot{x}^2 - x \dot{x}^2 + x^3 \]
\[ E = x \dot{x}^2 + x^3 \] (2)

Thm: "E is constant along soln's of (1)"

Check: \[ \frac{dE(x(t))}{dt} = 2x \ddot{x} \dot{x} + x^3 + 3x^2 \ddot{x} \]
\[ 2x \ddot{x} = -\dot{x}^2 + 3x^3 \]
\[ \frac{dE}{dt} = x (-\dot{x}^2 - 3x^3) + \dot{x}^3 + 3x^2 \ddot{x} = 0 \]
Write \( H \) as an equivalent (1st order) Hamiltonian system —

**Soln:** write \( E \) as a fn of \((x, p)\) by solving \( \dot{x} = f(x, p) \) and

\[
E(x, \dot{x}) = E(x, f(x, p)) = H(x, p)
\]

\[
\dot{x} = H_p
\]

\[
p = -H_x
\]

\[
\text{I.e., } p = \frac{\partial L}{\partial \dot{x}} = 2x \dot{x} \Rightarrow \dot{x} = \frac{p}{2x}
\]

\[
E(x, \dot{x}) = x \dot{x}^2 + x^3 = x \frac{p^2}{4x^2} + x^3 = \frac{p^2}{4x} + x^3
\]

\[
H(x, p) = \frac{p^2}{4x} + x^3
\]

\[
\dot{x} = H_p = \frac{2p}{4x} = \frac{p}{2x}
\]

\[
p = -H_x = -\frac{p^2}{4x^2} = 3x^2
\]

1st order system \( \equiv (1) \)
Check:

\[ 2 \times \dot{x} = -\ddot{x}^2 - 3x^3 \]

\[ \therefore \dot{x} = \frac{p}{2x} \iff (3) \]

\[ \therefore x = \frac{p}{2x} - \frac{p}{2x^2} \cdot \dot{x} \]

\[ = \frac{p}{2x} - \frac{p}{2x^2} \cdot \frac{p}{2x} = \frac{p}{2x} - \frac{p}{2x^2} \cdot \frac{p}{2x} \]

\[ 2 \times \left( \frac{p}{2x} - \frac{p^2}{2x^2 \cdot 2x} \right) = -\left( \frac{p}{2x} \right)^2 - 3x^3 \]

\[ \dot{p} = \frac{p^2}{2x^2} - \frac{p^2}{4x^2} - 3x^3 \]

\[ \frac{p}{4x^2} \]