Every ODE can be written as a 1st order system:

\[ \dot{x}(t) = f(x(t), t) \]  \( \text{(*)} \)

To solve: "find a function \( x(t) \) such that (*) holds at every \( t \)"

Ex:

\[ \ddot{x}(t) + x(t) \dot{x}(t)^2 + x(t) \dot{x}(t) = \sin t \]

Suppress dependence of \( x \) on \( t \):

\[ \ddot{x} + x \dot{x}^2 + x \dot{x} + x = \sin t \]

Solve for highest order deriv:

\[ \dddot{x} = -x \dot{x} - x \dot{x}^2 - x \ddot{x} + \sin t = g(x, \dot{x}, t) \]

\[ x_1 = x \quad \dot{x}_1 = (x_1) = (x_2) = f(x, t) \]

\[ x_2 = \dot{x} \quad \dot{x}_2 = g(x, \dot{x}, t) \]

\[ x_2 = (x_1, x_2) \]
Linear: \[ \dot{x}(t) = A(t) x(t) + b(t) \]

"coeff's of \( x_1, \ldots, x_n \) are functions of \( t \)."

\[ \text{Linear Homogeneous: } \dot{x}(t) = A(t) x(t) \quad b = 0 \]

\[ \text{Linear Homogeneous C.C.: } \dot{x} = A \bar{x} \]

\[
\begin{pmatrix}
\dot{x}_1(t) \\
\vdots \\
\dot{x}_n(t)
\end{pmatrix}
= 
\begin{pmatrix}
\ddots \\
\ddots \\
\ddots \\
0 \\
\ddots \\
0 \\
\ddots \\
0 \\
\ddots \\
0
\end{pmatrix}
\begin{pmatrix}
x_1(t) \\
\vdots \\
x_n(t)
\end{pmatrix}
\begin{pmatrix}
\dot{x}_1(t) \\
\vdots \\
\dot{x}_n(t)
\end{pmatrix}
\]
Nonlinear autonomous: \( \dot{x}(t) = f(x) \)

Ex: Harmonic Oscillator —

\[ x'' + kx = 0 \quad \Rightarrow \quad x = -kx \]

\[
\begin{align*}
\dot{x} &= x \\
y &= \dot{x} \\
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= 
\begin{pmatrix} y \\ -kx \end{pmatrix} = 
\begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} 
\begin{pmatrix} x \\ y \end{pmatrix}
\end{align*}
\]

\[
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = 
\begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} 
\begin{pmatrix} x \\ y \end{pmatrix}
\]

2x1 2x2 2x1

"Harmonic oscillator is equivalent to a 2x2 homo eqn"
Existence & Uniqueness Theorem - Motivation

It's best to visualize a general 1st order ODE \( \dot{x} = f(x) \) as a 2x2 autonomous system:

\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix} = \begin{pmatrix}
  x(t) \\
  y(t)
\end{pmatrix}
\]

\[
\begin{pmatrix}
  \dot{x}(t) \\
  \dot{y}(t)
\end{pmatrix} = \begin{pmatrix}
  f_1(x, y) \\
  f_2(x, y)
\end{pmatrix} = f(x)
\]

Defn: \( f(x) \) is called a vector field

It gives an arrow at each pt \((x,y)\)
\( \dot{x}(t) = f(x) : \) "Find a curve \( x(t) = (x(t), y(t)) \) whose tangent vector is \( f(x, y) \) at each point."

- If we start at \( x_0 = (x_0, y_0) \) at \( t=0 \), there should exist a curve whose tangent vector is \( f(x(t)) \) at each \( x(t) = (x(t), y(t)) \).
- But: it might not exist for all \( t \) when nonlinear because \( \dot{y} = y^2 \Rightarrow y(t) \to \infty \) as \( t \to t^* \).
Existence & Uniqueness Theorem - Stated in terms of 1st order systems

Theorem 1: Let $\dot{X} = f(X,t)$ be a general n x n system, $X(t) = (X_1(t), \ldots, X_n(t))^T$.
Assume $f$ is Lipschitz continuous in $X$ in some interval containing $X_0$.
Then a unique solution $X(t)$ exists in some interval $t \in (t_0-\varepsilon, t_0+\varepsilon)$ satisfying

$$\dot{X}(t) = f(X(t), t) \quad (1)$$

$$X(t_0) = X_0 \quad (2)$$
Defn. \( f(x, t) \) is Lipschitz continuous in \( x \) if there exists a constant \( K > 0 \) such that

\[
\| f(x_2, t) - f(x_1, t) \| \leq K \| x_2 - x_1 \|
\]

for all \( x_1 \) and \( x_2 \).

\[
\| x \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}
\]

Theorem 2: Suppose \( f(x, t) \) is Lipschitz continuous in \( x \) with the same \( K \) for all \( a \leq t \leq b \). Then a unique solution \( x(t) \) of (1) \& (2) exists \( \forall t \in [a, b] \).

Note: Thm's 1 \& 2 apply to all ODE's of any order \& any size. (No such unifying thms exist for PDE's.)
Exercise 8.10: Prove that \( x = kx \) has a unique solution for all time.

(We know \( x(t) = x_0 e^{kt} \). Are there others?)

Solution:

\[ f(x_0, t) = kx = f(x) \]

\( \Rightarrow \) linear, homogeneous, autonomous, etc.

\[ ||x|| = \sqrt{||x||^2} = ||x|| \]

\[ |f(x_2) - f(x_1)| = |kx_2 - kx_1| \]

\[ = k ||x_2 - x_1|| \]

So \( f \) is Lipschitz continuous with constant \( k = k \) for all \( t \) (independent of \( t \)).
By Theorem 2: \( \dot{x} = kx \) has a unique
\[ x(0) = x_0 \]

Solution defined \( \forall t \) \( \checkmark \)

Ex 2: What does Thm 1 tell you about solutions of \( \dot{x} = x^2 \)?
\[ x(0) = x_0 \]

Solution:
\[ f(x) = x^2 \]
\[ |f(x_2) - f(x_1)| = |x_2^2 - x_1^2| = |x_2 + x_1||x_2 - x_1| \]
Not bounded by \( K \)

However: in \( (x_0 - \varepsilon, x_0 + \varepsilon) \)
\[ K = 2x_0 + 2\varepsilon > x_1 + x_2 \quad \forall \]
\[ x_1, x_2 \in (x_0 - \varepsilon, x_0 + \varepsilon) \]
Conclude: \(|f(x_i) - f(x_j)| \leq K |x_i - x_j|\) in some interval around \(x_0\). \(\Rightarrow\)

Thm0 implies \(\exists!\) soln of \(\begin{cases} \dot{x} = x^2 \\ x(0) = x_0 \end{cases}\) defined in some interval \(t \in (t_0 - \delta, t_0 + \delta)\)

We know we can't do much better because the soln \(x(t) = \frac{1}{1 - t}
\)

Solves \(\dot{x} = x^2\)

\(x(0) = x_0\)

\(\lim x(t) \to \infty \text{ as } t \to \frac{1}{x_0} \ll 1\) when \(x_0 \gg 1\)?
\[ \sqrt{x} \text{ (E! fails)} \quad x = \sqrt{x} \quad \text{a: does there exist a unique soln} \]
\[
\int_0^x \frac{dx}{\sqrt{x}} = \int_0^t dt \quad \implies 
2x^{1/2} \int_0^x dt = t \quad 2x^{1/2} = t
\]
\[ x = \left(\frac{t}{2}\right)^2 \quad \boxed{\text{check: } x(0) = 0 \vee x = t^{1/2} = \sqrt{x} \vee} \]

\[ \boxed{\text{But: } x(t) = 0 \text{ also a soln. What goes wrong?}} \]
\[
|f(x_2) - f(x_1)| = |\sqrt{x_2} - \sqrt{x_1}| \leq k |x_2 - x_1|
\]
for \( x_1, x_2 \) near \( x = 0 \)? NO! try \( x_1 = 0 \)
\[
|\sqrt{x_2}| \leq k|x_2|
\]
always \( \sqrt{x} > kx \) for \( x \) suf\( f \) small \( \Rightarrow \) not Lipschitz cont? \]