Q: How much energy is stored in a car of mass \( M \) moving at velocity \( v \)? I.e., how much work would be done on you if you abruptly came to a stop?

Ans: \( KE = \frac{1}{2} M v^2 \)

0. How do we know this is correct?

2 main answers —

0. Work Done = \( \int_a^b F \cdot ds = \int_a^b ma \, ds \)

\[
= \int_a^b m \frac{dv}{dt} \, ds = \int_a^b m \frac{ds}{dt} \, dv
\]

\[
= \int_{V_a}^{V_b} mvdv = \frac{1}{2} M v^2 \quad \text{or} \quad \Delta KE
\]

\[
= \int_{V_a}^{V_b} mvdv = \frac{1}{2} M v^2 \quad \text{or} \quad \Delta KE
\]
So, the work done is the change in kinetic energy.

2. Assume mass \( m \) moves in a force field.

\[ F = ma = -mg \]

E.g., \( F = ma = -mg \)

Rewrite: \( ma = - \frac{d}{dx} (mgx) \)

"\( mgx \) = potential energy"

More generally, assume \( F = -U'(x) \)

\[ F = ma = -U'(x) \]

\( U(x) \) = "potential"

Claim: \( E = \frac{1}{2}mv^2 + U(x) \) is constant along solutions.

R.F. \[ \frac{d}{dt} E(x(t)) = \frac{d}{dt} \left\{ \frac{1}{2}m(\frac{dx}{dt})^2 + U(x(t)) \right\} = \frac{1}{2}m \cdot 2 \frac{dx}{dt} \frac{d^2x}{dt^2} + U'(x(t)) \frac{dx}{dt} \]
\[ E(x(t)) = \left[ m \frac{d^2x}{dt^2} + U'(x(t)) \right] \frac{dx}{dt} = 0 \]

\[ ma + U' = 0 \text{ by equation!} \]

"The energy is constant in a conservative force-field where \( F = -\frac{d}{dx}U(x) \),

\[ E = \frac{1}{2}mv^2 + U(x) = \text{const} \]

"All along the solution KE is stored & released as potential energy."

\[ \text{Note: same is true when } x \text{ is a vector & equation is} \]

\[ m \ddot{x} = -\nabla U(x) \]

\[ \Rightarrow E = \frac{1}{2}m|\dot{x}|^2 + U(x) \text{ is constant along soln's. This describes every conservative force field.} \]
\textbf{Note:} only special ODE's are conservative. E.g. if \( x(t) \in \mathbb{R} \) and
\[
m\ddot{x} = -U(x)
\]
is the eqn:

Write as 1st order system: \( \dot{x} = y \)

\[
\begin{pmatrix}
\dot{x} \\
y
\end{pmatrix}
= \begin{pmatrix}
y \\
-U(x)
\end{pmatrix}
\]

Very special 1st order system.

Ex: How much energy is stored in a 100 kg mass sitting 1000 meters above the ground?

\[
E = \frac{1}{2}mv_0^2 + mgx_0
\]
t=0 \( v_0 = 0 \) \( E = mgx_0 \)
t=t_0 \( x_0 = 0 \) \( E = \frac{1}{2}mv_x^2 \)
Conclude from $E = \text{const}$:

$$\frac{1}{2} m v^2 = \text{energy of impact} = mg x_0$$

$$= 100 \text{ kg} \times 9.8 \frac{\text{meter}}{s^2} \times 1000 \text{ meter}$$

$$= 980000 \frac{\text{kg \ meter^2}}{s^2} \checkmark$$

Also:

$$v = \sqrt{2g x_0} = \sqrt{2 \times 9800} \frac{\text{meter}}{s}$$

$$\approx 140 \frac{\text{meters}}{s} \checkmark$$
Example from Book - Force with Friction: 2.6-2.7

- Friction is a non-conservative force proportional to minus velocity:

\[ m \ddot{x} = \text{Force} = -U'(x) - kx \]

Conservative non-conservative force of part of force of friction

Equation:

\[ m \ddot{x} = -U'(x) - kx \]
Eg: Spring with Friction:

Force of spring \( \propto \) minus displacement

\[ F_s = -a x = -\frac{d}{dx} \left( \frac{a}{2} x^2 \right) \]

↑

spring const

\( U(x) \) the potential

Force of Friction:

\[ F_f = -k V = -k x \]

Equation:

\[ m \ddot{x} = -\frac{d}{dx} \left( \frac{a}{2} x^2 \right) - k x \]

\[ \underbrace{\text{conservative force}} \]

\[ \text{force} \]
Overdamped: When friction is much larger than acceleration and mass

\[ \dot{x} = -\frac{d}{dx} \left( \frac{g}{2k} x^2 \right) - \frac{m}{k^2} \ddot{x} \]

when this is really small

\[ \approx \text{overdamped} \]

\[ \approx 0 \]

Solve for \( x \):

\[ \dot{x} = -\frac{d}{dx} U(x) \]

\[ x = v \]

Equation for overdamped system with potential \( U(x) \)
\[ x' = -\frac{d}{dx} U(x) \] is a 1st order scalar eqn an interpretation of an overdamped motion in potential.

- Solve by phase portrait -

- Rest pts \(= \max/\min\) of \(U(x)\)
- Solutions head straight for the minima of the potential -