Name: Solutions

Student ID#: 38 students present

Section: ______________

Final Exam

Wednesday March 21, 2012

MAT 119A-W12, Temple

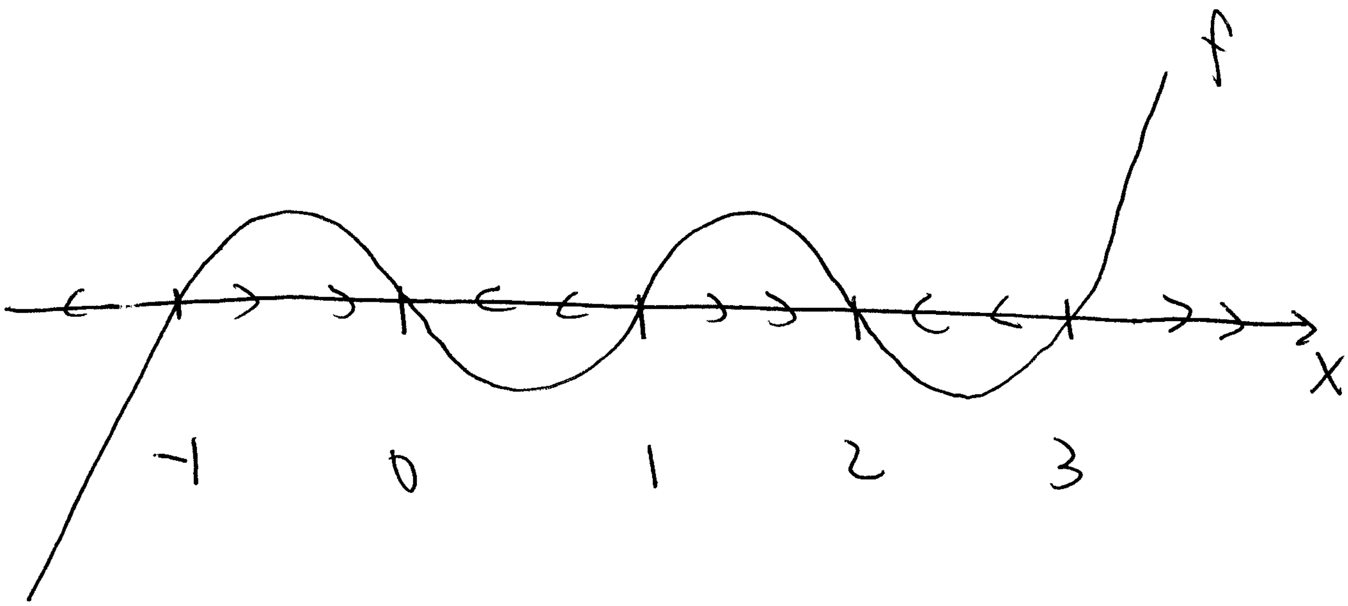
Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		25
2		25
3		25
4		25
5		25
6		25
7		25
8		25
Total		200

Problem #1: **(i)** Draw the phase portrait for the ODE

$$\dot{x} = f(x) = 2x(x+1)(x-1)(x-2)(x-3)$$

on the x-axis along with the graph of f. Which rest points are stable?



Rest pts 0, 2 are stable.

(ii) Use the approximation $f(x) \approx f(1) + f'(1)(x-1) + O(|x-1|^2)$ for x near $x = 1$ to deduce the linearized equations valid near rest point $x = 1$. Solve the linearized equations and use the solution to *deduce* the stability/instability of the rest point $x = 1$.

For x near $x=1$, the equations are

$$\dot{x} \approx f'(1)(x-1)$$

$$\Leftrightarrow \quad \dot{\overbrace{(x-1)}} = f'(1)(x-1)$$

$$f'(x) = 2x(x+1)(+1)(x-2)(x-3) + \{-\}(x-1)$$

$$f'(1) = 2 \cdot 1 \cdot 2 \cdot (+1)(-1)(-2) = +8$$

$$\dot{\overbrace{(x-1)}} = +8(x-1)$$

$$(x-1) = (x_0 - 1)e^{+8t} \xrightarrow{t \to \infty} \pm\infty$$

$\therefore$ $x(t) \to \pm\infty$ as $t \to \infty$ $\Rightarrow$ unstable

Problem #2: Assume $k > 1$, and recall the non-dimensionalized equation for the nonlinear pendulum with friction

$$\ddot{\phi} = \sin\phi\,(k\cos\phi - 1) - \mu\dot{\phi}.$$

(i) Write as a first order system in variables $x = \phi$, $y = \dot{\phi}$, and determine the rest points of the system.

$$x = \varphi,\ y = \dot{\varphi} \Rightarrow \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} y \\ \sin x\,(k\cos x - 1) - \mu y\end{pmatrix} = f(\underset{\sim}{x})$$

$$f(\underset{\sim}{x}) = 0 \Rightarrow y = 0,\ \sin x\,(k\cos x - 1) - \mu \underbrace{y}_{0} = 0$$

$$\boxed{y = 0 \quad \begin{matrix} x = n\pi \\ \cos x = \frac{1}{k}\end{matrix}}$$

(ii) **T** or **F**: If all eigenvalues of a linearized equation at a rest point have negative real part, then all solutions of the *nonlinear* equations starting sufficiently close to the rest point, will tend to the rest point as $t \to \infty$.

T or **F**: If all eigenvalues of a linearized equation have negative real part at a rest point, then all solutions of the *linearized* equations at the rest point will tend to the rest point as $t \to \infty$.

(iii) For each rest point in the range $0 \le x \le \pi$, find the linearized equations, find the eigenvalues, and determine which rest points are (linearly) asymptotically stable.

Rest pts $(0,0)$, $(0,\pi)$, $\cos\bar{x} = \frac{1}{k}$

$$Df(x,0) = \begin{pmatrix} -\nabla f_1- \\ -\nabla f_2- \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ k\cos^2 x - k\sin^2 x - \cos x & -\mu \end{pmatrix}$$

$$\lambda^2 - \tau\lambda + \Delta = \lambda^2 + \mu - (k(\cos^2 x - \sin^2 x) - \cos x) = 0$$

$$\lambda_\pm = \left(-\mu \pm \sqrt{\mu^2 + 4(k(\cos^2 x - \sin^2 x) - \cos x)}\right)\frac{1}{2}$$

$(0,0)$: $$\lambda_\pm = \frac{-\mu \pm \sqrt{\mu^2 + 4(k-1)}}{2} = -\frac{\mu}{2}\left\{1 \pm \sqrt{1 + \frac{4(k-1)}{\mu^2}}\right\}$$

one pos one neg $\Rightarrow$ saddle

$(0,\pi)$: $$\lambda_\pm = \frac{-\mu \pm \sqrt{\mu^2 + 4(k+1)}}{2} = -\frac{\mu}{2}\left\{1 \pm \sqrt{1 + \frac{4k+1}{\mu^2}}\right\}$$

one pos one neg $\Rightarrow$ saddle

$(0,\bar{x})$: $$\lambda_\pm = \frac{-\mu \pm \sqrt{\mu^2 + 4\left(k\left(\frac{1}{k^2} - 1 + \frac{1}{k^2}\right) - \frac{1}{k}\right)}}{2} \Rightarrow \text{stable node}$$

$$= \frac{-\mu \pm \sqrt{\mu^2 - 4k + \frac{8}{k} - \frac{4}{k}}}{2} = \frac{-\mu \pm \sqrt{\mu^2 + 4\left(\frac{1}{k} - k\right)}}{2} < 0$$

Problem #3: Find a basis of two real independent solutions $\mathbf{x}_1(t), \mathbf{x}_2(t)$ of the following linear homogeneous first order system, determine whether the rest point (0,0) is stable or asymptotically stable, classify the rest point, and graph the phase portrait in the (x, y)-plane:

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda^2 - \tau\lambda + \Delta = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3) = 0 \quad \lambda = -1, 3$$

Saddle
unstable

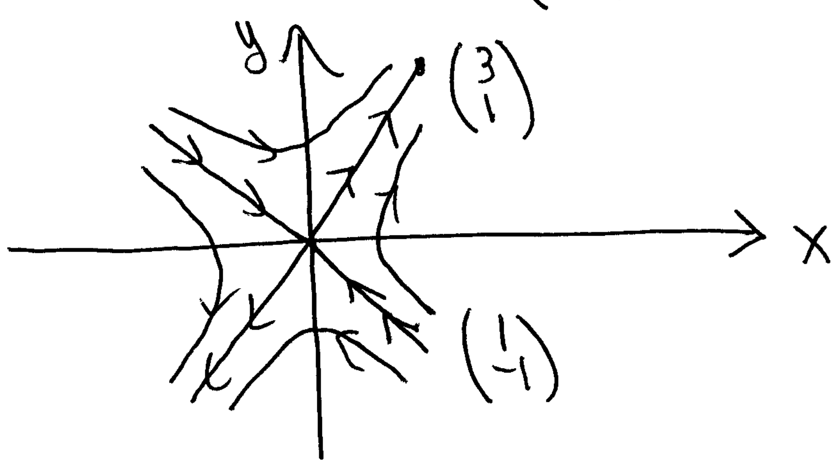
$$\begin{bmatrix} 2+1 & 3 \\ 1 & 0+1 \end{bmatrix} \begin{bmatrix} 1 \\ r \end{bmatrix} = 0 \qquad 3 + 3r = 0 \quad r = -1$$

$$\left(-1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \text{ e-pair}$$

$$\begin{bmatrix} 2-3 & 3 \\ 1 & 0-3 \end{bmatrix} \begin{bmatrix} 1 \\ r \end{bmatrix} = 0 \qquad -1 + 3r = 0 \quad r = 1/3$$

$$\left(3, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right) \text{ e-pair}$$

General soln: $\underset{\sim}{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{3t}$



Problem #4: Find a basis of two real independent solutions $\mathbf{x}_1(t), \mathbf{x}_2(t)$ of the following linear homogeneous first order system, determine whether the rest point (0,0) is stable or asymptotically stable, classify the rest point, and graph the phase portrait in the (x, y)-plane:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\lambda^2 - \tau\lambda + \Delta = \lambda^2 + 2 = 0 \quad \lambda = \pm i\sqrt{2}$ center, stable, not asymptotically stable

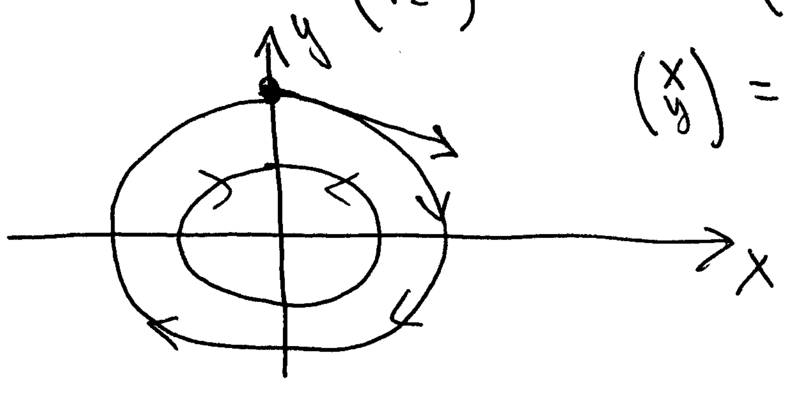
$\begin{bmatrix} 1 - i\sqrt{2} & 3 \\ -1 & -1 - i\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ r \end{bmatrix} = 0, \quad r = \frac{-1 + \sqrt{2}i}{3}$

Complex soln: $\begin{pmatrix} 3 \\ \frac{-1+\sqrt{2}i}{\cancel{3}} \end{pmatrix} e^{i\sqrt{2}t}$

$= \left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} \right\} \left(\cos\sqrt{2}t + i \sin\sqrt{2}t \right)$

real part: $\underset{\sim}{x}_1(t) = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \cos\sqrt{2}t - \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} \sin\sqrt{2}t$

imag part: $\underset{\sim}{x}_2(t) = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} \cos\sqrt{2}t + \begin{pmatrix} 3 \\ -1 \end{pmatrix} \sin\sqrt{2}t$



$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$\Rightarrow$ clockwise

Problem #5: Assume the Lagrangian is given by $L(x, \dot{x}) = x\dot{x}^2 + x^2$.
(i) Find the second order Euler Lagrange equations $\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L - \frac{\partial}{\partial x}L = 0$.

$$\frac{d}{dt}(2x\dot{x}) - \dot{x}^2 - 2x = \cancel{2}\dot{x}^2 + 2x\ddot{x} - \cancel{\dot{x}^2} - 2x = 0 \quad \boxed{\dot{x}^2 + 2x\ddot{x} - 2x = 0}$$

(ii) Find the generalized momentum $p = \frac{\partial}{\partial \dot{x}}L$.

$$p = \frac{\partial L}{\partial \dot{x}} = 2x\dot{x}$$

(iii) Find the generalized energy $E(x, \dot{x}) = \dot{x}\frac{\partial}{\partial \dot{x}}L - L$.

$$E = \dot{x}(2x\dot{x}) - x\dot{x}^2 - x^2 = \cancel{2}x\dot{x}^2 - \cancel{x\dot{x}^2} - x^2 = x\dot{x}^2 - x^2$$

(iii) Find the Hamiltonian $H(x, p)$. $\dot{x} = \frac{p}{2x}$

$$H(x,p) = E\left(x, \frac{p}{2x}\right) = x\left(\frac{p}{2x}\right)^2 - x^2 = \frac{p^2}{4x} - x^2$$

(iv) Find the first order Hamilton's equations $\dot{x} = H_p$, $\dot{p} = -H_x$.

$$\dot{x} = H_p = \frac{2p}{4x} = \frac{p}{2x}$$
$$\dot{p} = H_x = -\ p^2/4x^2 - 2x$$

(v) Prove that H is constant along solutions of Hamilton's equations.

$$\frac{d}{dt}H(x,p) = H_x\dot{x} + H_p\dot{p} = H_x(+H_p) + H_p(-H_x) = 0$$

Problem #6: A nonlinear pendulum solves the equations

$$mr\ddot{\phi} = a\sin\phi + bt^2\dot{\phi},$$

where m is the mass, r is the length of the pendulum, and ϕ is the angle the pendulum makes with the downward vertical.

(i) Find $[mr\ddot{\phi}]$, $[a]$ and $[b]$ where $[X]$ denote the *dimensions of* X in terms of the fundamental dimensions L=length, T=time, M=mass.

$$[mr\dddot{\varphi}] = \frac{ML}{T^2}$$

$$[a] = [mr\ddot{\varphi}] = \frac{ML}{T^2}$$

$$[bt^2\dot{\varphi}] = \frac{ML}{T^2} \Rightarrow [b][t^2][\dot{\varphi}] = \frac{ML}{T^2}$$

$$[b]T^x\frac{1}{\cancel{T}} = \frac{ML}{T^2} \qquad [b] = \frac{ML}{T^3}$$

(ii) Non-dimensionalize the equation $mr\ddot{\phi} = a\sin\phi + bt^2\dot{\phi}$ of Part **(i)**.

$$\tau = \frac{t}{T_0} \qquad \ddot{\varphi} = \frac{\varphi''}{T_0^2} \qquad ' \equiv \frac{d}{d\tau}$$

$$\frac{mr\varphi''}{T_0^2} = a\sin\varphi + bt^2\frac{1}{T_0}\varphi'$$

$$\varphi'' = \underbrace{\frac{aT_0^2}{mr}}\sin\varphi + \frac{bT_0^2\tau^2 T_0^2}{T_0 mr}\varphi'$$

$$\frac{aT_0^2}{mr} = 1 \Rightarrow T_0 = \frac{mr}{a}$$

$$k = \frac{b\left(\frac{mr}{a}\right)^3}{mr} = b\frac{(mr)^2}{a^3}$$

Dim'less Equn: $\boxed{\varphi'' = \sin\varphi + k\tau^2\varphi'}$

Problem #7: Recall the global existence theorem:

Theorem 1 *The ivp for the first order system* $\dot{\mathbf{x}} = f(\mathbf{x})$, $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{x} = (x, y) \in \mathcal{R}^2$ *has a unique solution* $\mathbf{x}(t)$ *defined for all* t *so long as* f *is uniformly* Lipschitz continuous.

Definition 1 f *is uniformly* Lipschitz continuous *if there exists constants* δ, K *such that* $\|f(\mathbf{x}_2) - f(\mathbf{x}_1)\| \leq K\|\mathbf{x}_2 - \mathbf{x}_1\|$ *for all* $\mathbf{x}_1, \mathbf{x}_2$ *in* $\mathcal{R}^2$. *(Here,* $\|\mathbf{x}\| = \sqrt{x^2 + y^2}$ *denotes Euclidean norm.)*

(i) Use the this theorem to prove that the nonlinear pendulum $\ddot{\phi} = \sin\phi$ has a global solution for any initial data $(\phi(0), \dot{\phi}(0)) = (x_0, y_0)$.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ \sin x \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\|f(\underset{\sim}{x}_2) - f(\underset{\sim}{x}_1)\| = \sqrt{(y_1 - y_2)^2 + (\sin x_1 - \sin x_2)^2}$$

But $\sin x_2 - \sin x_1 \underset{\text{MVT}}{=} \cos x^* (x_2 - x_1)$

$$\leq \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \|\underset{\sim}{x}_2 - \underset{\sim}{x}_1\|$$

$K = 1$

$\therefore$ f is Lipshitz continuous ✓

(ii) Prove by counterexample that not every nonlinear system has a global solution for all initial data.

Set $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x^2 \\ 0 \end{pmatrix}$

Then $\dot{x} = x^2 \Rightarrow x(t) = \dfrac{1}{\frac{1}{x_0} - t} \xrightarrow{t \to \frac{1}{x_0}} \infty$

Problem #8: Assume a scalar function $U(\mathbf{x})$ is given and defined for $\mathbf{x} = (x, y) \in \mathcal{R}^2$, and consider the gradient system $\dot{\mathbf{x}} = -\nabla U(\mathbf{x})$.

(i) Prove $U(\mathbf{x}(t))$ is strictly decreasing in time on solutions.

$$\frac{dU(\underset{\sim}{x}(t))}{dt} = \nabla U \cdot \dot{\underset{\sim}{x}} = \nabla U(-\nabla U) = -\|\nabla U\|^2 < 0$$

$\Rightarrow$ U decreases

(ii) Prove (by contradiction) the system has no periodic solutions.

Assume $\exists$ periodic solution $\underset{\sim}{x}(t)$. Then $\underset{\sim}{x}(t) = \underset{\sim}{x}(t+s)$ for some $s > 0$. But U decreases along soln's so

$$U(\underset{\sim}{x}(t)) > U(\underset{\sim}{x}(t+s)) \quad ⨳$$

because $\underset{\sim}{x}(t)$ & $\underset{\sim}{x}(t+s)$ are the same point.