3.1.4 \( x = r + \frac{1}{2} x - \frac{x}{1+x} \)

- \( r < r_1 \): 2 fixed points.
- \( r = r_1 \): one 1/2-stable fixed point.
- \( r_1 < r < r_2 \): no fixed points.
- \( r = r_2 \): one 1/2-stable fixed point.
- \( r > r_2 \): 2 fixed points.
We know saddlenode bifurcations occur when \( r + \frac{1}{2} x = \frac{x}{1+x} \).

There are 2 solutions to this equation.
\( a) \dot{x} = y^2 - x^2 \)  

Fixed points: \( x^* = \pm \sqrt{r} \)

\( x^* = r \) is stable if \( r > 0 \)

(UNSTABLE if \( r < 0 \))

\( x^* = -r \) is unstable if \( r > 0 \)

(Stable if \( r < 0 \))
b) $x = r^2 + x^2$  
due only fixed point  
is $x^* = 0$ when $r = 0$  
 Otherwise, flow is always in the positive $x$ direction.
3.2.6 Eliminating the cubic term

\[ x = x^2 - x^4 + a x^3 + o(x^4), A \neq 0 \]

Let \( x = \sqrt{2} x - x^2 + o(x^4) \)

Let \( x = x + b x^3 + o(x^4) \)

( near identity transformation )

\[ a) \quad x = X + b X^3 + o(X^4) \]

if \( X = x + c x^3 + o(x^4) \)

\[ x = \left( X + b X^3 + o(X^4) \right) + b \left( X + b X^3 + o(X^4) \right) + o(X^4) \]

Note: can absorb \( o(X^4) \) terms into one \( o(X^4) \)

\[ X = (X + b x^3) + c (X + b x^3)^3 + o(X^4) \]

\[ X = x + b x^3 + c \left( b^3 x^9 + 3 b x^7 + 3 X^5 + x^3 \right) + o(X^4) \]

absorb higher order terms into \( o(X^4) \)

\[ X = x + b x^3 + c x^3 + o(X^4) \]

\[ \Rightarrow \quad c = b x^3 + c x^3 + o(X^4) \]

\[ \Rightarrow \quad c = -b \]
b) \[ x = \bar{x} + 3b \bar{x}^2 \bar{x} + o(\bar{x}^4) \]
\[ = (\bar{x} + b \bar{x}^3) + 3b \bar{x}^2 (\bar{x} + c \bar{x}^3) + o(\bar{x}^4) \]
\[ = \bar{x} - b \bar{x}^3 + o(\bar{x}^4) - (x - b \bar{x}^2 + o(\bar{x}^4)) + (a + 3b \bar{x}) (x - b \bar{x}^2 + o(\bar{x}^4)) + o(\bar{x}^4) \]
\[ = \bar{x} - b \bar{x}^3 - \bar{x}^2 + (a + 3b \bar{x}) \bar{x}^3 + o(\bar{x}^4) \]
\[ = \bar{x} - b \bar{x}^2 - \bar{x}^2 + a \bar{x}^3 + 3b \bar{x} \bar{x}^3 + o(\bar{x}^4) \]
\[ = \bar{x} - \bar{x}^2 + a \bar{x}^3 + 2b \bar{x} \bar{x}^3 + o(\bar{x}^4) \]
\[ = \bar{x} - \bar{x}^2 + (a + 2b \bar{x}) \bar{x}^3 + o(\bar{x}^4) \]

c) \[ a + 2b \bar{x} = 0 \implies b = -\frac{a}{2\bar{x}} \]

d) It is not necessary to make the assumption that \( R \neq 0 \) because if \( R = 0 \) we have \( \bar{x} = \bar{x}^2 (1 - a \bar{x}) + o(\bar{x}^4) \).
3.27 \[ X = RX - X^2 + a_n X^n + O(X^{n+1}) \]

We need to consider the effect of a mean identity transformation on this equation. Consider the new variable:

(1) \[ X = T(X) = X + b_n X^n + O(X^{n+1}) \]

When \( T(X) \) is the transformation function.

Let us write the inverse of the transform.

(2) \[ X = S(x) \]

(3) \[ \dot{X} = \frac{dS}{dx} \cdot \dot{x} \]

So \( X \) becomes:

(4) \[ \left( \frac{dS}{dx} \right) \dot{x} = R S(x) - [S(x)]^2 + a_n [S(x)]^n + O(S(x)^{n+1}) \]

There are different ways to simplify this expression, but the brute force straight out calculation.

The question is to find \( S(x) \), the inverse of the transformation. So we can assume a Taylor series expansion for small \( X \), or \( x \).
(5) \( X = S(x) = x + \sum_{i=2}^{\infty} c_i X^i \)

Notice that we have already assumed that the inverse transformation is also a near identity transformation. Ultimately, we will only need the first term in the series - we don't yet know what the exponent or coefficient is so we simplify:

(6) \( X = x + c_m X^m + \mathcal{O}(X^{m+1}) \)

Now substitute (1) in (6):

\[
X = [X + b_n X^n + \mathcal{O}(X^{n+1})] + c_m [X + b_n X^n + \mathcal{O}(X^{n+1})]^m + \mathcal{O}(X^{m+1})
\]

where I have used \( \mathcal{O}(X^{m+1}) = \mathcal{O}(X^{m+1}) \).

Simplifying, we find:

\[
X = X + b_n X^n + c_m X^m + mb_n c_m X^{m-1} X^n + \mathcal{O}(X^{m+1})
\]

To solve for \( m - n \) we find:

\[
X = X + b_n X^n - c_n X^n + \mathcal{O}(X^{n+1})
\]
(2) \[ x = x - b_n x^n + O(x^{n+1}) = 5(x) \]
\[
\frac{ds}{dx} = 1 - nb_n x^{n-1} + O(x^n) 
\]

(4) becomes

(8) \[ \dot{x} = R[x - b_n x^n + O(x^{n+1})] \]
\[
- \left[x - b_n x^n + O(x^{n+1})\right]^2 
\]
\[
+ a_n \left[x - b_n x^n + O(x^{n+1})\right]^2 \left[1 - nb_n x^{n-1} + O(x^n)\right] 
\]

Remark: \(O(x^{n+1})\) means all terms with powers \(\geq n+1\). Simplify (8)

\[ \dot{x} = R[x - b_n x^n + nb_n x^{n-1} + O(x^{n+1})] \]
\[
- x^2 + 2b_n x^{n+1} + O(x^{n+2}) 
\]
\[
+ a_n x^n + O(x^{2n-1}) 
\]

We are still free to choose \(b_n\).

(9) \[ R b_n (n-1) + a_n = 0 \]
This choice makes all terms $\leq x^n$ cancel, so

$$\begin{aligned}
\delta \left[ x = Rn - x^2 + O(x^{n+1}) \right] 
\end{aligned}$$
\[ \dot{N} = u N N - k N \quad \forall i, k, f > 0 \]
\[ \dot{N} = -u N - P N F \]

a) \( \dot{N} \approx 0 \)
   \[ \Rightarrow 0 \approx -u N N - P N F \]
   \[ \Rightarrow N = \frac{P}{u F} \]
   \[ \Rightarrow \dot{N} = u N \left( \frac{P}{u F} \right) - k N \]

b) \( f(n) = \frac{P L n}{G n + f} - k n \)
   \[ \Rightarrow \frac{df}{dn}(n) = \frac{P G L f}{(G n + f)^2} - k = \frac{P G - P F}{f} \]
   \[ \Rightarrow \frac{df}{dn} \left( n^* \right) = \frac{P G L f}{f^2} - k = \frac{P G - P F}{f} \]

So, \( n^* \) is stable if \( \frac{P G - P F}{f} < 0 \) \( \Rightarrow P < \frac{p F}{G} = P_c \)
and it is unstable when \( P > P_c \)

C) transcritical bifurcation.
\[ n^* = \frac{P G - P F}{k L} \]

D)
3.3.1 (a) The only way to solve this is by first non-dimensionalizing:

1. \( \frac{dn}{dt} = GnN - dhn \)

2. \( \frac{dN}{dt} = -GnN - fN + p \)

Set \( N = N_0 \hat{N}, \quad n = n_0 \hat{n}, \quad t = t_0 \hat{t} \)

So \( 1 \) becomes:

\[ \frac{n_0}{t_0} \frac{d\hat{n}}{d\hat{t}} = n_0 N_0 GnN - kN \hat{N} \]

\[ \frac{d\hat{N}}{d\hat{t}} = GN_0 t_0 fN - kN \hat{N} \]

Define \( k_{t_0} = 1 \) \( \Rightarrow \quad t_0 = k^{-1} \)

\( GN_0 t_0 = 1 \) \( \Rightarrow \quad N_0 = k/G_0 \)

or \( 2 \) becomes:

\[ \frac{n_0}{t_0} \frac{d\hat{n}}{d\hat{t}} = -Gn_0 N_0 fN - N_0 t_0 fN + p \]
\[
\frac{dN}{dt} = -\left(\frac{Gn_0}{k}\right) \hat{N}^2 - \left(\frac{f}{k}\right) \hat{N} + \left(\frac{pg}{k^2}\right)
\]

- When we have used \( \Theta \) and \( e \)
- Now we define

\[ e = \frac{f}{k} \]
\[ G = \frac{n_0}{e} \Rightarrow n_0 = \frac{k^2}{Gf} \]
\[ \hat{p} = \frac{pg}{k^2} \]
\[ \frac{dN}{dt} = -\hat{N} \hat{N} - \hat{N} + \frac{\hat{p}}{e} \]

So we can reduce to a first order system when
(a) \( e \) is small
(b) \( \hat{p} \) is not being a small
(c) the initial conditions \( \hat{N}(0), N(0) \) are \( O(1) \)
\[ \Rightarrow \frac{e}{f} \] is small \( \Rightarrow \]
3.4.2 \[ \dot{x} = rx - \sinhx \]

Stable p.p., \( x^* = 0 \) is still stable, but the convergence is slower.

Bifurcation diagram:

PFB @ \( r = 1 \).
3.4.4 \( \dot{x} = x + \frac{rx}{1 + x^2} \)

Fixed point: \( x = 0 \)

\[ x \left(1 + \frac{r}{1 + x^2}\right) = 0 \]

\[ x^* = 0 \]

\[ 1 + \frac{r}{1 + x^2} = 0 \Rightarrow 1 + x^2 + r = 0 \]

\[ x^* = \pm \sqrt{-1 - r} \]

can exist for \( r < -1 \)

So,

- for \( r > -1 \)
  - \( 1 + x^2 + r = 0 \)

- for \( r = -1 \)
  - 1 + x^2 + r = 0

- for \( r < -1 \)
  - 3 f.s.

\[ \begin{align*}
\text{unstable} \\
\overset{\leftarrow}{\downarrow} \quad \overset{\rightarrow}{\downarrow}
\end{align*} \]

\[ \begin{align*}
\text{unstable} \\
\overset{\leftarrow}{\downarrow} \quad \overset{\rightarrow}{\uparrow}
\end{align*} \]

\[ r > -1 \quad r = -1 \quad r < -1 \]
\[ r < -1 \]

Bifurcation diagram

Subcritical pitchfork
$$x = rx - \frac{x}{1+x^2}$$

$$F.P. \ x = 0, \ rx - \frac{x}{1+x^2} = 0$$

$$\Rightarrow x^* = 0, \ r - \frac{1}{1+x^2} = 0$$

$$\Rightarrow x^* = \pm \sqrt{\frac{1}{r} - 1} \text{ exists for } 0 < r \leq 1$$

So,

1. f.p. for \(r > 1\)
2. f.p. for \(0 < r \leq 1\)

\text{Stability: } f'(x) = r - \frac{1-x^2}{(1+x^2)^2}

$$f'(0) = r - 1 \begin{cases} < 0, & r < 1 \Rightarrow \text{**is stable for** } r < 1 \\ > 0, & r > 1 \Rightarrow \text{**is unstable for** } r > 1 \end{cases}$$
\[ f'(\pm \sqrt{v-1}) = r - \frac{1 - (\pm -1)}{1 + (\pm -1)} \]

\[ = 2r > 0 \text{ for } 0 < r < 1 \]

\[ \Rightarrow x^* = \pm \sqrt{v - 1} \text{ are unstable when they exist.} \]

Bifurcation Diagram

Stable

Unstable

Subcritical pitchfork

at \( x^* = 0, \ r = 1 \)
3.4.11 \[ X = rX - \sin X \]

(a) \( r < 0 \) \( \Rightarrow X = -\sin X \)

Fixed points: \( X^* = n\pi \) for \( n \in \mathbb{Z} \)

- When \( n \) is even, \( X^* = n\pi \) is stable
- "Odd" \( \Rightarrow \) unstable

\[
\begin{array}{c|c|c|c}
X & 2\pi & \pi & 0 \\
2\pi & \pi & 0 & \pi \\
\pi & 0 & \pi & 2\pi \\
0 & \pi & 2\pi & \pi \\
-\pi & 2\pi & \pi & 0 \\
-2\pi & \pi & 0 & \pi \\
\end{array}
\]

(b) \( r > 1 \) \( \Rightarrow \) the only fixed point is \( X^* = 0 \)

It is unstable.

c) There will be a subcritical PFB at \( r = 1 \) and infinitely many SNB's as \( r \) decreases from 1 to 0.

d) 

e) As \( r \) decreases from 0 to \(-\infty\), we get infinitely many SNB's.
There are 2 ways to do this problem:

1. Graphically: \( x = r x - \sin(x) \)

Notice the bifurcations occur when you get a horizontal line as \( r \to 0 \) more and more bifurcations occur. They happen closer and closer to the points \( x^* = \left(2n + \frac{1}{2}\right) \pi \) as \( n \to \infty \)

so \( r x^*_n = \sin(x^*_n) \)

\[
r = \frac{1}{\pi (2n + \frac{1}{2})} \text{ for large } n \in \mathbb{N}
\]

(2) Algebraically

All of these bifurcations have the feature that \( x^*_n \) is a double root of the equation

\( f(x) = r x_n - \sin(x) = 0 \)
So \( x_0 \) satisfies \( f(x_0) = 0 \) 

\[
\frac{df}{dx} \bigg|_{x=x_0} = 0
\]

(a) \( x_0 = \sin(x_0) \) \( x_0 \) with \( \sin(x_0) x_0 \)

(b) \( x_0 = \cos(x_0) \)

So divided \( \tan(x_0) = x_0 \)

The intersection of these curves goes very close to \( x_0 = \frac{\pi}{2} (2n+\frac{1}{2}) \frac{\pi}{2} \) because \( \frac{3\pi}{2} \), \( \frac{7\pi}{2} \) are not admissible if \( \sin(x_0) \leq 0 \) there.

So \( x_0 > 0 \) \( \frac{1}{\sin(x_0)} \) as \( x_0 \to 0 \)
34.12 "Quad function"

\[ \dot{x} = (x^2 - x_1 r)(x^2 - x_2 r), \ x_2 > x_1 \geq 0 \]

Fixed points

\[ x^* = \pm \sqrt{x_1 r}, \pm \sqrt{x_2 r} \]

\( \Rightarrow r < 0, \) no fixed points

\( r > 0, \) 4 fixed points

Generalization

(i) \[ \dot{x} = (x^2 - x_1 r)(x^2 - x_2 r) - (x^2 - x_N r) \]

\( 2N > 2n \rightarrow \Rightarrow x_1 > 0 \)

\( r < 0, \) no fixed points

\( r > 0, \) 2N fixed points
ii) $x = x \left( x^{\frac{1}{r}} - x_{2} r \right) \left( x^{\frac{1}{r}} - x_{2} r \right) - (x^{\frac{1}{r}} - x_{N} r)$

$x_{N} > x_{N-1} > \ldots > x_{1} \geq 0$

$r < 0$, one fixed pt exists, $x^{*} = 0$

$r > 0$, 2N+1 fixed pts. exist, i.e. pitchfork-like bifurcation.
3.4.14 Subcritical Pitchfork

\[ x = (x + x^2 - x^4) - x^5 \]

a) \( x = 0 \)

\[ \Rightarrow x (x + x^2 - x^4) = 0 \]

\[ x^* = 0, \quad r + x^2 - x^4 = 0 \]

\[ x^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ = 1 \pm \sqrt{1 + 4r} \quad \text{exists for } r > -\frac{1}{4} \]

\[ \Rightarrow x^* = \pm \sqrt{\frac{1 \pm \sqrt{1 + 4r}}{2}} \quad \text{for } r > -\frac{1}{4} \]

\[ N_{Hee}: \]

1 f.p. for \( r < -\frac{1}{4} \)

5 f.p. for \( -\frac{1}{4} < r < 0 \)

3 f.p. for \( r > 0 \), \( r = -\frac{1}{4} \)
b) Vector fields

\[ r \leq -\frac{1}{4} \]

\[ \frac{1}{4} < r < 0 \]

\[ r = 0 \]

\[ r > 0 \]
Note that the vector fields for the different values of \( r \) match the bifurcation diagram on page 59 (Figure 3.4-2) of the text.

c) \( r_5 = -\frac{1}{4} \).
35.7 Non-dimensionalizing the logistic equation
\[ \dot{N} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0. \]

a) 
\( r \) = growth rate, dimensions are population/sec.
\( K \) = carrying capacity, dimensions are population size.
\( N \) = population size at time \( t \), dimensions are population size.

b) To non-dimensionalize time we set
\[ \tau = rt, \quad \text{which cancels out seconds.} \]
To non-dimensionalize \( N \) we set
\[ X = \frac{N}{K}. \]

Thus,
\[ \frac{dX}{dt} = \frac{dN}{dt}/K. \]
\[ \frac{dN}{dt} = rN \left( 1 - \frac{N}{K} \right) \]

\[ \Rightarrow \quad \frac{1}{r} \frac{dN}{dt} = N \left( 1 - \frac{N}{K} \right) \]

\[ \frac{dN}{d\tau} = N \left( 1 - \frac{N}{K} \right) \]

\[ x = \frac{N}{K}, \quad N = Kx \]

\[ \frac{dx}{d\tau} |_c = Kx \left( 1 - \frac{(Kx)/K}{K} \right) \]

\[ \Rightarrow \quad \frac{dx}{d\tau} = x \left( 1 - x \right) \]

\[ x(0) = \frac{N(0)}{K} = \frac{N}{K} \]
c) Different nondimensionalization

\[ c = r + \]
\[ u = \frac{N}{N_0} \Rightarrow u(0) = \frac{N(0)}{N_0} = \frac{N_0}{N_0} = 1 \]

Thus,

\[ \frac{du}{dc} = \frac{dN}{dc} / N_0 \]

\[ \Rightarrow \frac{du}{dc} N_0 = \frac{dN}{dc} \]

\[ N_0 u = N \]

\[ \Rightarrow \frac{dN}{dt} = r N (1 - \frac{N}{k}) \]

\[ \frac{1}{r} \frac{dN}{dt} = N (1 - \frac{N}{k}) \]

\[ \frac{dN}{dc} = N (1 - \frac{N}{k}) \]

\[ \frac{du}{dc} N_0 = u N_0 (1 - \frac{N_0}{k}) \]
\[
\frac{du}{dc} = u \left(1 - \frac{N_0}{c} u\right)
\]

(d) Well, one advantage is that the second nondimensionalization has initial condition \( u_0 = 1 \), while the first has \( x_0 = \frac{N_0}{k} \). \( u(0) = 1 \) could make it easier to find \( u(t) \).