\[ \dot{\theta} = \sin \theta + \cos \theta = f(\theta) = \sin \theta - (-\cos \theta) \]

F. P. E.
\[ f'(\theta) = 0 \]
\[ \sin \theta^* = -\cos \theta^* \]
\[ \theta^* = \frac{3\pi}{4}, \frac{7\pi}{4} \]

Stability
\[ f'(\theta) = \cos \theta - \sin \theta \]
\[ \theta^* = \frac{3\pi}{4} \quad f'(\frac{2\pi}{4}) = \cos \frac{3\pi}{4} - \sin \frac{3\pi}{4} < 0 \Rightarrow \theta^* = \frac{3\pi}{4} \text{ is STABLE.} \]
\[ \theta^* = \frac{7\pi}{4} \quad f'(\frac{7\pi}{4}) = \cos \frac{7\pi}{4} - \sin \frac{7\pi}{4} > 0 \Rightarrow \theta^* = \frac{7\pi}{4} \text{ is UNSTABLE.} \]
\[ \dot{\theta} = \mu \sin \theta - \sin 2\theta = f(\theta; \mu) \]

Bifurcation when curves are tangent to one another.

Critical pitchfork bifurcation at \( \Theta^* = \pi \)
- \( \mu = \pm 2 \)

Critical pitchfork bifurcation at \( \Theta^* = 0, 2\pi \)
- \( \mu = \pm 2 \)

Bif. pts when \( f(\theta) = 0 \) includes \( \Theta^* = 0, \pi \)

\( f'(\theta) = 0 \) \Rightarrow \( \Theta^* = 0 \) at \( \mu = 2 \)

\( f'(-\pi) = \mu \cos(-\pi) - 2 \cos(-\pi) = -\mu - 2 = 0 \) \Rightarrow \( \Theta^* = \pi \) at \( \mu = -2 \)
\[ f(\theta^* = 0, \mu = 2) = 0 \]
\[ \frac{\partial f}{\partial \theta}(\theta^* = 0, \mu = 2) = 0 \]
\[ \frac{1}{2} \frac{\partial^2 f}{\partial \theta^2}(\theta^* = 0, \mu = 2) = 0 \]
\[ \frac{1}{6} \frac{\partial^3 f}{\partial \theta^3}(\theta^* = 0, \mu = 2) = \frac{-\mu + 3}{6} \frac{\mu - 2}{6} = \frac{6}{6} = 1 \]

\[ \Rightarrow \Theta \approx (\mu - 2)\Theta + \Theta^3 = r\Theta + \Theta^3 \]

which is the normal form for a SUBCRITICAL PITCHFORK BIFURCATION (at \( \mu = 2, \theta^* = 0 \)).

\[ r = 0 \]

\[ \Rightarrow \dot{\theta} = - (\mu + 2)(\theta - \pi) + (\theta - \pi)^3 \]

\[ y = \theta - \pi \Rightarrow \dot{y} = r - y + y^3 \]

\[ r = \mu + 2 \]

which is the normal form for a SUBCRITICAL PITCHFORK BIF.

at \( y = 0, r = 0 \) i.e. \( \mu = -2, \theta^* = \pi \)

\[ \Rightarrow \dot{\theta} = - (\mu + 2)(\theta - \pi) + (\theta - \pi)^3 \]

\[ y = \theta - \pi \Rightarrow \dot{y} = r - y + y^3 \]

which is the normal form for a SUBCRITICAL PITCHFORK BIF.

at \( y = 0, r = 0 \) i.e. \( \mu = -2, \theta^* = \pi \)
\[ \dot{\theta} = \mu + \sin \theta, \quad \theta \in (-\pi, \pi] \]

\(\mu \) slightly less than 1
\[ (x)' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} (x) \]

A)

\[ x'y' - y'x' = 0 \]

\[ \text{i.e. } x(-y) - y(-x) = 0. \]

\[ x \frac{dy}{dx} = y \frac{dx}{dy} \]

\[ \int x \frac{dx}{dy} \, dy = \int y \frac{dy}{dx} \, dx \]

\[ \int x \, dx = \int y \, dy \]

\[ \frac{x(t)^2 - x(o)^2}{2} = \frac{y(t)^2 - y(o)^2}{2} \]

\[ \Rightarrow \quad x(t)^2 - y(t)^2 = c \]

\[ C = x(o)^2 - y(o)^2 \]
\[
\begin{bmatrix}
-\lambda & -1 \\
-1 & -\lambda
\end{bmatrix} - I = 0 \quad \Rightarrow \quad \lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda_+ = 1, \quad \lambda_- = -1
\]

\[\lambda_+ = 1 \quad \begin{bmatrix}
-1 & -1 \\
-1 & -1
\end{bmatrix} \begin{bmatrix} v_+ \\ v_+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad -1 - v_+ = 0 \quad v_+ = 1\]

\[\lambda_- = -1 \quad \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix} v_- \\ v_- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad 1 - v_- = 0 \quad v_- = 1\]

\[\Rightarrow \text{eigenvalue} \quad \lambda_+ = 1, \quad v_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\]

\[\lambda_- = -1, \quad v_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\]

1. **Stable manifold of saddle pt is along** \(\overline{v}_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\)
   - \(\dot{y} = -x\)

2. **Stable manifold of saddle pt is along** \(\overline{v}_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\)
   - \(\dot{y} = x\)
\[
\begin{align*}
  x &= \gamma - y \\
  y &= x - y
\end{align*}
\Rightarrow \quad \chi &= \frac{u - y}{z} \\
\dot{\gamma} &= \frac{u - y}{z}
\]

\[
\begin{align*}
\frac{du}{dt} &= \frac{dx}{dt} + \frac{dy}{dt} = -u - \chi = -u \\
\frac{dv}{dt} &= \frac{dx}{dt} - \frac{dy}{dt} = -y + \chi = v
\end{align*}
\]

\[
\left\{
\begin{align*}
\frac{du}{dt} &= -u \\
\frac{dv}{dt} &= v
\end{align*}
\right. 
\Rightarrow \quad u(t) &= u_0 e^{-t} \\
v(t) &= v_0 e^{t}
\]

Where $u(0) = u_0$ \\
v(0) = v_0

\text{E)}

UNSTABLE MANIFOLD
\Rightarrow \quad u = 0

STABLE MANIFOLD
\Rightarrow \quad v = 0

\text{F)}

\begin{align*}
\chi(t) &= \frac{u(t) + v(t)}{2} = \frac{u_0 e^{-t} + v_0 e^{t}}{2} \\
&= \frac{x_0 + y_0}{2} e^{-t} + \frac{\gamma_0 - y_0}{2} e^{t}
\end{align*}

\begin{align*}
\frac{\gamma(t)}{2} &= \frac{u(t) - v(t)}{2} = \frac{x_0 + y_0}{2} e^{-t} - \frac{\gamma_0 - y_0}{2} e^{t}
\end{align*}
$$(y)' = \begin{bmatrix} 4 - \lambda & -1 \\ 2 & -\lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \lambda' = \lambda \gamma$$

a) Eigenvalues and eigenvectors

\[ \det \begin{bmatrix} 4-\lambda & -1 \\ 2 & -\lambda \end{bmatrix} = (4-\lambda)(-\lambda) - 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2) = 0 \]

\[ \lambda_1 = 3, \quad \lambda_2 = 2 \]

\[ \lambda_1 = 3 \Rightarrow \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

\[ \lambda_2 = 2 \Rightarrow \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ \Rightarrow \{ \lambda_1 = 3, \quad \overline{y}_+ = (1) \}, \{ \lambda_2 = 2, \quad \overline{y}_- = (1,0) \} \]

\[ \text{General solution of } y = A\overline{y} : \]

\[ \begin{bmatrix} y(1) \\ y(0) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

b) \( \lambda_1 = 3, \lambda_2 = 2 \) \( \Rightarrow (0,0) \text{ is an unstable node.} \)
d) \( \bar{X}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \)

\[
\Rightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]

\( c_2 = 1 \), \( c_1 = 2 \)

\[
\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \alpha e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]
5.2.2 \[ \begin{cases} x' = x - y \\ y' = x + y \end{cases} \]

a) \[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

\[ x' = A \begin{bmatrix} x \end{bmatrix} \]

\[ \det \begin{bmatrix} 1 - \lambda & -1 \\ 1 & 1 - \lambda \end{bmatrix} = (\lambda - 1)^2 + 1 \]

\[ = \lambda^2 - 2\lambda + 2 = 0 \]

\[ \lambda = 2 \pm \sqrt{4 - 4(2)} = 1 \pm i \]

\[ \begin{bmatrix} 1 - (1 \pm i) & -1 \\ 1 & 1 - (1 \pm i) \end{bmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 0 \]

\[ \begin{bmatrix} \mp i & -1 \\ 1 & \mp i \end{bmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 0 \quad \Rightarrow \begin{pmatrix} v \end{pmatrix} = \mp i \]

Eigenpairs:

\[ \lambda_1 = 1 + i, \quad \begin{pmatrix} 1 \\ i \end{pmatrix} \]

or \( \begin{pmatrix} i \\ 1 \end{pmatrix} \)

\[ \lambda_2 = 1 - i, \quad \begin{pmatrix} 1 \\ -i \end{pmatrix} \]

or \( \begin{pmatrix} -i \\ 1 \end{pmatrix} \)
\[ \overline{X}(t) = c_1 e^{(1+i)t} \left( \frac{1}{1-i} \right) + c_2 e^{(1-i)t} \left( \frac{1}{i} \right) \]

where \( c_1, c_2 \in \mathbb{C} \)

\[ \overline{X}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1 \left( \frac{1}{i} \right) + c_2 \left( \frac{1}{i} \right) \]

\[ \Rightarrow c_1 + c_2 = x_0 \iff \text{Im}(c_1) + \text{Im}(c_2) = 0 \]
\[ \text{Re}(c_1) + \text{Re}(c_2) = x_0 \]
\[ -ic_1 + ic_2 = y_0 \iff -\text{Re}(c_1) + \text{Re}(c_2) = 0 \]
\[ \text{Im}(c_1) - \text{Im}(c_2) = y_0 \]

\[ c_1 = \frac{x_0 + i y_0}{2}, \quad c_2 = \frac{x_0 - i y_0}{2} \]

\[ \overline{X}(t) = e^t \left( \cos t + i \sin t \right) \begin{pmatrix} x_0 + i y_0 \\ y_0 - i x_0 \end{pmatrix} \]
\[ + \frac{e^t}{2} \left( \cos t - i \sin t \right) \begin{pmatrix} x_0 - i y_0 \\ y_0 + i x_0 \end{pmatrix} \]

\[ = \frac{e^t}{2} \left( \begin{pmatrix} x_0 \cos t - y_0 \sin t + i x_0 \sin t + i y_0 \cos t \\ y_0 \cos t + x_0 \sin t - i x_0 \cos t + i y_0 \sin t \end{pmatrix} \right) \]
\[ + \frac{e^t}{2} \left( \begin{pmatrix} x_0 \cos t - y_0 \sin t - i x_0 \sin t - i y_0 \cos t \\ y_0 \cos t + x_0 \sin t + i x_0 \cos t - i y_0 \sin t \end{pmatrix} \right) \]

\[ \overline{X}(t) = e^t \begin{pmatrix} x_0 \cos t - y_0 \sin t \\ y_0 \cos t + x_0 \sin t \end{pmatrix} \]
5.2.4 \[
\begin{align*}
\dot{x}' &= 5x + 10y \\
\dot{y}' &= -x - y
\end{align*}
\]

- eigenvalues: \[
\det \begin{bmatrix} 5-\lambda & 10 \\ -1 & -1-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 5 = 0
\]
  \[
  \lambda = 2 \pm i
\]
  \[
  \implies (0,0) \text{ is an UNSTABLE SPIRAL.}
\]

* note: eigenvectors will be complex.
\[ \begin{align*}
\begin{cases}
x' &= -3x + 4y \\
y' &= -2x + 3y
\end{cases}
\end{align*} \]

**Eigenvalues:**
\[
\det \begin{bmatrix} -3 - \lambda & 4 \\ -2 & 3 - \lambda \end{bmatrix} = \lambda^2 - 1 = 0
\]
\[\lambda_1, \lambda_2 = \pm 1\]

**Eigenvectors:**
\[
\lambda_1 = 1 \quad \begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]
\[
\lambda_2 = -1 \quad \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \quad \Rightarrow \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

\[
\Rightarrow \quad \left\{ \lambda_1 = 1, \ v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \left\{ \lambda_2 = -1, \ v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}
\]

\((0, 0)\) is a **SADDLE POINT**

![Diagram showing unstable and stable manifolds]

5.2.13 \quad m \ddot{x} + b \dot{x} + kx = 0 \quad b > 0, \ m, k > 0

(a) \quad \dot{x} = v
\quad \dot{v} = -\frac{k}{m} x - \frac{b}{m} v

\quad (x')' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x' \\ v' \end{bmatrix} \Rightarrow \begin{bmatrix} x' \\ v' \end{bmatrix} = A \begin{bmatrix} x \\ v \end{bmatrix}

(b) calculate eigenvalues of A

\quad \text{det} \begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{b}{m} -\lambda \end{bmatrix} = \lambda^2 + \frac{b}{m} \lambda + \frac{k}{m} = 0

\quad \lambda_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}

because \quad b, k, m > 0, \quad \text{Re} (\lambda_{1,2}) < 0

\quad \Rightarrow (0, 0) \ is \ always \ STABLE

(i) \quad \text{if} \quad \left(\frac{b}{2m}\right)^2 > \frac{k}{m}, \ \text{then}

\quad \lambda_1 = -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} < \lambda_2 = -\frac{b}{m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} < 0

\Rightarrow (0, 0) \ is \ a \ STABLE \ NODE.
Note that the eigenvectors of $A$ are

$$\vec{v}_{1,2} = \left( \begin{array}{c} 1 \\ \lambda_{1,2} \end{array} \right) = \left( \begin{array}{c} 1 \\ \frac{-b \pm \sqrt{(b/2m)^2 - \frac{k}{m}}} {2m} \end{array} \right)$$

$$\vec{v}_1 = \left( \begin{array}{c} 1 \\ \lambda_1 \end{array} \right)$$

$$\lambda_1 = -\frac{b}{2m} - \sqrt{(\frac{b}{2m})^2 - \frac{k}{m}}$$

$$\vec{v}_2 = \left( \begin{array}{c} 1 \\ \lambda_2 \end{array} \right)$$

$$\lambda_2 = -\frac{b}{2m} + \sqrt{(\frac{b}{2m})^2 - \frac{k}{m}}$$

assuming $\vec{v}(0) = 0$ $\chi(0) > 0$

"overdamped"
if \( \left( \frac{b}{2m} \right)^2 = \frac{k}{m} \), then \( \lambda_1 = \lambda_2 = \lambda = -\frac{b}{m} < 0 \)

with only one eigenvector \( \vec{v} = \begin{pmatrix} 1 \\ -\frac{b}{\lambda} \end{pmatrix} \)

\( \Rightarrow (0, 0) \) is a **STABLE DEGENERATE NODE**.

Note that as \( \left( \frac{b}{2m} \right)^2 - \frac{k}{m} \) approaches 0 from above (going from case (i) to case (iii)), the eigenvectors \( \vec{v}_1 \) and \( \vec{v}_2 \) are squeezed together until they coalesce when \( \left( \frac{b}{2m} \right)^2 - \frac{k}{m} = 0 \).

"CRITICALLY DAMPED"
(iii) \( \frac{(b^2)}{2m} < \frac{k}{m} \), then \( \lambda_{1,2} \in \mathbb{C} \)

\[
\lambda_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b^2}{2m}\right)^2 - \frac{k}{m}} \\
= \alpha \pm i \omega
\]

\( \alpha = \text{Re} (\lambda_{1,2}) = -\frac{b}{2m} \)

\( \omega = \text{Im} (\lambda_{1,2}) = \pm \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \)

\((0,0)\) is a \textbf{STABLE FOCUS.}

\textit{damped oscillations}

"\textit{UNDER DAMPED}"

---

\[
\begin{align*}
\lambda_{1,2} &= -\frac{b}{2m} \pm \sqrt{\left(\frac{b^2}{2m}\right)^2 - \frac{k}{m}} \\
\alpha &= \text{Re} (\lambda_{1,2}) = -\frac{b}{2m} \\
\omega &= \text{Im} (\lambda_{1,2}) = \pm \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}
\end{align*}
\]