MIDTERM EXAM
Math 119A
Temple-Winter 2012

- Print your name, section number and put your signature on the upper right-hand corner of this exam. Write only on the exam.

- Show all of your work, and justify your answers for full credit.

SCORES

#1

#2

#3

#4

#5

#6

TOTAL:
1. (16 pts) Use separation of variables to derive exact formulas for the solutions of the following fundamental initial value problems:

(a) $\dot{x} = kx$, $x(t_0) = x_0 > 0$.

$$\frac{dx}{dt} = kx \Rightarrow \int \frac{dx}{x} = \int k \, dt \Rightarrow \ln|x| = kt + \hat{c}$$

$$\Rightarrow x = ce^{kt} \quad \text{then} \quad x(t_0) = x_0$$

$$\Rightarrow x_0 = c \frac{e^{kt_0}}{e^{kt_0}} \Rightarrow c = \frac{x_0}{e^{kt_0}}$$

$$\Rightarrow x(t) = \frac{x_0}{e^{kt_0}} \cdot e^{kt} = x_0 \cdot e^{kt} \cdot e^{-kt_0} = x_0 \cdot e^{k(t-t_0)}$$

(b) $\dot{x} = x^2$, $x(t_0) = x_0 > 0$.

$$\frac{dx}{dt} = x^2 \Rightarrow \int \frac{dx}{x^2} = \int dt \Rightarrow -\frac{1}{x} = t + c$$

$$\Rightarrow x = \frac{-1}{t + c} \quad \text{then} \quad x(t_0) = x_0$$

$$\Rightarrow x_0 = \frac{-1}{t_0 + c} \Rightarrow t_0 + c = -\frac{1}{x_0}$$

$$\Rightarrow c = \frac{-1}{x_0} - t_0$$

$$x = \frac{1}{t + \frac{1}{x_0} - t_0}$$
2. (15 pts) Consider the ivp for the harmonic oscillator,

\[ \ddot{x} + kx = 0 \]
\[ x(0) = a, \quad \dot{x}(0) = b. \]  

(a) Write the general solution: (you need not derive it).

\[ x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \Rightarrow \dot{x} = c_1 \omega \sin(\omega t) + c_2 \omega \cos(\omega t) \]
\[ x''(t) = -c_1 \omega^2 \cos(\omega t) + c_2 \omega^2 \sin(\omega t) \Rightarrow \omega = \sqrt{\frac{k}{m}} \]

(b) Write (1) as a 2 x 2 first order system in matrix form.

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -kx
\end{align*}
\]

\[
\begin{bmatrix}
\dot{x} \\
\dot{v}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-k & 0
\end{bmatrix}
\begin{bmatrix}
x \\
v
\end{bmatrix}
\]

* Note that it is silly to introduce a new variable \( X_1 = x \) since that accomplishes nothing.

(c) Circle all correct answers: Equation (1) is

linear / homogeneous / autonomous / constant-coefficient
3. (18 pts) Recall the local existence theorem:

**Theorem 1** The (scalar) ivp $\dot{x} = f(x)$, $x(t_0) = x_0$ has a unique solution $x(t)$ defined for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$ so long as $f$ is Lipschitz continuous in a neighborhood of $x_0$.

**Definition 1** $f$ is Lipschitz continuous in a neighborhood of $x_0$ if there exists constants $\delta, K$ such that $|f(x_2) - f(x_1)| \leq K|x_2 - x_1|$ for all $x_1, x_2$ within a distance $\delta$ of $x_0$, (i.e., $|x_1 - x_0| < \delta$).

(a) Use $\dot{x} = x^2$, $x(0) = x_0 > 0$ to show $\epsilon$ can depend on $x_0$.

This has no bearing on the argument:

1. $|f(x) - f(y)| = |x^2 - y^2| \leq |x + y||x - y| = K|x - y|, K = |x + y| \implies f$ is Lipschitz for every pair $x, y < \infty$, but $K = K(x, y)$.

2. From (1) $x = x^2, x(0) = x_0 \implies -\frac{1}{x} = t + C \implies C = -\frac{1}{x_0}$

3. If $t = \frac{1}{x_0}$ then $x(t)$ f. N. E.

Since $x(t)$ only exists until $t = \frac{1}{x_0}$, $x(t)$ is only guaranteed to exist for $t \in (-\epsilon, \epsilon) \subseteq (-\frac{1}{x_0}, \frac{1}{x_0})$

*E.g.* $\epsilon = f(x_0)$.

Note: $f$ is Lipschitz for any $f < \infty$, so this says nothing about $\epsilon = f(x_0)$.
(b) Use $\dot{x} = x^{1/2}$, $x(0) = 0$ to show that continuity of $f$ alone is not enough to imply uniqueness of solutions.

\[
\begin{align*}
(1) \quad \int_0^t \frac{dx}{x^{1/2}} &= \int_0^t dt \quad \Rightarrow \quad 2x^{1/2} \bigg|_0^x = t \quad \Rightarrow \quad 2x^{1/2} = t \quad \Rightarrow \quad x = (\frac{t}{2})^2 \\
(2) \quad x(t) = 0 \quad \text{is also a solution} \\
\Rightarrow \quad f \text{ is continuous for all } x > 0, \text{ but there is not a unique solution to the IVP.}
\end{align*}
\]

* Showing not Lipschitz or explaining what more is needed is extra.

* Also showing continuity $\Rightarrow$ Lipschitz is largely irrelevant.
4. (20 pts) (a) Draw the phase portrait for the ode

\[ \dot{x} = f(x) = 2(x + 3)(x + 1)x(2x - 1)(x - 1)(x - 2) \]

on the x-axis along with the graph of \( f \).

\[ X \cap C \]

\[ x = -3, -1, 0, \frac{1}{2}, 1, 2 \Rightarrow 6 \text{ roots} \]

get it backwards!

\[ \text{M et C} \]
(b) Use the approximation \( f(x) \approx f(1) + f'(1)(x - 1) \) for \( x \) near \( x = 1 \) to approximate solutions of \( x = f(x) \) near \( x = 1 \), and deduce the stability/instability of the rest point.

\[
\dot{x} = f(x) \approx f(1) + f'(1)(x - 1) \quad \text{sufficiently close to } x = 1,
\]

\[
\Rightarrow |f(x) - f(1)| \leq |f'(1)| |x - 1|
\]

\[
\Rightarrow \delta_0 < f'(1) \quad \text{where } \delta = (x - 1) \text{ is a perturbation about } x = 1.
\]

Stable if \( f'(1) < 0 \) \( \text{b/c } \delta \rightarrow 0 \) as \( t \rightarrow 0 \)

unstable if \( f'(1) > 0 \) \( \text{b/c } \delta \rightarrow \infty \) \( \text{as } t \rightarrow 0 \)

Now in this case:

\[
\dot{x} = f(1) = (x - 1)^2 (2x + 3)(x + 1) x (2x - 1) (x - 2) \bigg|_{x=1}
\]

\[
= x^2 (2x + 3)(x + 1) x (2x - 1) (x - 2) \bigg|_{x=1}
\]

\[
= -4(1)(2)(1)(1)(-1)
\]

\[
= -16
\]

\[
\Rightarrow f'(1) < 0 \Rightarrow \delta = x_0 e^{-16t} \rightarrow 0
\]

Big idea.

Using LSA = +5

Explain why = +5

using \( f(1) \approx f(1) (x - 1) \)
in almost any way....
5. (16 pts) Draw the phase portrait for each $r = \text{constant}$ and construct the bifurcation diagram for each of the following ODE's. Justify and label (saddle-node, transcritical, pitchfork).

(a) $\dot{x} = r + x^2$

$$r + x^2 = 0 \quad \Rightarrow \quad x^2 = -r \quad \Rightarrow \quad x = \pm \sqrt{-r}$$

![Phase portraits for different values of r: r < 0, r = 0, r > 0, illustrating saddle-node, transcritical, and pitchfork bifurcations.](image-url)
(b) \( \dot{x} = rx - x^2 \)

\[
\begin{align*}
\dot{x} &= 0 \\
rx - x^2 &= 0 \\
x(r - x) &= 0 \\
x &= 0 \quad \text{or} \quad x = r
\end{align*}
\]

\[r < 0 \quad r = 0 \quad r > 0\]
6. (15 pts) Draw the phase portrait for each $\gamma = constant$ and construct the bifurcation diagram for the equation

$$\dot{\phi} = -\sin \phi + \gamma \sin \phi \cos \phi,$$

(the overdamped bead on a rotating hoop.)