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MIDTERM EXAM
Math 119A
Temple-Winter 2012

–Print your name, section number and put your signature on the upper right-hand corner of this exam. Write only on the exam.

–Show all of your work, and justify your answers for full credit.

SCORES

#1

#2

#3

#4

#5

#6

TOTAL:

1. (16 pts) Use separation of variables to derive exact formulas for the solutions of the following fundamental initial value problems:

(a) $\dot{x} = kx$, $x(t_0) = x_0 > 0$.

$\frac{dx}{dt} = kx \Rightarrow \int \frac{dx}{x} = \int k\,dt \Rightarrow \ln|x| = kt + \hat{c}$

$\Rightarrow x = ce^{kt}$ then $x(t_0) = x_0$

$\Rightarrow x_0 = ce^{kt_0} \Rightarrow c = \frac{x_0}{e^{kt_0}}$

-2 for $t_0 = 0$

max -3 if do it twice

$\Rightarrow x(t) = \frac{x_0}{e^{kt_0}} e^{kt} = x_0 e^{kt} e^{-kt_0} = x_0 e^{k(t-t_0)}$

(b) $\dot{x} = x^2$, $x(t_0) = x_0 > 0$.

$\frac{dx}{dt} = x^2 \Rightarrow \int \frac{dx}{x^2} = \int dt \Rightarrow -\frac{1}{x} = t + c$

$\Rightarrow x = -\frac{1}{t+c}$ then $x(t_0) = x_0$

$\Rightarrow x_0 = \frac{-1}{t_0 + c} \Rightarrow \Rightarrow t_0 + c = -\frac{1}{x_0}$

$c = -\frac{1}{x_0} - t_0$

$\Rightarrow x = \frac{-1}{t - \frac{1}{x_0} - t_0}$

$= \frac{1}{t_0 + \frac{1}{x_0} - t}$

6 min

2. (15 pts) Consider the ivp for the harmonic oscillator,

$$\ddot{x} + kx = 0 \tag{1}$$
$$x(0) = a, \quad \dot{x}(0) = b.$$

5 (a) Write the general solution: (you need not derive it).

Fullpoints

$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \xrightarrow{\text{i.c.}} a = c_1$

$x'(t) = -c_1 \omega \sin(\omega t) + c_2 \omega \cos(\omega t) \xrightarrow{\text{i.c.}} b = c_2 \omega \Rightarrow c_2 = \frac{b}{\omega}$

$x''(t) = -c_1 \omega^2 \cos(\omega t) - c_2 \omega^2 \sin(\omega t)$

$\Rightarrow -c_1 \omega^2 \cos(\omega t) - c_2 \omega^2 \sin(\omega t) + kc_1 \cos(\omega t) + kc_2 \sin(\omega t) = 0$

$\Rightarrow k = \omega^2 \Rightarrow \omega = \pm\sqrt{k}$

$\Rightarrow x(t) = a\cos(\sqrt{k}t) + \frac{b}{\sqrt{k}} \sin(\sqrt{k}t)$

this is extra

6 (b) Write (1) as a 2×2 first order system in matrix form.

$$\begin{cases} \dot{x} = v \\ \dot{v} = \ddot{x} = -kx \\ x(0) = a \\ \dot{x}(0) = b \end{cases} \Rightarrow \begin{bmatrix} \frac{dx}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

* note that it just silly to introduce a new variable $x_1 = x$ ... that accomplishes nothing.

4 (c) Circle all correct answers: *Equation (1) is*

linear / homogeneous / autonomous / constant–coefficient

3. (18 pts) Recall the local existence theorem:

Theorem 1 *The (scalar) ivp $\dot{x} = f(x)$, $x(t_0) = x_0$ has a unique solution $x(t)$ defined for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$ so long as f is* Lipschitz continuous *in a neighborhood of x_0.*

Definition 1 *f is* Lipschitz continuous *in a neighborhood of x_0 if there exists constants δ, K such that $|f(x_2) - f(x_1)| \leq K|x_2 - x_1|$ for all x_1, x_2 within a distance δ of x_0, (i.e., $|x_i - x_0| < \delta$).*

(9)

(a) Use $\dot{x} = x^2$, $x(0) = x_0 > 0$ to show ϵ can depend on x_0.

This has no bearing on the argument:

optional → $|f(x) - f(y)| = |x^2 - y^2| \leq |x+y||x-y| = K|x-y|$, $K = |x+y|$

(I gave 2pts if you did this and nothing else...)

$\Rightarrow$ f is Lipschitz for every pair $x, y < \infty$, but $K = K(x,y)$.

From (1b) $\dot{x} = x^2$, $x(0) = x_0 \Rightarrow -\frac{1}{x} = t + c \Rightarrow c = -\frac{1}{x_0}$

$$\Rightarrow x(t) = -\frac{1}{t - \frac{1}{x_0}}$$

If $t = \frac{1}{x_0}$ then $x(t)$ D.N.E.

Since $x(t)$ only exists until $t = \frac{1}{x_0}$, $x(t)$ is only guaranteed to exist for:

$$t \in (-\varepsilon, \varepsilon) = \left(-\frac{1}{x_0}, \frac{1}{x_0}\right)$$

e.g. $\varepsilon = f(x_0)$.

enough for full points

note: f is Lipschitz for any $\delta < \infty$, so this says nothing about $\varepsilon = f(x_0)$!

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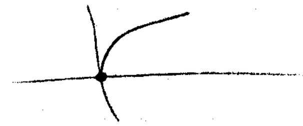


(9)

(b) Use $\dot{x} = x^{1/2}$, $x(0) = 0$ to show that continuity of f alone is not enough to imply uniqueness of solutions.

(1) $$\int_0^x \frac{dx}{x^{1/2}} = \int_0^t dt \Rightarrow 2x^{1/2}\Big|_0^x = t \Rightarrow 2x^{1/2} = t \Rightarrow x = \left(\frac{t}{2}\right)^2$$

(2) $x(t) = 0$ is also a solution

$\therefore \Rightarrow f$ is continuous for all $x > 0$, but there is not a unique solution to the IVP.

* Showing not Lipschitz or explaining what more is need is extra.

* Also showing continuity $\not\Rightarrow$ Lipschitz is largely irrelevant.

4. (20 pts) (a) Draw the phase portrait for the ode

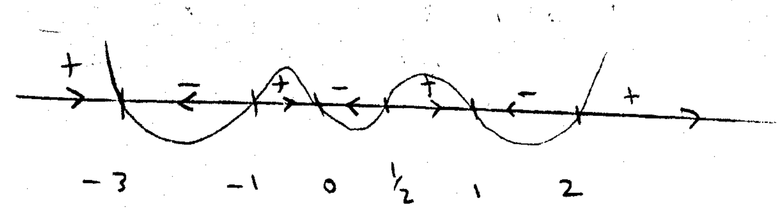
$$\dot{x} = f(x) = 2(x+3)(x+1)x(2x-1)(x-1)(x-2)$$

(10) on the x-axis along with the graph of f.

$+x^6$

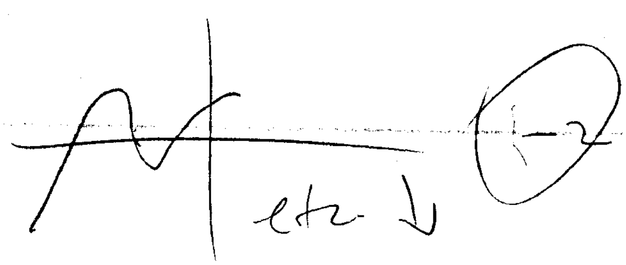
$x = -3, -1, 0, \frac{1}{2}, 1, 2 \Rightarrow 6$ roots

$\Rightarrow$



S us. S us. S us.

get it backwards:



(10)

(b) Use the approximation $f(x) \approx f(1) + f'(1)(x-1)$ for x near $x = 1$ to approximate solutions of $\dot{x} = f(x)$ near $x = 1$, and deduce the stability/instability of the rest point.

$\dot{x} = f(x) \approx f(1) + f'(1)(x-1)$ sufficiently close to $x=1$.

$\Rightarrow |f(x) - f(1)| \leq |f'(1)||x-1|$

$\Rightarrow \dot{\xi}_{error} \leq f'(1)\xi$ where $\xi = (x-1)$ is a perturbation about $x=1$.

$\therefore$

Stable if $f'(x) < 0$ b/c $\xi \to 0$ as $t \to \infty$

u.s. if $f'(x) > 0$ b/c $\xi \to \infty$ " "

Now in this ex.

other terms will all be zero

$f'(1) = (x-1)' \, | \, 2(x+3)(x+1) \times (2x-1)(x-2) \, |_{x=1}$

$= x \cdot 2(x+3)(x+1)\times(2x-1)(x-2) \, |_{x=1}$

$= 2(4)(2)(1)(1)(-1)$

$= -16$

$\Rightarrow f'(1) < 0 \Rightarrow \xi = x_0 e^{-16t} \xrightarrow{t \to \infty} 0$

Big idea.

Using LSA $= +5$

explaining it $= +5$

using $f(x) \approx f(1)(x-1)$ in almost any way....

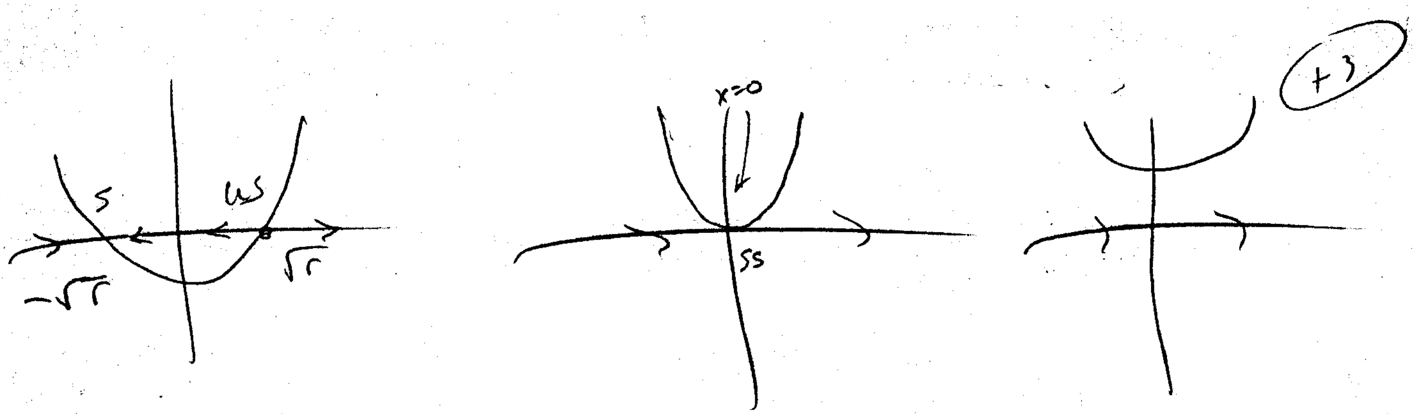
5. (16 pts) Draw the phase portrait for each $r = constant$ and construct the bifurcation diagram for each of the following ODE's. Justify and label (saddle-node, transcritical, pitchfork).

(8)

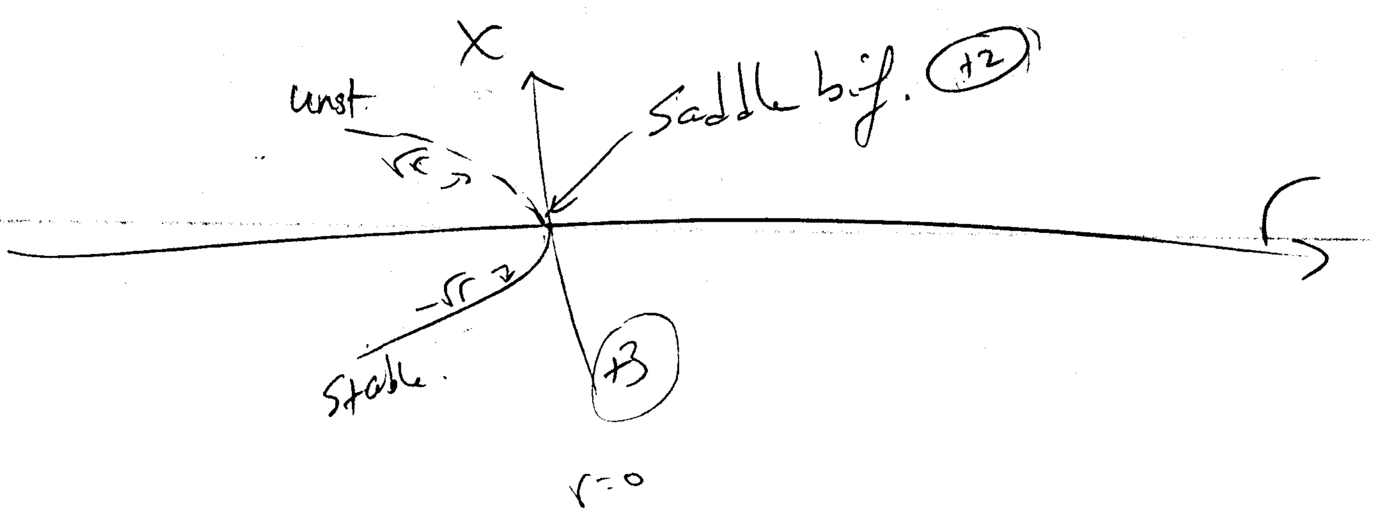
(a) $\dot{x} = r + x^2$

$r + x^2 = 0 \Rightarrow x^2 = -r$

$\Rightarrow x = \pm\sqrt{-r}$



$r < 0$ $r = 0$ $r > 0$



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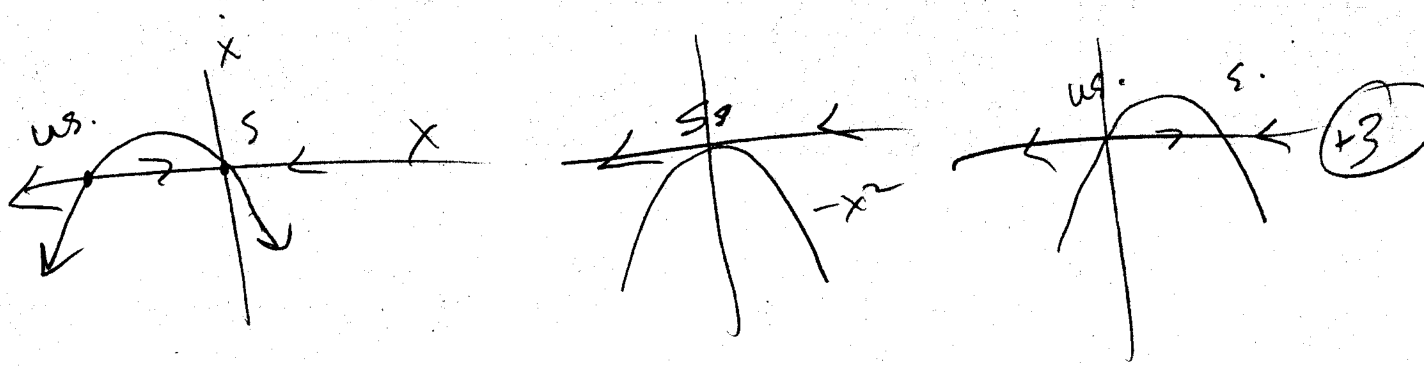
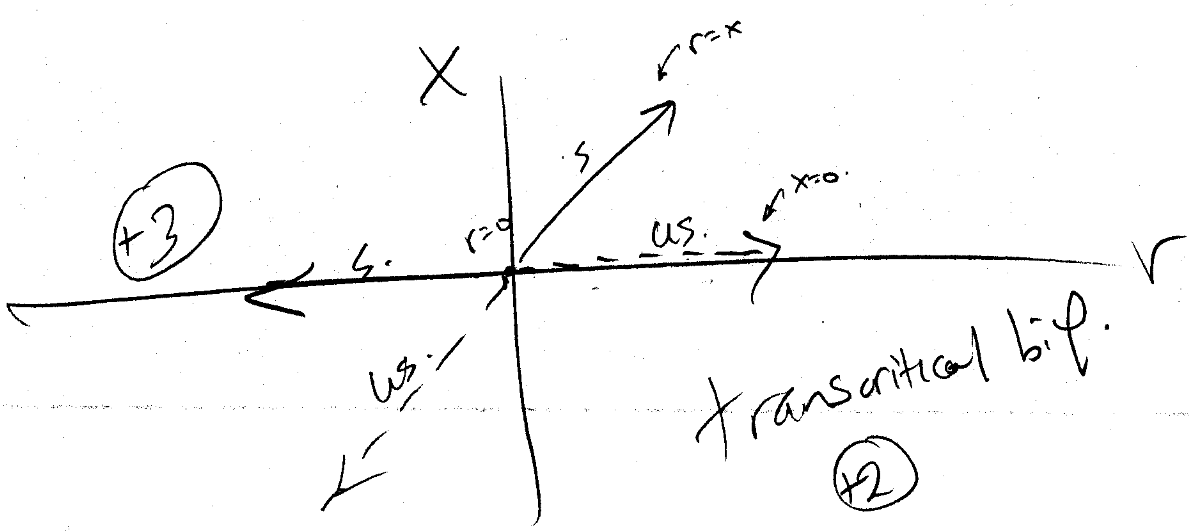
(b) $\dot{x} = rx - x^2$

$\dot{x} = 0 \Rightarrow rx - x^2 = 0 \Rightarrow x(r - x) = 0$

$x = 0$ or $x = r$



$r < 0$ $\qquad$ $r = 0$ $\qquad$ $r > 0$



transcritical bif.

+2

8

$* \ \gamma \geq 0$ is o.k.
"physical solution"

6. (15 pts) Draw the phase portrait for each $\gamma = \textit{constant}$ and construct the bifurcation diagram for the the equation

$$\dot{\phi} = -\sin\phi + \gamma \sin\phi \cos\phi,$$

(the overdamped bead on a rotating hoop.)

$$0 = -\sin\phi + \gamma \sin\phi\cos\phi$$

$$= \sin\phi\,(\gamma\cos\phi - 1) \qquad \text{take } \phi \in [-\pi, \pi]$$

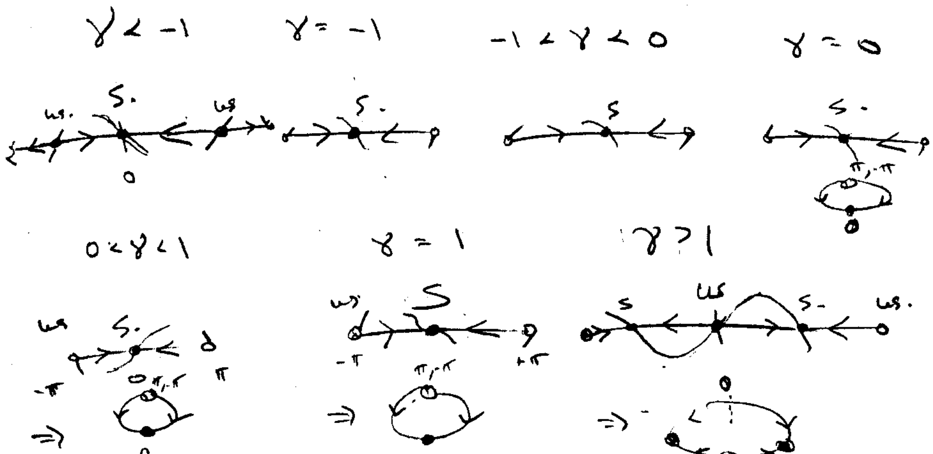
$$\Rightarrow \sin\phi = 0 \quad \text{or} \quad \cos\phi = \frac{1}{\gamma}$$

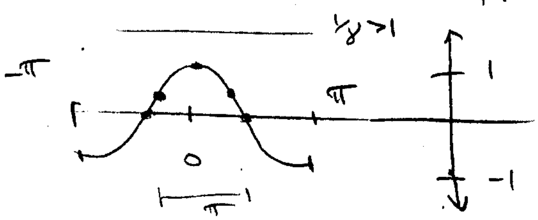
$\phi = 0, \pm\pi$ ↑ same point.

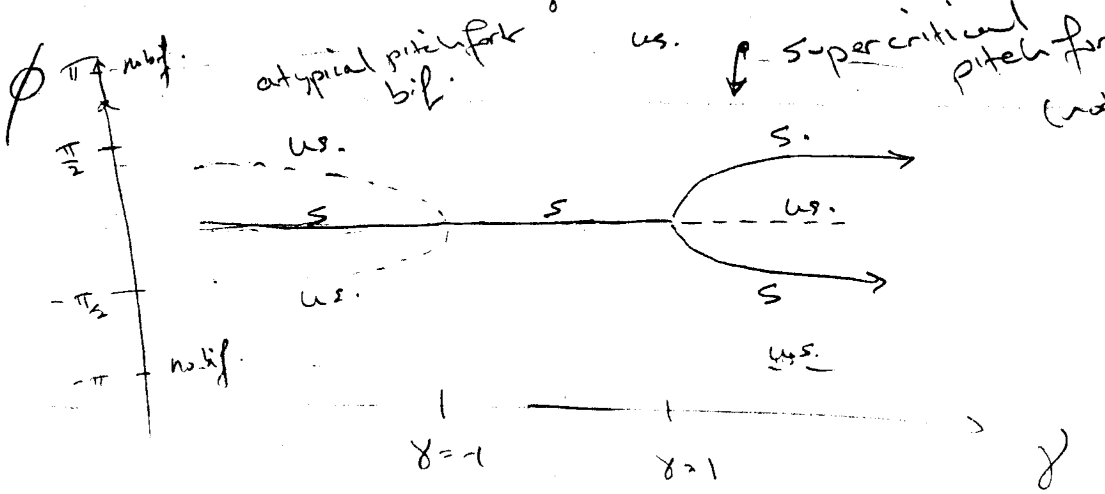
$\Rightarrow |\gamma| < 1$ + none

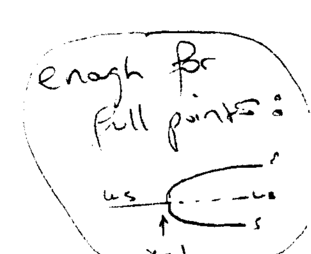
$|\gamma| = 1$ + 1 f.p.

$|\gamma| > 1$ + 2 f.p.









(with phase portraits for $\gamma \geq 0$)