HW 1

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1.4.10: a) True

b) False, counterexample:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 3 \\ 0 & 4 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 1 & -1 \end{pmatrix}$$

$$AB = \begin{pmatrix} -1 & -4 \\ 6 & 4 \\ 6 & 14 \end{pmatrix}$$

c) True

d) False: $(AB)^2 = ABAB \neq A^2B^2$

example: any 2 non-commuting matrices (s.t. $AB \neq BA$)

1.4.12: A, B - lower-triangular

$$(AB)_{ij} = \sum_k A_{ik} B_{kj},$$

$$A_{ik} = 0 \text{ for } i<k, \quad B_{kj} = 0 \text{ for } k<j$$

$$\Rightarrow (AB)_{ij} = \sum_{j \le k \le i} A_{ik} B_{kj} = 0 \text{ for } i<j$$

HW1

1.4.13: a) $A^2 = -I$, A - real

let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $\cancel{A^2 = \left(\begin{matrix} a_{11}^2 + a_{12}a_{21} + a_{11}a_{12} \end{matrix}\right)}$

$$A^2 = \begin{pmatrix} a_{11}^2 + a_{12}a_{21} & (a_{11}+a_{22})a_{12} \\ (a_{11}+a_{22})a_{21} & a_{22}^2 + a_{12}a_{21} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow$ let $a_{22} = -a_{11}$, $a_{11}^2 + a_{12}a_{21} = -1$

ex.: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = A$

b) $B^2 = 0$, $B \neq 0$: ex.: $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

c) $CD = -DC$, $CD \neq 0$

ex: $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$CD = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $DC = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -DC$

d) $EF = 0$, no entries of E or $F = 0$

ex.: $E = F = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

or $E = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, $F = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

HW 1

1.4.9: a) a_{11}, b) $l_{i1} = \frac{a_{i1}}{a_{11}}$,

c) $a_{ij} - l_{i1} a_{1j} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}$

d) $a_{22} - \frac{a_{21}}{a_{11}} a_{12}$

1.4.24:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{pmatrix}, \quad E_{32}E_{31}E_{21}A = U$$

$$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}, \quad E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M = E_{32}E_{31}E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{pmatrix} = L^{-1}, \quad A = LU$$

1.4.42: a) True, b) False: if A is $n \times m$ matrix, B – $m \times n$ matrix, then AB – $n \times n$ and BA – $m \times m$ matrix

c) True

d) False, B^{-1} may not exist

ex.: $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, $AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = B$

HW1

1.5.5:

$$Ax = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} = c$$

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}, \quad E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = LU, \quad L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$U = L^{-1}A = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{pmatrix}, \quad c' = L^{-1}c = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$Ux = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

1.5.11:

$$LUx = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = b$$

1) $Ly = b$: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

$y_1 = 2$
$y_2 = -2$
$y_3 = 0$

2) $Ux = y$: $\begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$

$w = 0, \quad v = -2, \quad 2u - 8 = 2 \Rightarrow u = 5$

$$x = (5, -2, 0)^T$$

HW 1

1.5.12: $A = UL$, $E_{12}E_{13}E_{23}A = L$

or $A E_{21} E_{31} E_{32} = U$,

E_{ij} - corresponding elimination matrices

$A = \tilde{L}\tilde{U}$: $E_{32}E_{31}E_{21}A = \tilde{U}$

In general, $\tilde{L} \neq L$, $\tilde{U} \neq U$

1.5.13: $Ax = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ -2 & -8 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 32 \end{pmatrix} = c$

$E_{21} = I$, $E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $E_{32} = I \Rightarrow L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

$U = L^{-1}A = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{pmatrix}$, $c' = L^{-1}c = \begin{pmatrix} -2 \\ 1 \\ 28 \end{pmatrix}$

$\Rightarrow (u, v, w) = (2, -3, 4)$

$Ax = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = c \rightarrow$ permute rows 1,2

$\rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$U = L^{-1}A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $c' = L^{-1}c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow (u, v, w) = (+1, -1, 1)$

HW1

1.6.1: $A_1 = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$, $A_3 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$A_1^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix}$, $A_2^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{pmatrix}$, $A_3 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

1.6.2: a) $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

б) If P has 1 in (i,k) place then P^T has 1 in (k,i) place

$$(PP^T)_{ij} = \sum_k P_{ik} P^T_{kj} = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases}$$

1.6.4: a) $AB = AC$, A - invertible

$\Rightarrow A^{-1}AB = B = A^{-1}AC = C$

б) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, let $B = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$

$\Rightarrow AB = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$, $AC = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$

$AB = AC$, $B \neq C$

HW1

1.6.5: $A^2B = BA^2 = I$

$A^2B = A\cdot(AB)$, ~~$BA^2 = (BA)A$~~

$A\cdot(AB) = I \Rightarrow AB = A^{-1}$

1.6.10:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}, \quad A_1^{-1} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{pmatrix}, \quad A_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{pmatrix}$$

$$A_3 = \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hline 0 & 0 & a & b \\ 0 & 0 & c & d \end{array}\right),$$ if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \cdot \frac{1}{ad-bc}$$

$$\Rightarrow A_3^{-1} = \left(\begin{array}{cc|cc} d & -b & 0 & 0 \\ -c & a & 0 & 0 \\ \hline 0 & 0 & d & -b \\ 0 & 0 & -c & a \end{array}\right) \cdot \frac{1}{ad-bc}$$ – diagonal block-matrix