HW 2

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1.4.1:

$$\underbrace{\begin{pmatrix} -2 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & -1 & -2 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{4} \end{pmatrix}}_{D} \cdot \underbrace{\begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{L^T}$$

$$\det A = \det D = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} = 5.$$

1.4.3:

$$A_0 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \text{ then } A_0 \begin{pmatrix} c \\ c \\ c \\ c \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

first row of A_0: $u_0 + 2u_1 - u_2 = h^2 f(h)$, $u_0 = u_1 \Rightarrow$

$\Rightarrow \quad u_1 - u_2 = h^2 f(h)$

fifth row of A_0: $-u_4 + 2u_5 - u_6 = h^2 f(5h)$, $u_6 = u_5 \Rightarrow$

$\Rightarrow \quad -u_4 + u_5 = h^2 f(5h)$

the other rows are not modified by boundary conditions

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1.7.4:

$u_0 = u_4 = 0 \quad , \quad h = \frac{1}{4} \; :$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + h^2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = h^2 \begin{pmatrix} h \\ 2h \\ 3h \end{pmatrix}$$

or

$$\begin{pmatrix} +2+h^2 & -1 & 0 \\ -1 & 2+h^2 & 0 \\ 0 & -1 & 2+h^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = h^2 \begin{pmatrix} h \\ 2h \\ 3h \end{pmatrix} ,$$

numerically

$$\begin{pmatrix} \frac{33}{16} & -1 & 0 \\ -1 & \frac{33}{16} & 0 \\ 0 & -1 & \frac{33}{16} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{64} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

1.7.5:

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{\pi^2}{4} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

From 1.7.1: $L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$

$$U \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = L^{-1} \begin{pmatrix} \pi^2/4 \\ 0 \\ -\pi^2/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} \pi^2/4 \\ 0 \\ -\frac{\pi^2}{4} \end{pmatrix} = \begin{pmatrix} \pi^2/4 \\ \pi^2/8 \\ -\frac{\pi^2}{6} \end{pmatrix}$$

$\Rightarrow u_3 = -\frac{\pi^2}{8} , \; u_2 = 0 , \; u_1 = \frac{\pi^2}{8}$

$u_{exact} = \sin 2\pi x \; : \; u(\tfrac{1}{4}) = 1 , \; u(\tfrac{1}{2}) = 0 , \; u(\tfrac{3}{4}) = -1 , \; \frac{|\Delta u_1|}{|u(\frac{1}{4})|} \approx \frac{\pi^2}{8} - 1 \approx 0.23$

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1.7.5 (continued): $u_{exact}(x) = \sin(2\pi x)$

$$u\left(\tfrac{1}{4}\right) = 1, \ u\left(\tfrac{1}{2}\right) = 0, \ u\left(\tfrac{3}{4}\right) = -1$$

$$\Rightarrow \frac{|\Delta u_1|}{|u(\frac{1}{4})|} = \frac{\pi^2}{8} - 1 \approx 0.23, \quad \frac{|\Delta u_3|}{|u(\frac{3}{4})|} = \frac{\pi^2}{8} - 1 \approx 0.23$$

2.1.2: a) subspace: $c_1(0, b_2, b_3) + c_2(0, b_2', b_3') =$
$= (0, c_1 b_2 + c_2 b_2', c_1 b_3 + c_2 b_3')$

b) <u>not</u> a subspace, c) <u>not</u> a subspace

d) $c_1(1,1,0) + c_2(2,0,1) = (c_1 + 2c_2, c_1, c_2)$ – a subspace

e) (b_1, b_2, b_3): $b_3 - b_2 + 3b_1 = 0$ – a subspace

2.1.4: Symmetric <u>and</u> lower-triangular = diagonal –
– diagonal matrices are the largest subspace

all symmetric + all lower-triangular = all matrices
$\Rightarrow$ all 3×3 matrices – smallest subspace
containing ~~both~~ all sym. and all l-tr. matrices

2.1.7: a) <u>no</u>, can't multiply by negative number

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2.1.7 (continued): b) yes

c) $x_{j+1} \leq x_j$ for each j — no, can't multiply by negative number

d) yes, e) yes, linear combination of arithmetic progressions is arithmetic progression

f) no, linear combination of geometric progressions in general is not geometric

2.1.11: a) a subspace of matrices of the form $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ contains $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ but not $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

b) $I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A - B \Rightarrow$ yes

c) matrices of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, b, c – any numbers

2.1.14: a) $\mathbb{R}^2$: 1) Lines, 2) $\mathbb{R}^2$ itself, 3) point $(0,0)$

b) $\mathbb{R}^4$: 1) 3-dimensional hyperplanes, 2) 2-dimensional planes, 3) 1-dimensional lines, 4) $\mathbb{R}^4$ itself, 5) point $(0,0,0,0)$

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2.1.22: a) $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ —

— solvable for $(b_1, b_2, b_3) = c\cdot(1,2,-1)$ only.

b) $\begin{pmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ — solvable for

$(b_1, b_2, b_3) = (b_1, b_2, -b_1)$ only.

2.1.25: ... unless b is in the column space of A. In the last case $Ax = b$ is solvable.

2.2.1: $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ — no solutions

$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 + 2x_2 + x_3 = 0$

All solutions have form $(x_1, x_2, -x_1 - 2x_2)$

or $x_n = x_1\cdot(1, 0, -1) + x_2\cdot(0, 1, -2)$

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2.2.2: $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \xrightarrow{\text{row } 3 - \text{row } 1} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

the echelon form

2 pivots

$\Rightarrow \operatorname{rank}(A) = 2$

$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \Rightarrow$ x_1, x_2 - pivot variables
x_3, x_4 - free variables

$$\begin{cases} x_1 + 2x_2 + x_4 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_3 - x_4 \\ x_2 = -x_3 \end{cases}$$

$\Rightarrow$ all solutions $x = x_3 \cdot \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$Б = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{\text{row } 2 - 4 \cdot \text{row } 1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{\text{row } 3 - 7 \cdot \text{row } 1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \rightarrow$$

$$\xrightarrow{\text{row } 3 - 2 \cdot \text{row } 2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$ - the echelon form,

2 pivots $\Rightarrow \operatorname{rank}(Б) = 2$

$Б \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow$ x_1, x_2 - pivot var-s,
x_3 - free variable

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2.2.2 (continued): $\begin{cases} x_1+2x_2+3x_3=0 \\ -3x_2-6x_3=0 \end{cases} \Rightarrow \begin{cases} x_1=x_3 \\ x_2=-2x_3 \end{cases}$

$\Rightarrow$ all solutions are $x = x_3 \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

2.2.3: $A = \begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = b$

echelon form: $\begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, 1 pivot, x_2 - pivot var.,

x_1, x_3, x_4 - free var-s, $b_2 = 2b_1$ to have solutions

Then $x_2 + 3x_4 = b_1$, $x_2 = b_1 - 3x_4$

Complete solution: $x = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ b_1 \\ 0 \\ 0 \end{pmatrix}$

($(0, b_1, 0, 0)$ - a special solution)

2.2.5: $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ pivots

$\Rightarrow w = 2$, v-free, $u = -3 - 2v$

$$x = x_p + x_n = \underbrace{\begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}}_{x_p} + \underbrace{v \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}_{x_n}$$

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2.2.5 (continued): $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \to$

$\to \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad 0 \neq 2 \Rightarrow$ <u>no</u> solution

2.3.2: v_1, v_2, v_3 are linearly independent

$v_4 = v_2 - v_1,\ v_5 = v_3 - v_1,\ v_6 = v_3 - v_2$

$\Rightarrow$ 3 independent vectors among

$v_1, v_2, v_3, v_4, v_5, v_6$. 3 is the <u>dimension</u> of the space spanned by the v's.

2.3.7: $v_1 = w_2 - w_3,\ v_2 = w_1 - w_3,\ v_3 = w_1 - w_2$

$v_1 - v_2 + v_3 = 0 \quad \Rightarrow v_1, v_2, v_3$ - dependent

2.3.11: a) $(-1,-1,1) = -1\cdot(1,1,-1) \Rightarrow$ <u>a line</u>

b) $(0,1,1), (1,1,0), (0,0,0)$ - 2 indep. vectors + 0 vector

$\Rightarrow$ <u>a plane</u>

c) 2 3-dimensional column vectors are independent

$\Rightarrow$ <u>a plane</u> in $\mathbb{R}^3$

d) all $\mathbb{R}^3$

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2.3.17: If elimination gives one or more zero rows, the rows of A are linearly dependent.

2.3.20: a) vectors of form $(c,c,c,c) \Rightarrow$
$\Rightarrow$ 1 vector $(1,1,1,1)$ is a basis

b) (a_1, a_2, a_3, a_4) s.t. $a_1 + a_2 + a_3 + a_4 = 0$
$\Rightarrow a_4 = -a_1 - a_2 - a_3$, 3 independent vectors
A basis: $(1,0,0,-1), (0,1,0,-1), (0,0,1,-1)$

c) vectors (a_1, a_2, a_3, a_4) s.t.

$$\begin{cases} 1\cdot a_1 + 1\cdot a_2 + 0\cdot a_3 + 0\cdot a_4 = 0 \\ 1\cdot a_1 + 0\cdot a_2 + 1\cdot a_3 + 1\cdot a_4 = 0 \end{cases} \Rightarrow \begin{cases} a_2 = -a_1 \\ a_4 = -a_1 - a_3 \end{cases}$$

$\Rightarrow$ 2 independent components (2 indep. vectors in basis), a basis: $(1,-1,0,-1), (0,0,1,-1)$

d) $U = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$, Column space in $\mathbb{R}^2$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ - a basis

Nullspace in $\mathbb{R}^5$: $\begin{cases} x_1 + x_3 + x_5 = 0 \\ x_2 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_5 = -x_1 - x_3 \\ x_4 = -x_2 \end{cases}$

$\Rightarrow$ a basis of 3 independent vectors: $(1,0,0,0,-1), (0,1,0,-1,0), (0,0,1,0,-1)$

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2.3.26: a) True (add vector $\perp S$)

b) False: basis vectors for $\mathbb{R}^6$ may not be in S.

2.3.34: V, W - 3 dim. subspaces of $\mathbb{R}^5$

Let v_1, v_2, v_3 be a basis in V; w_1, w_2, w_3 be a basis in W. 6 vectors $v_1, v_2, v_3, w_1, w_2, w_3$ in $\mathbb{R}^5$ cannot be independent. Let, say,

$w_3 = a v_1 + b v_2 + c v_3 + d w_1 + e w_2$, then

$a v_1 + b v_2 + c v_3 = w_3 - d w_1 - e w_2$ is the common vector in V and W (it is non-zero).

2.3.35: a) False, may be no solution

b) True, 7 vectors in $\mathbb{R}^5$ are always dependent.