HW 3   by TA Igor Rumanov

2.4.2:

$A = \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{pmatrix}$:

1) Column space $C(A)$: $\dim C(A) = 1$, a basis: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

2) Nullspace $N(A)$: $x_2 + 4x_3 = 0$, $4x_3 = -x_2$

$\dim N(A) = 4 - 1 = 3$, a basis: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

3) Row space $C(A^T)$: $\dim C(A^T) = 1 = \operatorname{rank}(A)$,

a basis: $(0, 1, 4, 0)$

4) Left nullspace $N(A^T)$: $x_1 + 2x_2 = 0$, $2x_2 = -x_1$

$\dim N(A^T) = 2 - 1 = 1$, a basis: $(2, -1)$

$U = \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$:

1) $C(A)$: $\dim C(A) = 1$, basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2) $N(A)$: $x_2 + 4x_3 = 0$, $\dim N(A) = 4 - 1 = 3$, $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ - basis

3) $C(A^T)$: $\dim C(A^T) = 1$, basis $(0, 1, 4, 0)$

4) $N(A^T)$: $\dim N(A^T) = 2 - 1 = 1$, basis $(0, 1)$

2.4.3:

$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$

1) $\dim C(A) = \operatorname{rank}(A) = 2$, a basis: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

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2.4.3 (continued): 2) $\dim N(A) = 4 - 2 = 2$,

a basis: $\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

3) $\dim C(A^T) = 2$, a basis: $(1,2,0,1), (0,1,1,0)$

4) $\dim N(A^T) = 3 - 2 = 1$, a basis: $(1,0,-1)$

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1) $\dim C(U) = 2$; $(1,0,0), (0,1,0)$ - a basis

2) $\dim N(U) = 2$; $(0,-1,1,0), (-1,0,0,1)$ - a basis

3) $\dim C(U^T) = 2$; a basis: $(1,2,0,1), (0,1,1,0)$

4) $\dim N(U^T) = 1$; a basis: $(0,0,1)$

2.4.6: a) $AA^{-1} = A^{-1}A = I \Rightarrow m = n = r$

b) $Ax = b$ has infinitely many solutions for every b when $n > m = r$

2.4.7: Row space and Nullspace are orthogonal so have no common vector

2.4.16: $AB = I \Rightarrow A^T AB = A^T \not\Rightarrow$

$\not\Rightarrow B = (A^TA)^{-1}A^T$, (A^TA) may be not invertible

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2.4.27: a) No solution $\Rightarrow r < m$
(also $r \le n$ always)

b) $A^T y = 0$: $\dim N(A^T) = m - r > 0$ by a)
$\Rightarrow$ there is a non-zero vector y in $N(A^T)$
$\Rightarrow$ $A^T y$ has a nonzero solution

2.4.29: $A = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

a basis of $C(A)$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

a basis of $N(A)$: $(0, 1, -2, 1)$

a basis of $C(A^T)$: $(1,2,3,4), (0,1,2,3), (0,0,1,2)$

$N(A^T) = (0, 0, 0)$

2.6.1: Rotation: $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix}$,

Projection: $P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \boxed{P_x R = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}}$

$P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\boxed{P_y P_x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$.

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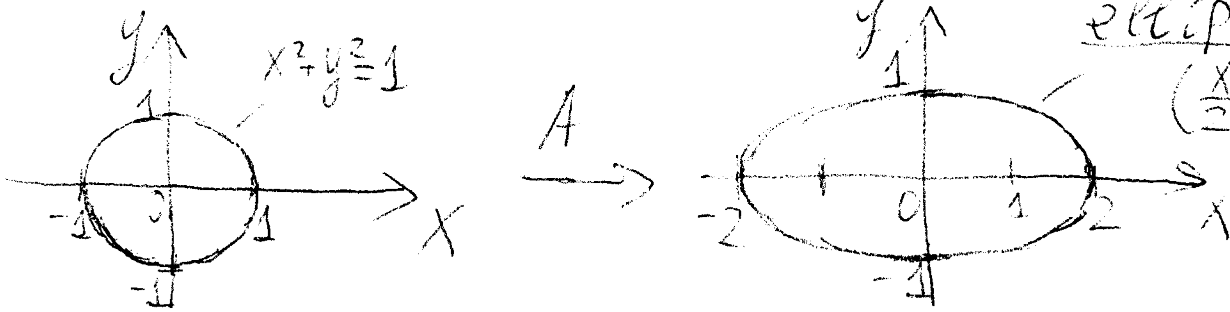
2.6.3: $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y \end{pmatrix}$

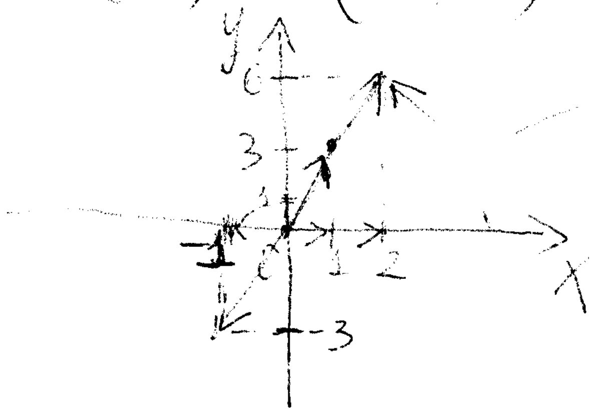


ellipse
$\left(\frac{x}{2}\right)^2 + y^2 = 1$

2.6.4: $z = \frac{x+y}{2} \Rightarrow Az = A\left(\frac{x+y}{2}\right) = \frac{Ax + Ay}{2}$

2.6.5: $A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$: $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $A\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, $A\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$



x-axis turns to $y = 3x$

2.6.7: $\frac{d}{dt}\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2t \\ 3t^2 \end{pmatrix} = A\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$

$\Rightarrow A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$, $\frac{d^2}{dt^2}\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 6t \end{pmatrix} = B\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$

$\Rightarrow \boxed{B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{pmatrix}}$, $C(B) = c_1\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $N(B) = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

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2.6.7 (continued): $\cancel{N(B) + C(B) = c_1\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}$

2-nd derivatives of linear functions $= 0$,
2-nd derivatives of cubic polynomials are linear.

2.6.8: $(2+3t)(a_0 + a_1 t + a_2 t^2 + a_3 t^3) =$

$$= 2a_0 + (3a_0 + 2a_1)t + (3a_1 + 2a_2)t^2 + (3a_2 + 2a_3)t^3 + 3a_3 t^4$$

$$\Rightarrow A: \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 2a_0 \\ 3a_0 + 2a_1 \\ 3a_1 + 2a_2 \\ 3a_2 + 2a_3 \\ 3a_3 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

2.6.7: $\frac{d^2}{dt^2}(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = 2a_2 + 6a_3 t$

$$\Rightarrow A: \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 2a_2 \\ 6a_3 \\ 0 \\ 0 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow C(A) = N(A) = c_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

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2.6.9: $\frac{d^2u}{dt^2} = u \quad u(t) = c_1 e^t + c_2 e^{-t}$

e^t and e^{-t} - 2 independent solutions giving a basis.

2.6.10: $u(0) = x$, $\frac{du}{dt}(0) = y$,

$$u(t) = c_1 e^t + c_2 e^{-t} \Rightarrow \begin{cases} c_1 + c_2 = x \\ c_1 - c_2 = y \end{cases}$$
$$\frac{du}{dt}(t) = c_1 e^t - c_2 e^{-t} \Rightarrow$$

$$\Rightarrow c_1 = \frac{x+y}{2}, \quad c_2 = \frac{x-y}{2}, \quad u(t) = \frac{(x+y)}{2}e^t + \frac{(x-y)}{2}e^{-t}$$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+y}{2} \\ \frac{x-y}{2} \end{pmatrix} \Rightarrow A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

2.6.14: $T^2 = TT$, let v_1, v_2 be 2 vectors, a, b - numbers:

$$T(av_1 + bv_2) = a \cdot Tv_1 + b \cdot Tv_2 \quad (T\text{-linear})$$

$$\Rightarrow T^2(av_1 + bv_2) = T(a \cdot Tv_1 + b \cdot Tv_2) = a \cdot T^2 v_1 + b \cdot T^2 v_2$$

$\Rightarrow T^2$ is also linear.

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2.6.16: $A\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}x_2\\x_3\\x_4\\x_1\end{pmatrix}$

$$\Rightarrow A=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\Rightarrow A^2=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},$$

$$A^3=\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}=A^T=A^{-1}, \quad A^4=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=I$$

A^2 moves every entry 2 positions

2.6.22: $v=(v_1,v_2)$

a) $T(v)=(v_2,v_1) \Rightarrow T=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ - linear

b) $T(v)=(v_1,v_1) \Rightarrow T=\begin{pmatrix}1&0\\1&0\end{pmatrix}$ - linear

c) $T(v)=(0,v_1) \Rightarrow T=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ - linear

d) $T(v)=(0,1)$ $\quad T(cv)=T(v)$ - <u>not</u> linear

2.6.50: Let $A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$, $x=\begin{pmatrix}x_1\\x_2\end{pmatrix}$

$$\Rightarrow Ax=\begin{pmatrix}a_{11}x_1+a_{12}x_2\\a_{21}x_1+a_{22}x_2\end{pmatrix}$$

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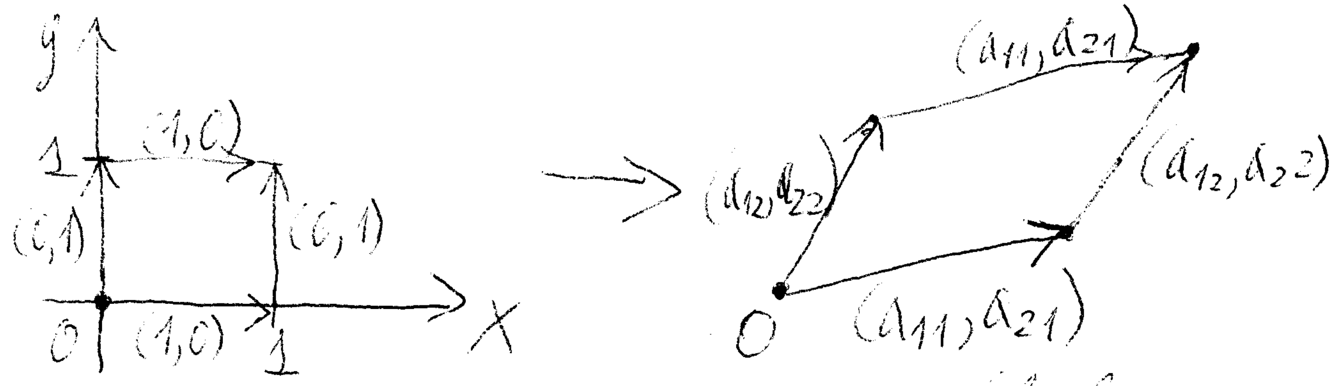
2.6.50 (continued): $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}$, $A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}$

a) $\Rightarrow$



a <u>parallelogram</u> on columns of A — on vectors (a_{11}, a_{21}) and (a_{12}, a_{22})

b) <u>Square</u>: 1) $(a_{12}, a_{22}) \perp (a_{11}, a_{21})$, i.e.

$$a_{11}a_{12} + a_{21}a_{22} = 0$$

2) $\|(a_{11}, a_{21})\| = \|(a_{12}, a_{22})\|$ or

$$a_{11}^2 + a_{21}^2 = a_{12}^2 + a_{22}^2$$

<u>orthogonal matrices</u>

<u>All matrices A satisfying 1) and 2)</u>

c) <u>Line</u>: All A s.t. $(a_{12}, a_{22}) = c\cdot(a_{11}, a_{21})$,
i.e. <u>columns are linearly dependent</u>

d) Area$(Ax) = 1$: $\|(a_{11}, a_{21}) \times (a_{12}, a_{22})\| = 1$

where $(a_{11}, a_{21}) \times (a_{12}, a_{22}) = a_{11}a_{22} - a_{21}a_{12}$,

so $\boxed{|\det A| = 1}$