# HW 3

**3.1.1:** $x = (1,4,0,2)$ , $y = (2,-2,1,3)$

$$\|x\| = \sqrt{1^2+4^2+0^2+2^2} = \sqrt{21}$$

$$\|y\| = \sqrt{2^2+2^2+1^2+3^2} = \sqrt{18} = 3\sqrt{2}$$

$$x^T y = 1\cdot 2 + 4\cdot(-2) + 0\cdot 1 + 2\cdot 3 = 0$$

**3.1.4:** $BB^{-1} = I$ , $(BB^{-1})_{ij} =$

$= (i\text{-th row of } B)\cdot(j\text{-th column of } B^{-1}) = 0$
for $i \neq j$.

**3.1.8:** For any $v \in V, w \in W$ : $v^T w = 0$

If $v \in V \cap W$ then $v \in V$ <u>and</u> $v \in W$

$\Rightarrow v^T v = 0 \Rightarrow v = 0$ ∎

**3.1.9:** $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\Rightarrow x_1 = x_2 = -x_3$

$\Rightarrow$ <u>line</u> $c \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is the orthogonal complement of the plane spanned by $(1,1,2)$ and $(1,2,3)$.

# HW 3

**3.1.11:** 1) $Ax = b$ or 2) $A^T y = 0$, $y^T b \neq 0$

Suppose 1) and 2) both have solutions

$$\Rightarrow y^T b = y^T A x = (A^T y)^T x = 0 \cdot x = 0 -$$

$-$ a contradiction

**3.1.15:** ?: $A \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$, $(1,2,1) \in C(A^T)$

But $(1,2,1)^T \cdot (1,-2,1) \neq 0 \Rightarrow$ no such $A$

**3.1.16:** $\begin{pmatrix} 1 & 4 & 4 & 1 \\ 2 & 9 & 8 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} 1 & 4 & 4 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = 0$

$$\Rightarrow \begin{cases} a_2 = 0, \quad a_1 + 4a_3 + a_4 = 0 \\ a_1 = -4a_3 - a_4 \end{cases}$$

$\Rightarrow$ all vectors of the form $a_3(-4,0,1,0) + a_4(-1,0,0,1)$

**3.1.26:** $AB = 0 \Rightarrow$ columns of $B \subset N(A)$ (nullspace),

rows of $A \subset N(B^T)$ - left nullspace of $B$

If $\text{rank}(A) = \text{rank}(B) = 2$, $A, B$ - $3 \times 3$ matrices

$\Rightarrow \dim C(B) = 2$, $\dim N(A) = 3 - 2 = 1 < \dim C(B)$, $B \subset N(A)$ -

$-$ a contradiction.

# HW 3

**3.1.31:** $A^T = A$

a) $\Rightarrow C(A) = C(A^T) \perp N(A)$

b) $Ax = 0$, $Az = 5z \Rightarrow x \in N(A)$ - nullspace of $A$,

$z \in C(A) = C(A^T) \Rightarrow x^T z = 0$

**3.1.32:** $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$

$\Rightarrow N(A) = c \cdot (-2, 1)$, $C(A) = c \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}$,

$C(A^T) = c \cdot (1, 2)$, $N(A^T) = c(-3, 1)$

$\Rightarrow N(B) = c \cdot (0, 1)$, $C(B) = c \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}$,

$C(B^T) = c \cdot (1, 0)$, $N(B^T) = c \cdot (-3, 1)$

**3.2.1:** a) $a = (\sqrt{y}, \sqrt{x})$, $b = (\sqrt{x}, \sqrt{y})$

$\Rightarrow \|a\| = \sqrt{y + x}$, $\|b\| = \sqrt{x + y}$, $a^T b = 2\sqrt{xy}$

$|a^T b| \leq \|a\| \cdot \|b\| \Rightarrow 2\sqrt{xy} \leq (x + y)$

$\Rightarrow \boxed{\frac{x+y}{2} \geq \sqrt{xy}}$

b) $\|x + y\| \leq \|x\| + \|y\| \Rightarrow \|x + y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$

$\|x + y\|^2 = (x+y)^T(x+y) = x^T x + x^T y + y^T x + y^T y = \|x\|^2 + \|y\|^2 + 2x^T y$

let $x^T y \geq 0 \Rightarrow |x^T y| \leq \|x\| \cdot \|y\|$

# HW 3

**3.2.2:** $p = \frac{a^T b}{a^T a} a = \frac{\|a\| \cdot \|b\| \cos\theta}{\|a\|^2} a =$

$= \|b\| \cdot \cos\theta \cdot \frac{a}{\|a\|} \Rightarrow \|p\| = \|b\| \cdot \cos\theta$

**3.2.3:** $p_1 = \frac{a^T b}{\|a\|^2} a = \frac{10}{3} a = \left(\frac{10}{3}, \frac{10}{3}, \frac{10}{3}\right)$

$p_2 = \frac{b^T a}{\|b\|^2} b = \frac{10}{36} b = \frac{5}{18} b = \left(\frac{5}{9}, \frac{10}{9}, \frac{10}{9}\right)$

**3.2.4:** $|a^T b| = \|a\| \cdot \|b\|$ iff $a = \alpha b$, where $\alpha$ is a real number

**3.2.5:** $a = (\underbrace{1, 1, \dots, 1}_{n})$, $e_k = (0, 0, \dots, 0, \underset{\uparrow k}{1}, 0, \dots, 0)$

$a^T e_k = 1$ for all $k = 1, 2, \dots, n$

$\cos\theta = \frac{a^T e_k}{\|a\| \cdot \|e_k\|} = \frac{1}{\|a\|} = \frac{1}{\sqrt{n}}$,

projection matrix $P = \frac{a a^T}{a^T a} = \begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$

($n \times n$ matrix with all entries $= \frac{1}{n}$)

**3.2.6:** Problem 3.2.1 gives

$|a_j| \cdot |b_j| \leq \frac{1}{2}(|a_j|^2 + |b_j|^2)$ for all $j$.

# HW 3

**3.2.10:** Not invertible, $\operatorname{rank}(P) < \operatorname{size}(P)$

**3.2.11:** $a = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $a^T a_\perp = 0$, let $a_\perp = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

orthogonal to $a$

a) $$P_1 = \frac{a a^T}{a^T a} = \frac{1}{10} \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$$

$$P_2 = \frac{a_\perp a_\perp^T}{a_\perp^T a_\perp} = \frac{1}{10} \begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix}$$

б) $$P_1 + P_2 = \frac{1}{10} \cdot \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = I$$ – sum of 2 complementary orthogonal projections

$$P_1 P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ – second projection is on the line orthogonal to the first projection.

**3.2.13:** $P = \frac{a a^T}{a^T a}$, $\operatorname{tr} P = 1$;

$$\operatorname{tr}(a a^T) = a^T a$$

# HW 3

**3.3.1:** $\begin{cases} 3x = 10 \\ 4x = 5 \end{cases}$ $\quad E^2 = (3x-10)^2 + (4x-5)^2$

$$\frac{dE^2}{dx} = 6(3x-10) + 8(4x-5)$$

$$\Rightarrow 50\hat{x} = 100 \Rightarrow \boxed{\hat{x} = 2}$$

$$E = (10 - 3\hat{x},\ 5 - 4\hat{x}) = (4, -3)$$

$$4 \cdot 3 - 3 \cdot 4 = 0 \quad \Rightarrow E \perp \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

**3.3.3:** $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad Ax = b$

$$E^2 = (1 - x_1)^2 + (1 - x_2)^2 + (0 - x_1 - x_2)^2$$

$$\hat{x} = (A^TA)^{-1}A^Tb, \quad A^TA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad A^Tb = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\hat{x} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$p = A\hat{x} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \quad b - p = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$(b - p) \perp C(A)$$

# HW3

**3.3.9:** a) $P = P^T P$

$\Rightarrow$ 1) $P^T = (P^T P)^T = P^T (P^T)^T = P^T P = P$,

2) ~~$P^2 = P^T P P^T P$, but $PP^T = P^T P P^T$~~

~~$\Rightarrow P^2 = PP^T P$~~

$\Rightarrow P = P^T P = P^2$

$\Rightarrow P$ - projection matrix

b) $P = 0$ projects onto $\{0\}$

**3.3.19:** $P_C = A(A^T A)^{-1} A^T$ - proj. onto column space of $A$

$P_R = A^T (AA^T)^{-1} A$ - projection onto row space of $A$ if rows of $A$ are independent

**3.3.27:** $a = \underbrace{(1, \dots, 1)}_{m}$, $b = (b_1, \dots, b_m)$

a) $a^T a = m$, $a^T b = b_1 + \dots + b_m$

$\Rightarrow \hat{x} = \frac{a^T b}{a^T a} = \frac{b_1 + \dots + b_m}{m}$ - the mean of $b$'s

b) $e = b - \hat{x} a$, $\|e\|^2 = \sum_{i=1}^{m} (b_i - \hat{x})^2$ - variance

c) $\hat{x} = 3 \Rightarrow e = b - \hat{x} a = (-2, -1, 3)$, $p = \hat{x} a = (3, 3, 3)$

# HW3

**3.3.27 (continued):** $p^T e = 3\cdot(-2)+3\cdot(-1)+3\cdot 3 = 0$

$\Rightarrow p \perp e$, projection matrix $P = \frac{aa^T}{a^Ta} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

**3.3.6:** $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$

$A^T = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix}$, $A^TA = \begin{pmatrix} 6 & -8 \\ -8 & 18 \end{pmatrix}$

$\Rightarrow (A^TA)^{-1} = \frac{1}{44}\cdot\begin{pmatrix} 18 & 8 \\ 8 & 6 \end{pmatrix}$

$P = A(A^TA)^{-1}A^T = A\cdot\frac{1}{44}\begin{pmatrix} 26 & 10 & -4 \\ 14 & 2 & 8 \end{pmatrix} =$

$= \frac{1}{44}\begin{pmatrix} 40 & 12 & 4 \\ 12 & 8 & -12 \\ 4 & -12 & 40 \end{pmatrix}$ - projection matrix,

$P = \frac{1}{11}\begin{pmatrix} 10 & 3 & 1 \\ 3 & 2 & -3 \\ 1 & -3 & 10 \end{pmatrix}$

$p = Pb = \frac{1}{11}\cdot\begin{pmatrix} 23 \\ -14 \\ 65 \end{pmatrix}$

$e = b - p = \begin{pmatrix} -\frac{12}{11} \\ \frac{36}{11} \\ \frac{12}{11} \end{pmatrix} = \frac{12}{11}\cdot\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \in N(A^T)$ - left null space