HW 4 by TA Igor Rumanov

3.4.2: $b = (0, 3, 0)$ $\quad a_1 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right),$

$a_2 = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$$p_1 = \frac{a_1^T \cdot b}{a_1^T \cdot a_1} a_1 = \frac{2}{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} \cdot \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \\ -2/3 \end{pmatrix}$$

$$p_2 = \frac{a_2^T \cdot b}{a_2^T \cdot a_2} a_2 = \frac{2}{1} \cdot \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 4/3 \\ 4/3 \end{pmatrix}$$

projection p onto plane of a_1, a_2:

$$p = p_1 + p_2 = \begin{pmatrix} 2/3 \\ 8/3 \\ 2/3 \end{pmatrix}$$

3.4.3: $a_3 = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$

$$\cancel{p_3 = 2 a_3 = (4/3, -2/3, 4/3)}$$

$$p_3 = -1 \cdot a_3 = (-2/3, 1/3, -2/3)$$

$P = a_1 \cdot a_1^T + a_2 a_2^T + a_3 a_3^T = I$ because the summands are projections onto three orthogonal directions ($p_1 + p_2 + p_3 = b$)

3.4.13: $a = (0,0,1)$, $b = (0,1,1)$, $c = (1,1,1)$

let $q_1 = a$, $q_2 = b - (q_1^T b) q_1 = (0,1,1) - (0,0,1) =$
$= (0,1,0)$

$q_3 = c - (q_1^T c) q_1 - (q_2^T c) q_2 = c - a - (0,1,0) = (1,0,0)$

$$A = \begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = QR = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} R =$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot R \qquad Q^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = Q,$$

$$R = Q^{-1} A = QA = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.4.27: subspace spanned by

$a_1 = (1,-1,0,0)$, $a_2 = (0,1,-1,0)$, $a_3 = (0,0,1,-1)$

$a_1^T a_1 = a_2^T a_2 = a_3^T a_3 = 2$

let $\boxed{q_1 = \frac{1}{(a_1^T a_1)^{1/2}} a_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right)}$, then

$q_2 = \frac{1}{\sqrt{2}} \left(a_2 - (q_1^T a_2) q_1\right) = \frac{1}{\sqrt{2}} \left((0,1,-1,0) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right)\right)$

$= \frac{1}{\sqrt{2}} \left(\frac{1}{2}, \frac{1}{2}, -1, 0\right)$

$q_3 = \frac{1}{\sqrt{2}} \left(a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2\right) = \frac{1}{\sqrt{2}} \left(a_3 - 0 + \frac{1}{\sqrt{2}} q_2\right) =$

5.1.1: $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$, $\operatorname{tr} A = 5$, $\det A = 6$

$\det(A - \lambda I) = 0$: $(1-\lambda)(4-\lambda) + 2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$ $\quad \lambda_{1,2} = 2 ; 3$ - eigenvalues

$A v_1 = \lambda_1 v_1$, $A v_2 = \lambda_2 v_2$ — v_1, v_2 - eigenvectors

$(A-2) v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, let $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(A-3) v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, let $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\operatorname{tr} A = \lambda_1 + \lambda_2 = 2+3 = 5$, $\det A = \lambda_1 \cdot \lambda_2 = 2 \cdot 3 = 6$

5.1.2: $\dfrac{du}{dt} = Au$, $u(0) = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$, $\lambda_1 = 2, \lambda_2 = 3$, $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$u(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

$u(0) = c_1 v_1 + c_2 v_2 = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

$c_2 = -c_1$ $\quad c_1 = 6$, $c_2 = -6$

$$\boxed{u(t) = 6 \cdot e^{2t} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (-6) e^{3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 6 \cdot \begin{pmatrix} e^{2t} - e^{3t} \\ -e^{2t} + 2e^{3t} \end{pmatrix}}$$

5.1.4:

$$\frac{du}{dt} = Pu = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} u \quad , \quad u(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Eigenvalues: $Pu = \lambda u \quad , \quad \det(P - \lambda I) = 0$

$$\left(\frac{1}{2} - \lambda\right)^2 - \frac{1}{4} = 0 \qquad \lambda^2 - \lambda = 0 \qquad \lambda_{1,2} = 0; 1$$

Eigenvectors: $Pv_1 = \lambda_1 v_1 = 0 \cdot v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad , \quad v_1 \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$Pv_2 = \lambda_2 v_2 = v_2 \qquad \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad v_2 \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$u(t) = c_1 v_1 + c_2 e^t \cdot v_2$$

$$u(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$c_1 = 1 \ , \ c_2 = 4 \qquad \boxed{u(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4e^t \begin{pmatrix} 1 \\ +1 \end{pmatrix}}$$

5.1.5:

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \det(A - \lambda I) = 0 : (3-\lambda)(1-\lambda)\lambda = 0$$

$$\lambda_{1,2,3} = 3; 1; 0$$

$$(A-3)v_1 = \vec{0} \qquad \begin{pmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_1 \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$(A-1)\,v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 \sim \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$Av_3 = \begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 \sim \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad \begin{aligned} \operatorname{tr} A &= 4 = \lambda_1 + \lambda_2 + \lambda_3 \\ \det A &= 0 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \end{aligned}$$

5.1.5 (continued) $\quad B = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

$\lambda^2 \cdot (2-\lambda) - 4(2-\lambda) = 0 \qquad (\lambda-2)^2 \cdot (\lambda+2) = 0$

$\lambda_{1,2,3} = 2; 2; -2 \qquad \operatorname{tr} B = 2 = \lambda_1 + \lambda_2 + \lambda_3, \ \det B = -8 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

$(B - 2 \cdot I) v_{1,2} = \begin{pmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix} v_{1,2} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$

can take $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$(B + 2 \cdot I) v_3 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad v_3 \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

5.1.8: $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$

Take $\lambda = 0 \Rightarrow \det A = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$

5.1.9: $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda) =$

$= (-\lambda)^n + (\lambda_1 + \lambda_2 + \ldots + \lambda_n)(-\lambda)^{n-1} + \ldots =$

$= \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{pmatrix} = (-\lambda)^n + (a_{11} + a_{22} + \ldots + a_{nn}) \cdot (-\lambda)^{n-1} + \ldots$

$\Rightarrow \operatorname{tr} A \equiv a_{11} + a_{22} + \ldots + a_{nn} = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$

5.1.24: $Ax = \lambda x$

a) multiply by A: $A \cdot Ax = A \cdot \lambda x = \lambda A x = \lambda^2 x$

$\Rightarrow A^2 x = \lambda^2 x \Rightarrow \lambda^2$ - eigenvalue of A^2

b) $\frac{A^{-1}}{\lambda} \cdot Ax = \frac{A^{-1}}{\lambda} \cdot \lambda x \Rightarrow \frac{1}{\lambda} x = A^{-1} x \Rightarrow$

$\Rightarrow \lambda^{-1}$ - eigenvalue of A^{-1}

c) $Ax + x = \lambda x + x \Rightarrow (A+I)x = (\lambda+1)x \Rightarrow$

$\Rightarrow (\lambda+1)$ is eigenvalue of $(A+I)$

5.1.6: let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; $(1-\lambda)^2 - 1 = 0$, $\lambda^2 - 2\lambda = 0$

$\lambda_{1,2} = 0; 2$

row 1 - row 2 $\rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow$ eigenvalues $\lambda_{1,2} = 0; 1$

A singular matrix remains singular