HW 5 by TA Igor Rumanov

5.4.1: $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to (\lambda+1)^2 - 1 = 0$

$\Rightarrow \lambda(\lambda+2) = 0 \quad \lambda_1 = 0, \lambda_2 = -2$

$A v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A + 2I) v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} : \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_2 \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$e^{At} = S e^{\Lambda t} S^{-1}, \quad \Lambda = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$\Rightarrow e^{\Lambda t} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2t} \end{pmatrix} \Rightarrow e^{At} = \frac{1}{2} \begin{pmatrix} 1+e^{-2t} & 1-e^{-2t} \\ 1-e^{-2t} & 1+e^{-2t} \end{pmatrix}$

5.4.4: P - projection matrix

$\Rightarrow P^2 = P \Rightarrow e^P = I + P + \frac{P^2}{2!} + \frac{P^3}{3!} + \dots =$

$= I + P\left(1 + \frac{1}{2!} + \frac{1}{3!} + \dots\right) = I + (e-1)\underline{P}$

$e - 1 \approx 1.718 \quad \Rightarrow e^P \approx I + 1.718 P$

5.4.5: a) $e^{A(t+T)} = S \cdot e^{\Lambda(t+T)} S^{-1} = S \cdot e^{\Lambda t} e^{\Lambda T} S^{-1} =$

$= S e^{\Lambda t} S^{-1} \cdot S e^{\Lambda T} S^{-1} = e^{At} \cdot e^{AT}$

b) $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} A^2 = A^3 = \dots = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ B^2 = B^3 = \dots = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{cases}$

5.4.5 c) (continued) $e^A = I + A + \frac{A^2}{2} + \dots = I + A$

$e^B = I + B$

$A + B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longrightarrow \lambda^2 + 1 = 0$

$\lambda_{\pm} = \pm i$, $v_+ = \begin{pmatrix} i \\ 1 \end{pmatrix}$, $v_- = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$A + B = S \Lambda S^{-1}$, $S = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $S^{-1} = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}$

$e^{A+B} = S \cdot e^{\Lambda} S^{-1} = \frac{1}{2} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^i & 0 \\ 0 & e^{-i} \end{pmatrix} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} =$

$= \begin{pmatrix} \frac{e^i + e^{-i}}{2} & -\frac{e^i - e^{-i}}{2i} \\ \frac{e^i - e^{-i}}{2i} & \frac{e^i + e^{-i}}{2} \end{pmatrix} = \begin{pmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{pmatrix}$

$e^A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $e^B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \Rightarrow e^A e^B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow e^{A+B} \neq e^A e^B$.

5.4.34: $B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow B^2 = 0$

$\Rightarrow e^{Bt} = I + Bt + \frac{B^2 t^2}{2} + \dots = I + Bt$

$e^{Bt} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$ $\qquad Be^{Bt} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

$\frac{d}{dt} e^{Bt} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = Be^{Bt}$.

5.4.6: $\quad y'' + y = 0: \quad \dfrac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -y \end{pmatrix}$

$$\frac{du}{dt} = Au, \quad u = \begin{pmatrix} y \\ y' \end{pmatrix}, \quad Au = A\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ -y \end{pmatrix}$$

$$\Rightarrow \begin{cases} a_{11} y + a_{12} y' = y' \\ a_{21} y + a_{22} y' = -y \end{cases} \Rightarrow \begin{matrix} a_{11} = 0, a_{12} = 1 \\ a_{21} = -1, a_{22} = 0, \end{matrix}$$

i.e. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longrightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda_\pm = \pm i$

$$v_+ = \begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad v_- = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad u(t) = c_1 v_+ e^{\lambda_+ t} + c_2 v_- e^{\lambda_- t}$$

$$u(0) = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad u(0) = c_1 v_+ + c_2 v_-$$

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad s.t. \quad u_2(t) = u_1'(t)$$

$$u_2(t) = c_1 e^{it} + c_2 e^{-it}, \quad u_1(t) = -ic_1 e^{it} + ic_2 e^{-it}$$

$$\Rightarrow u_1'(t) = c_1 e^{it} + c_2 e^{-it}$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -ic_1 + ic_2 \\ c_1 + c_2 \end{pmatrix} \Rightarrow \begin{cases} c_1 = i \\ c_2 = -i \end{cases} \qquad y(t) = u_1(t) = 1 \cdot e^{it} + e^{-it} \Rightarrow$$

$$\Rightarrow \boxed{y(t) = 2\cos t}$$

$\underline{5.2.2:}$ $\lambda_1 = 1, \lambda_2 = 4, \quad v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$A v_1 = \lambda_1 v_1, \quad A v_2 = \lambda_2 v_2$

$A = S \Lambda S^{-1}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad S = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$

$S = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \Rightarrow S^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

$\Rightarrow A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -5 & 18 \\ -3 & 10 \end{pmatrix}$

$\underline{5.2.4:}$ $U = \begin{pmatrix} 1 & x & x \\ 0 & 2 & x \\ 0 & 0 & 7 \end{pmatrix}, \quad \det(U - \lambda I) = 0:$

$(1-\lambda)(2-\lambda)(7-\lambda) = 0$

$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 7 \Rightarrow U = S \Lambda S^{-1}, \quad \boxed{\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}}$

$(\lambda_{1,2,3} \neq 0)$

$\lambda_1 \neq \lambda_2 \neq \lambda_3 \Rightarrow U$ can be diagonalized.

$\underline{5.2.7:}$ $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}, \quad (4-\lambda)(2-\lambda) - 3 = 0$

$\lambda^2 - 6\lambda + 5 = 0$

$\lambda_1 = 5, \lambda_2 = 1$

$v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(A - \lambda_1) v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}: \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(A - \lambda_2) v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}: \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow S = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$

5.2.7: (continued) $\Rightarrow S^{-1} = -\frac{1}{4}\cdot\begin{pmatrix} -1 & -1 \\ -1 & 3 \end{pmatrix}$

$A = S\cdot\Lambda\cdot S^{-1}$, $\Lambda = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$

$A^{100} = S\cdot\Lambda^{100}\cdot S^{-1} = -\frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5^{100} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 3 \end{pmatrix} =$

$= \frac{1}{4}\cdot\begin{pmatrix} 3\cdot 5^{100}+1 & 3\cdot 5^{100}-3 \\ 5^{100}-1 & 5^{100}+3 \end{pmatrix}$.

5.2.8: $A = u\,v^T$ (rank-1 matrix)

a) $Au = u\,v^T\cdot u = u\,(v^T u) = (v^T u)\,u \Rightarrow$

$\Rightarrow$ u - eigenvector with eigenvalue $\lambda = v^T u$

b) All other eigenvalues of A $= 0$,
because $\dim N(A) = \dim A - \operatorname{rank} A = \dim A - 1$

$\Rightarrow$ there are $(\dim A - 1)$ eigenvectors for $\lambda = 0$.

c) $\operatorname{tr} A = \sum_i \lambda_i = v^T\cdot u = \sum_i u_{ii}(v^T)_{ii}$

5.2.10: A with eigenvalues $\lambda = 1, 2, 4$

$\operatorname{tr} A^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 + 4 + 16 = 21$

$\det (A^{-1})^T = \det A^{-1} = \frac{1}{\det A} = \frac{1}{\lambda_1\lambda_2\lambda_3} = \frac{1}{8}$

<u>5.2.12:</u> the only eigenvectors of $A \sim x=(1,0,0)$

a) False (the repeated eigenvalue can $\neq 0$)

b) True , c) True

<u>5.2.20:</u> $S = I \Rightarrow S^{-1} = I \Rightarrow A = S\Lambda S^{-1} = \Lambda$,

i.e. A is diagonal

<u>5.2.25:</u> A with eigenvalues 2, 2, 5:

a) True

b) False

c) False (A may have 3 independent eigenvectors)

<u>5.2.38:</u> $A^2 = A \Rightarrow Ax_i = x_i$ for columns of A x_i $\Rightarrow$ <u>Column space</u> $C(A)$ contains eigenvectors with $\lambda = 1$

<u>Nullspace $N(A)$</u> contains eigenvectors with $\lambda = 0$

$\dim C(A) + \dim N(A) = \dim A$