5.3.4: $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$

$$\Rightarrow u_{k+1} = \begin{pmatrix} G_{k+2} \\ G_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} G_{k+1} + \frac{1}{2} G_k \\ G_{k+1} \end{pmatrix} = A u_k = A \begin{pmatrix} G_{k+1} \\ G_k \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

a) $\lambda(\lambda - \frac{1}{2}) - \frac{1}{2} = 0 \qquad \lambda_\pm = \frac{1}{4} \pm \frac{\sqrt{\frac{1}{4}+2}}{2}$

$$\Rightarrow \lambda_\pm = \frac{1}{4} \pm \frac{3}{4}, \quad \lambda_+ = 1, \quad \lambda_- = -\frac{1}{2}$$

$$(A - I) v_+ = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} v_+ \Rightarrow v_+ \sim \begin{pmatrix} 1 \\ +1 \end{pmatrix}$$

$$(A + \tfrac{1}{2} I) v_- = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix} v_- \Rightarrow v_- \sim \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

c) $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \Lambda^n = \begin{pmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^n \end{pmatrix}$

$$S = \begin{pmatrix} 1 & \frac{1}{2} \\ +1 & -1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 2/3 & 1/3 \\ +\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} S \Lambda^n S^{-1} = \cancel{\begin{pmatrix} 1 & \frac{1}{2} \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -2 & -2 \end{pmatrix}} =$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} = \qquad \cancel{= \begin{pmatrix} 1 & \frac{1}{2} \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}}$$

$$= \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}$$

5.3.4 (continued)

c) $G_0 = 0,\ G_1 = 1$; $\quad G_k \xrightarrow[k\to\infty]{} \frac{2}{3}$

$$\begin{pmatrix} G_{k+1} \\ G_k \end{pmatrix} = A^k \begin{pmatrix} G_1 \\ G_0 \end{pmatrix} \xrightarrow[k\to\infty]{} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

5.3.10:

$$\begin{cases} y_{k+1} = 0.8\, y_k + 0.3 \cdot z_k \\ z_{k+1} = 0.2\, y_k + 0.7 \cdot z_k \end{cases} \qquad \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$u_{k+1} = A u_k \ ; \ u_k = \begin{pmatrix} y_k \\ z_k \end{pmatrix}, \ A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

$$(0.8 - \lambda)(0.7 - \lambda) - 0.06 = 0$$

$$\lambda^2 - 1.5\lambda + 0.5 = 0 \qquad \lambda_{1,2} = \frac{3}{4} \pm \frac{\sqrt{9/4 - 2}}{2}$$

$$\lambda_{1,2} = \frac{3}{4} \pm \frac{1}{4} = 1 ; \frac{1}{2}$$

$$(A - I) v_1 = \begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$(A - \tfrac{1}{2} I) v_2 = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A^k = S \Lambda^k S^{-1} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (\frac{1}{2})^k \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \\ +2/5 & -\frac{3}{5} \end{pmatrix}$$

$$\lim_{k\to\infty} A^k = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{pmatrix}$$

$$\begin{pmatrix} y_k \\ z_k \end{pmatrix} = A^k \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 3(1 - (\frac{1}{2})^k) \\ 2 + 3/2^k \end{pmatrix} \Rightarrow \begin{matrix} y_\infty = 3 \\ z_\infty = 2 \end{matrix}$$

5.3.11:

$$A = \begin{pmatrix} 0.2 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{pmatrix}$$

a) column 1 + column 2 = 2·(column 3)

$\Rightarrow \lambda_1 = 0 \quad , \quad v_1 = (1, 1, -2)$

b) $(0.2-\lambda)^2(0.4-\lambda) + 0.048 + 0.048 - 0.12(0.2-\lambda)\cdot 2 -$

$- 0.16\cdot(0.4-\lambda) = 0$

$\Rightarrow -\lambda(\lambda^2 - 0.8\lambda + \overset{0.04+0.16-0.24-0.16}{\cancel{0.4 - 0.24 - 0.16}}) = 0$

$\Rightarrow \lambda_1 = 0 \quad , \quad \lambda_{2,3}^2 - 0.8\lambda_{2,3} - 0.2 = 0$

$\lambda_{2,3} = +0.4 \pm \frac{\sqrt{0.64+0.8}}{2} = +0.4 \pm \frac{1.2}{2} = -0.2, +1$

$\lambda_2 = 1 , \lambda_3 = -0.2 \qquad v_2 = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$(A - I)v_2 = (0,0,0) = (A + 0.2I)v_3$

c) $u_0 = (0, 10, 0) \quad , \quad (A-I)v_2 = (0,0,0)$

$u_K = A^K u_0 = c_1\lambda_1^K v_1 + c_2\lambda_2^K v_2 + c_3\lambda_3^K v_3$

$\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -0.2 \; : \; u_K = c_2 v_2 + c_3(-0.2)^K v_3$

$u_0 = c_2 v_2 + c_3 v_3 + c_1 v_1 , \quad \lim_{K\to\infty} u_K = c_2 v_2$

$\begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix} = \begin{pmatrix} 3c_2 + c_3 + c_1 \\ 3c_2 - c_3 + c_1 \\ 4c_2 - 2c_1 \end{pmatrix}, \quad \begin{matrix} c_1 = 2c_2 \\ c_3 = -5c_2 \\ c_2 = 1 \end{matrix} \qquad c_2 = 1$

$\Rightarrow \boxed{\lim_{K\to\infty} u_K = v_2 = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}}$

5.3.15: A - Markov $\iff$ each column adds to 1

$$\Rightarrow \sum_i (Ax)_i = \sum x_i$$

$$Ax = \lambda x,\ \lambda \neq 1 \Rightarrow \sum_i (Ax)_i = \sum x_i = \lambda \sum x_i$$

$$\lambda \neq 1 \Rightarrow \sum x_i = 0.$$

5.5.1: a) $3+4i$: $(3,4)$
$1-i$: $(1,-1)$ — points in the complex plane

b) $3+4i+1-i = 4+3i$, $3+4i-(1-i) = 2+5i$

c) $\overline{3+4i} = 3-4i$, $\overline{1-i} = 1+i$

$|3+4i| = \sqrt{3^2+4^2} = 5$, $|1-i| = \sqrt{1^2+1^2} = \sqrt{2}$

Both numbers are <u>outside</u> the unit circle

5.5.3: $x = 2+i$, $y = 1+3i$: $|x| = \sqrt{5}$, $|y| = \sqrt{10}$

$\bar{x} = 2-i$, $x\bar{x} = 4+1 = 5$, $xy = -1+7i$

$1/x = \frac{2-i}{5}$, $x/y = \frac{x\bar{y}}{y\bar{y}} = \frac{5-5i}{10} = \frac{1}{2} - \frac{1}{2}i$

$|xy| = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2} = \sqrt{5}\cdot\sqrt{10} = |x|\cdot|y|$

$|1/x| = \sqrt{\frac{4}{25}+\frac{1}{25}} = \frac{1}{\sqrt{5}} = \frac{1}{|x|}.$

5.5.6: $x = \begin{pmatrix} 2-4i \\ 4i \end{pmatrix}$, $y = \begin{pmatrix} 2+4i \\ 4i \end{pmatrix}$

$|x|^2 = |2-4i|^2 + |4i|^2 = 4+16+16 = 36$

$\Rightarrow |x| = 6$

$|y|^2 = |2+4i|^2 + |4i|^2 = 36$, $|y| = 6$

$\bar{x} \cdot y = (2+4i, -4i) \cdot \begin{pmatrix} 2+4i \\ 4i \end{pmatrix} = 4-16+16i+16 =$
$= 4+16i$

5.5.7: $A = \begin{pmatrix} 1 & i & 0 \\ i & i & 1 \end{pmatrix}$

$\Rightarrow A^H = \begin{pmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{pmatrix}$, $C = A^H A = \begin{pmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{pmatrix}$

$C^H = \begin{pmatrix} 2 & +i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{pmatrix} = C$ – always true since $(A^H A)^H = A^H A$

~~**5.5.18:** U – unitary: $U^H U = U U^H = I$~~

~~$\Rightarrow \det U^H \cdot \det U = 1$, but~~

~~$\det(U^H U) = |\det U|^2 \Rightarrow |\det U| = 1$~~

~~2×2 matrices: $U^H U = \begin{pmatrix} a & \bar{c} \\ \bar{c} & d \end{pmatrix} \begin{pmatrix} a & c \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+\bar{c}c & ac+\bar{c}d \\ a\bar{c}+d\bar{c} & \bar{c}c+d^2 \end{pmatrix}$~~

~~$= U U^H = \begin{pmatrix} a^2+\bar{c}c & a\bar{c}+\bar{c}d \\ ac+d\bar{c} & \bar{c}c+d^2 \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$~~

5.5.13: A is 3×3 matrix with eigenvalues $0, 1, 2$

a) eigenvalues are all different, so unit eigenvectors u, v, w are orthogonal to each other

b) the nullspace is spanned by u (e.v. $\lambda=0$); the left nullspace = nullspace

the column space = the row space — — spanned by v and w

c) $Ax = v + w$, $x = c_1 u + c_2 v + c_3 w$

$\Rightarrow Ax = c_2 Av + c_3 Aw = c_2 v + 2c_3 w = v + w$

$\Rightarrow c_2 = 1$, $c_3 = \frac{1}{2}$, but c_1 is arbitrary

general $\boxed{x = c_1 u + v + \frac{1}{2} w}$

d) $Ax = b$, $x = c_1 u + c_2 v + c_3 w$, ~~$A^TAx$~~

Ax is spanned by v, w only $\Rightarrow b^T u = 0$ for the system $Ax = b$ to have solutions

5.5.13 (continued)

e) $S = \begin{pmatrix} u & v & w \end{pmatrix}$ – orthogonal matrix (u, v, w – orthogonal by a)

$\Rightarrow S^{-1} = S^T$

$A = S \cdot \mathrm{diag}(0,1,2) \cdot S^{-1} \Rightarrow S^{-1} \cdot A S = \mathrm{diag}(0,1,2)$

5.5.18: U – unitary: $U^H U = U U^H = I$

$\det(U^H U) = |\det U|^2 = 1 \Rightarrow |\det U| = 1$

$|\det U^H| = |\det U| = 1$

2×2 matrices: let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$U^H = \begin{pmatrix} \bar a & \bar c \\ \bar b & \bar d \end{pmatrix}$ $\det U^H = \overline{\det U}$,

$\det U = e^{i\theta}$, $\det U^H = e^{-i\theta}$, θ – real

$I = U^H U = \begin{pmatrix} |a|^2 + |c|^2 & \bar a b + \bar c d \\ a \bar b + c \bar d & |b|^2 + |d|^2 \end{pmatrix} \Rightarrow$ $|a|^2 + |c|^2 = |b|^2 + |d|^2 = 1$, $a\bar b + c\bar d = \overline{(\bar a b + \bar c d)} = 0$

$I = U U^H = \begin{pmatrix} |a|^2 + |b|^2 & a\bar c + b\bar d \\ \bar a c + \bar b d & |c|^2 + |d|^2 \end{pmatrix} \Rightarrow$ $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$, $a\bar c + b\bar d = \overline{(\bar a c + \bar b d)} = 0$

$\Rightarrow |a| = |d|$, $|b| = |c|$, i.e. $a = \cos\varphi \cdot e^{i\alpha}$,

$b = \sin\varphi \cdot e^{i\beta}$, $c = \sin\varphi \cdot e^{i\gamma}$, $d = \cos\varphi \cdot e^{i\delta}$,

$\varphi, \alpha, \beta, \gamma, \delta$ – real, $e^{i(\alpha-\beta)} + e^{i(\gamma-\delta)} = 0 = e^{i(\alpha-\gamma)} + e^{i(\beta-\delta)}$

$\underline{\underline{5.5.36:}}$ $$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = U \Lambda U^H$$

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\lambda^2 - 2\cos\theta \cdot \lambda + 1 = 0$$

$$\lambda_\pm = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta$$

$$(Q - \lambda_+) v_+ = \begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{pmatrix} v_+ = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_+ = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$(Q - \lambda_-) v_- = \begin{pmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{pmatrix} v_- = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_- = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \Rightarrow \quad U = (v_+ \; v_-) = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$$

$$\Rightarrow U^H = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$$

$$|\lambda_\pm| = \cos^2\theta + \sin^2\theta = \underline{1}$$

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$