The Fundamental Subspaces of a Matrix - Purpose of § is to understand Fig 3.4

\[ Ax = b \]

\((m \times n)(n \times 1)(m \times 1)\)

As a transformation: \( A : \mathbb{R}^n \rightarrow \mathbb{R}^m \)

(Identify composed \( x \) with vector \( \Sigma x_i e_i \))

Picture (3.4) Pg 147 (Picture enables you to visualize matrix mult, \( A \mathbf{v} \), but not exact picture.

\[ \text{Row Span } C(A^T) \]

\[ \text{Colm Span } C(A) \]

\[ \text{Nullspace } N(A) \]

\[ \text{Left Nullspace } N(A^T) \]

\[ x = x_r + x_n \]

\[ \dim r \]

\[ \dim n-r \]

\[ \dim m-r \]
In $\mathbb{R}^n$: $x \in \mathbb{R}^n \Rightarrow x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

(vector $x$ always vertical, $x^T$ horizontal)

Inner product: $x \cdot z = \sum_{i=1}^n x_i z_i$

dot product

Write as matrix mult:

$x \cdot z = x^T \cdot z = (x_1 \ldots x_n) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$

dot matrix mult

Length: $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2} = \sqrt{x \cdot x}$

Thm: in $\mathbb{R}^n$: $x \cdot y = 0$ iff $x \perp y$

"Pf": $x \perp y$ iff $\|x\|^2 + x^T y = (x-y) \cdot (x-y) = x \cdot x - 2x \cdot y + y \cdot y$

iff $x \cdot y = 0$
Proof: The proof relies on the Gramm–Schmidt procedure which shows that every subspace of \( \mathbb{R}^n \) of dimension \( r \) has an orthonormal basis of size \( r \). (Topic of Sect 3.4) The ON basis on \( V \) together with ON basis on \( V^\perp \) together form a basis for \( \mathbb{R}^n \), so \( \dim(V^\perp) \) must be \( n-r \). Rest you get by taking dot products of ON representation of a vector.

**Defn:** Let \( V \subseteq \mathbb{R}^n \) be a subspace. Then 
\[
V^\perp = \{ x \in \mathbb{R}^n : x \cdot v = 0 \ \forall \ v \in V \} = \{ x \in \mathbb{R}^n : x \cdot v = 0 \ \forall \ v \in V \}
\]

We write: \( \mathbb{R}^n = V \oplus V^\perp \) to indicate

**Thm:** For any subspace \( V \subseteq \mathbb{R}^n \), \( \dim V = r \), we have \( \dim (V^\perp) = n-r \), and \( \forall \ x \in \mathbb{R}^n \) can be uniquely written as 
\[
X = V + V^\perp
\]

for \( v \in V \), \( v^\perp \in V^\perp \)
Example: Consider $V \in \mathbb{R}^2$, $V = \text{Span}\{ (c,s) \}$ = "line with \( \theta \)"

$V^\perp = \text{Span}\{ (-s,c) \}$

$x = \text{Proj}_{(c,s)} x + \text{Proj}_{(-s,c)} x$
In fact: \[ \text{Proj}_{(c,s)} x = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ \text{Proj}_{(-s,c)} x = \begin{bmatrix} (-s)^2 & -sc \\ -sc & c^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

"This gives a visualization of \( V \oplus V^\perp = \mathbb{R}^n \) for any \( n \) — not a real picture in any dim!"

In \( \mathbb{R}^3 \): \( V = \text{plane } 2x - 3y + z = 0 \implies V^\perp \) is the line \( \{ t(2, -3, 1) \mid t \in \mathbb{R} \} \)
Def.: We say $V \& W$ are orthogonal if $v \cdot w = 0$ for all $v \in V \& w \in W$, but $V^\perp$ is every vector $w$ s.t. $w \cdot v = 0$ for all $v$.

Eg.: $V = \{ t(1,0,0) \in \mathbb{R}^3 \} = \text{Span}\{ (1,0,0) \}$

$W = \text{Span}\{ (0,1,0) \}$

$W \perp V$ but $W \neq V^\perp = \text{Span}\{ (0,1,0), (0,0,1) \}$

Claim: A $A_{mxn}$ Matrix Must have visualization(1)
• \( N(A) = \text{Row}(A) = \text{Col}(A^T) = \text{Col}(A^T) \)

\[ P.F.: \quad \begin{bmatrix} -r_1 & \cdots & x_1 \\ \vdots & \ddots & \vdots \\ -r_m & \cdots & x_n \end{bmatrix} = \begin{bmatrix} r_1 \cdot x \\ \vdots \\ r_m \cdot x \end{bmatrix} \]

\[ A = (x \cdot x^T) \]

* If \( x \cdot r_i = 0 \quad \forall \quad i = 1, \ldots, m \), then \( A \cdot x = 0 \)

Let \( V = \text{Row}(A) = \text{Col}(A^T) = \text{Span} \{ r_1, \ldots, r_m \} \)

Then \( x \cdot r_i = 0 \iff x \perp r_i \quad \forall \quad i = 1, \ldots, m \)

\[ \iff A \cdot x = 0 \]

But \( x \cdot r_i = 0 \iff x \cdot (\sum a_i \cdot r_i) = \sum a_i \cdot x \cdot r_i = 0 \)

\[ \iff x \cdot v = 0 \quad \forall \quad v \in V \]

\[ \iff N(A) = V^\perp = \text{Row}(A)^\perp \]
Similarly: \( N(A^T) = \text{Col}(A)^\perp = \text{C}(A)^\perp \)

pf

\[
\begin{bmatrix}
y_1 & \cdots & y_m
\end{bmatrix}
\begin{bmatrix}
c_1 & \cdots & c_n
\end{bmatrix}
= \begin{bmatrix}
y_1 c_1 & \cdots & y_m c_n
\end{bmatrix}
\]

\((1 \times m)\quad A_{mxn} = b_{nxn}\)

\(\forall i\), If \( y \cdot c_i = 0 \quad i = 1, \ldots, n \) then \( y^TA = 0 \)

Let \( V = \text{Col}(A) = \text{Span} \{c_1, \ldots, c_n\} \)

Then \( y \cdot c_i = 0 \quad \forall i \) iff \( y \perp c_i \quad \forall i \)

iff \( yA = 0 \)

But \( y \cdot c_i = 0 \quad \forall i \) iff \( x \cdot \sum a_i c_i = \sum a_i x \cdot c_i = 0 \)

iff \( x \cdot v = 0 \quad \forall v \in V \)

\(\therefore \quad N(A^T) = V^\perp = \text{Col}(A)^\perp \)
The picture is now justified:

- $A$ takes $\text{Row}(A)$ (1-1 & onto) $\text{Col}(B)$.
- $B$ it takes $\text{Row}(A) = N(A)$ to zero.
- This tells how every $x \in \mathbb{R}^n$ gets mapped over to $\mathbb{R}^n$ because every $x$ can be uniquely written as

$$x = x_r + x_n \rightarrow b$$
No 2 elements of $\text{Row}(A)$ can map to the same $b$ ($A$ is 1-1 on $\text{Row}(A) \to \text{Col}(A)$)

I.e.

$$A x_r = b \quad \& \quad A \bar{x}_r = b$$

$$\Rightarrow \quad A (x_r - \bar{x}_r) = 0$$

$$\Rightarrow \quad x_r - \bar{x}_r \in \text{Row}(A)^\perp = \text{Ker}(A)$$

So

$$x_r = \bar{x}_r + (x_r - \bar{x}_r) = \bar{x}_r + \text{Ker}(A)$$

Therefore

$$x_r = \bar{x}_r \checkmark$$

Evey $b \in \text{Col}(A)$ is "hit" by some $x_r$ (only)

I.e. $b \in \text{Col}(A) \quad \Rightarrow \quad b = \sum_{i=1}^{n} x_i e_i = A x \checkmark$

Note: That $z = v + w$ for $v$ in $V$ and $w$ in $V^\perp$, just use the ON basis and dot with $e_i$ to get unique representation in terms of ON basis. That means there is a unique way to write a vector in terms of $v$ and its Orthogonal complement.
Note: By symmetry, $A^T : \text{Col}(A) \rightarrow \text{Row}(A)$ but $A^T \neq A^{-1}$ in general.

Picture $A^T A$:

Here: if $x_r = A^T A x_r$ then $A^T A \equiv \text{id} \& A^T = \text{left inverse of } A \text{ on } \text{Col}(A)$

Conclude: $A^T A$ is a "bijection" from $\text{Row}(A) \rightarrow R(\text{row}(A))$