The Fundament Subspaces of a Matrix –

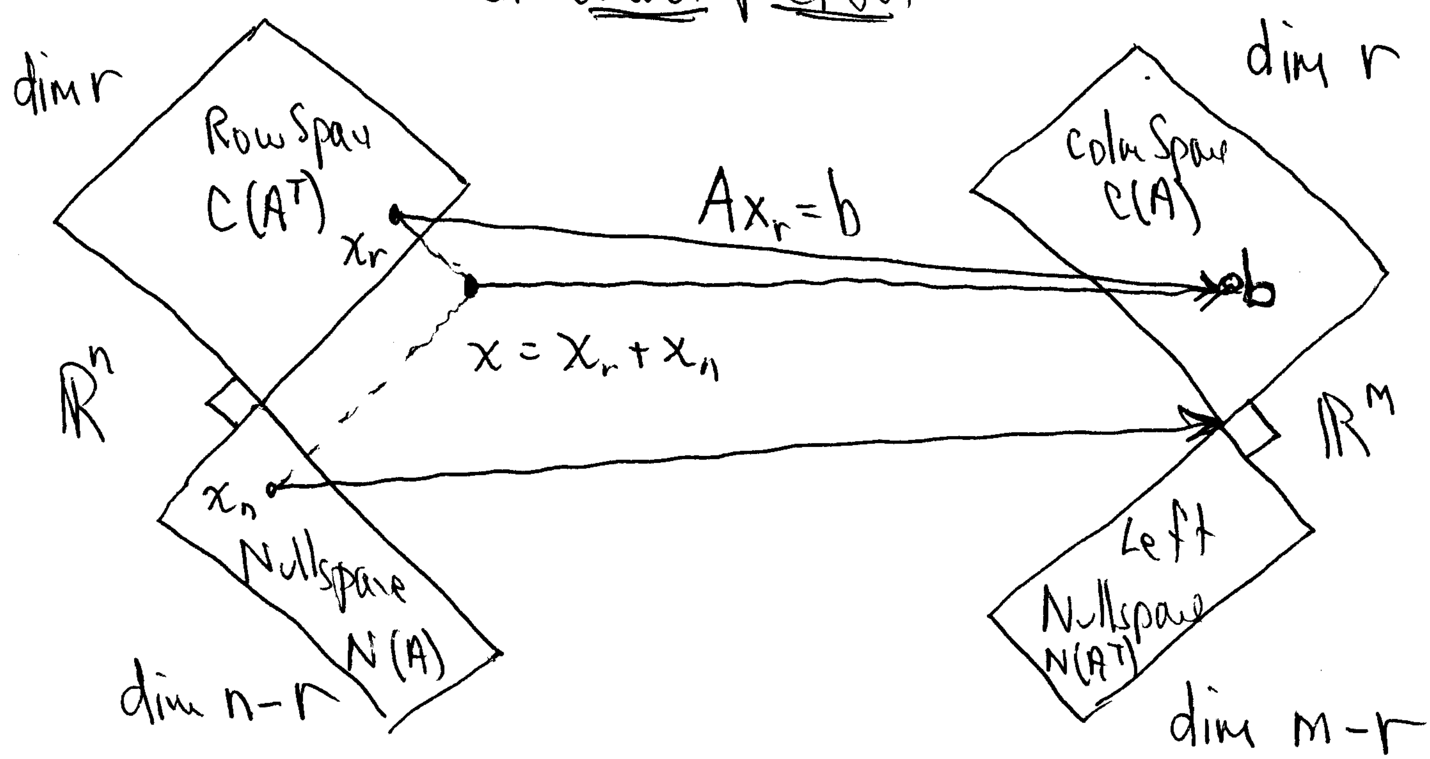
Purpose of § is to understand Fig 3.4

$$\underset{(m\times n)}{A}\ \underset{(n\times 1)}{x} = \underset{(m\times 1)}{b}$$

As a transformation: $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

(identify components x with vector $\sum x_i e_i$!)

Picture: (3.4) Pg 147 (Picture enables you to visualize Matrix Mult; $\forall m, n$, but not exact picture)



- In $\mathbb{R}^n$: $x, z \in \mathbb{R}^n \Rightarrow x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$

(vector x alway vertical, x^T horizontal) $\quad x = \vec{x} = \underset{\sim}{x}$

- Inner product: $\underbrace{x \cdot z}_{\text{dot product}} = \sum_{i=1}^{n} x_i z_i$

Write as matrix mult:

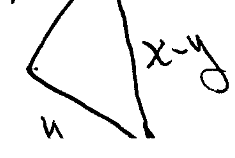
$$\underbrace{x \cdot z}_{\text{dot}} = \underbrace{x^T \cdot z}_{\text{matrix mult}} = (x_1, \dots, x_n) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

- Length: $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{x \cdot x}$

<u>Thm</u>: in $\mathbb{R}^n$: $x \cdot y = 0$ iff $x \perp y$

"Pf" $x \perp y$ iff $\|x\|^2 + \|y\|^2 = \|x-y\|^2 = (x-y) \cdot (x-y) = x \cdot x - 2x \cdot y + y \cdot y$

iff $x \cdot y = 0$ ✓



• <u>Defn</u>: Let $V \subseteq \mathbb{R}^n$ be a subspace. Then

$V^\perp$ = "orthogonal Complement of V in $\mathbb{R}^n$"

$$= \{ x \in \mathbb{R}^n : x \cdot v = 0 \ \forall \ v \in V \}$$

We write: $\mathbb{R}^n = V \oplus V^\perp$ to indicate

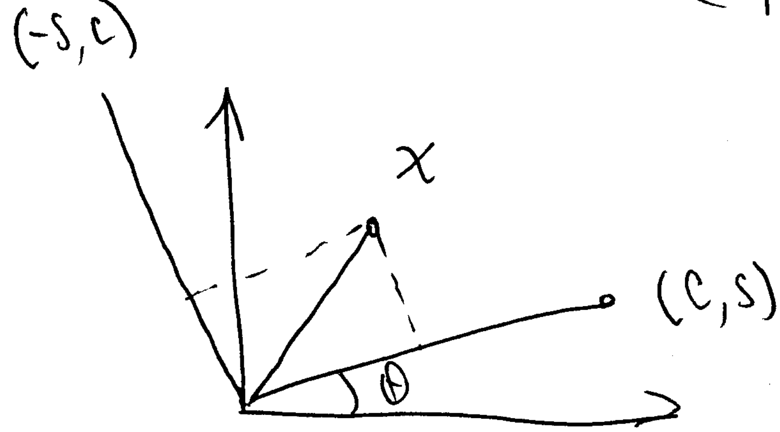
<u>Thm</u>: For any subspace $V \subseteq \mathbb{R}^n$, $\dim V = r$, we have $\dim(V^\perp) = n-r$, & $\forall \ x \in \mathbb{R}^n$ can be uniquely written as

$$x = v + v^\perp$$

for $v \in V$, $v^\perp \in V^\perp$

Proof: The proof relies on the Gramm-Schmidt procedure which shows that every sub-space of R^n of dimension r has an orthonormal basis of size r. (Topic of Sect 3.4) The ON basis on V together with ON basis on V^{perp} together form a basis for R^n, so dim(v^{perp}) must be n-r. Rest you get by taking dot products of ON represenation of a vector.

Ex: Consider $V \subseteq \mathbb{R}^2$ $V = \text{Span}\{(c, s)\}$
= "line with $\angle \theta$"



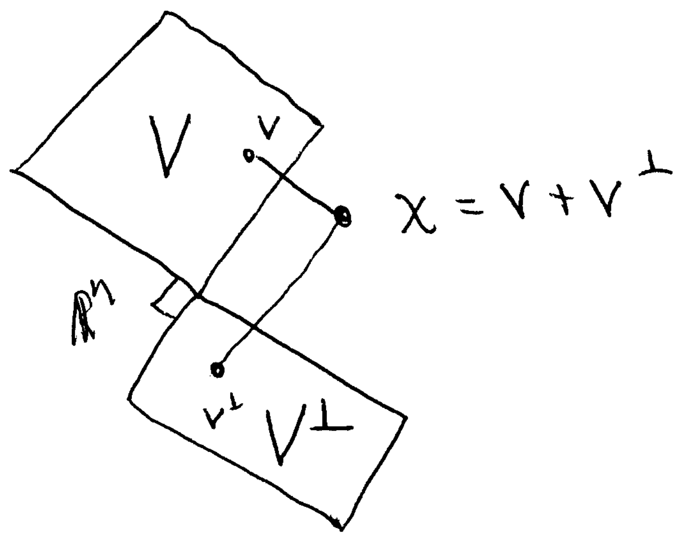
$V^\perp = \text{Span}\{(-s, c)\}$

$$x = \text{Porj}_{(c,s)} x + \text{Proj}_{(-s,c)} x$$

In Fact: $\text{Proj}_{(c,s)} x = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\text{Proj}_{(-s,c)} x = \begin{bmatrix} (-s)^2 & -sc \\ -sc & c^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Picture:



"This gives a visualizat. of $V \oplus V^\perp = \mathbb{R}^n$ for any n — not a <u>real</u> <u>picture</u> in any dim!"

In $\mathbb{R}^3$: V = plane $2x - 3y + z = 0$ $\Rightarrow$

$V^\perp$ is the line $\{t(2,-3,1)\ t \in \mathbb{R}\}$

$(V \perp W)$

Defn: We say V & W are othogonal $(V \perp W)$ if $v \cdot w = 0$ $\forall\, v \in V$ & $w \in W$, but $V^\perp$ is every w st $w \cdot v = 0$ $\forall v$!

Eg: $\bar{V} = \{t(1,0,0)\ t \in \mathbb{R}\} = \text{Span}\{(1,0,0)\}$

$\bar{W} = \text{Span}\{(0,1,0)\}$

$\bar{W} \perp \bar{V}$ but $\bar{W} \neq V^\perp = \text{Span}\{(0,1,0),(0,0,1)\}$

Claim: $\forall\ A_{m \times n}$, Matrix Mult has visualization(1)

- $N(A) = Row(A)^{\perp} = C(A^T)^{\perp}$

"Pf."

$$\underset{A_{(m\times n)}}{\begin{bmatrix} - r_1 - \\ \vdots \\ - r_m - \end{bmatrix}} \underset{x_{(n\times 1)}}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} = \underset{b_{(m\times 1)}}{\begin{bmatrix} r_1 \cdot x \\ \vdots \\ r_m \cdot x \end{bmatrix}}$$

$\therefore$ If $x \cdot r_i = 0 \ \forall i = 1, \ldots, m$, then $Ax = 0$

Let $V = Row(A) = C(A^T) = \text{Span}\{r_1, \ldots, r_m\}$

Then $x \cdot r_i = 0 \ \forall i$ iff $x \perp r_i \ \forall i = 1, \ldots, m$

iff $Ax = 0$

But $x \cdot r_i = 0 \ \forall i$ iff $x \cdot (\sum a_i r_i) = \sum a_i x \cdot r_i = 0$

iff $x \cdot v = 0 \ \forall v \in V$

$\therefore$ $N(A) = V^{\perp} = Row(A)^{\perp}$

- Similarly: $N(A^T) = Col(A)^\perp = C(A)^\perp$

"pf"

$$[y_1, \cdots, y_m] \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix} = [y \cdot c_1, \cdots, y \cdot c_n]$$

$$(1 \times m) \qquad A_{m \times n} \qquad = \qquad b_{1 \times n}$$

$\therefore$ If $y \cdot c_i = 0$ $i = 1, \ldots, n$, then $y^T A = 0$

Let $V = Col(A) = Span\{c_1, \ldots, c_n\}$

Then $y \cdot c_i = 0 \; \forall i$ iff $y \perp c_i \; \forall i$

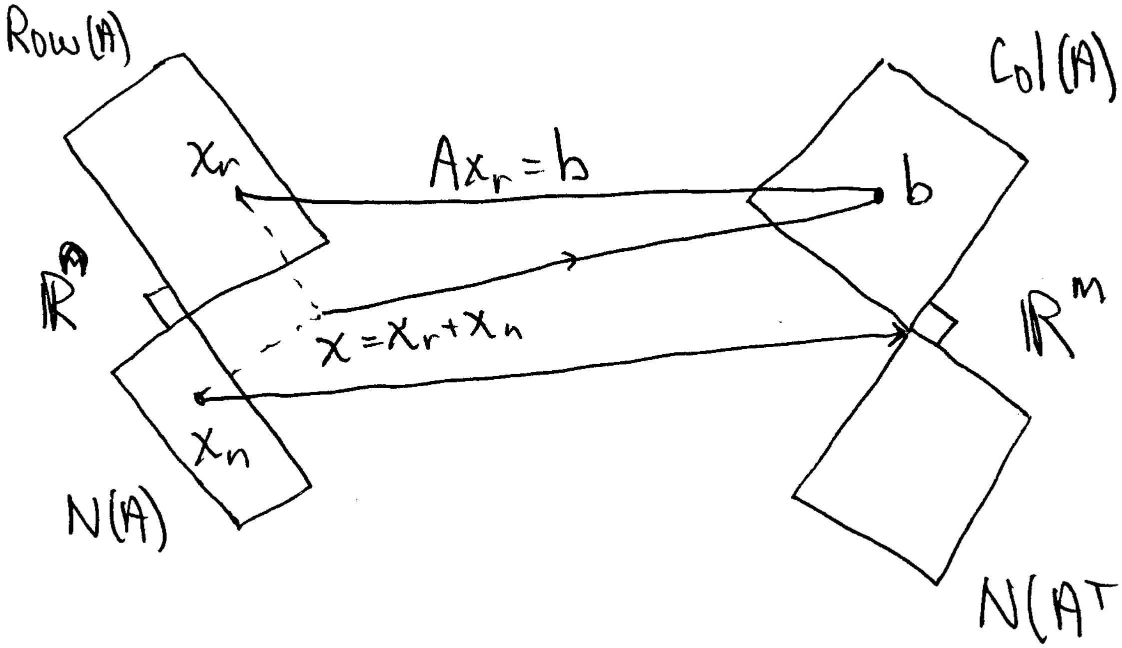
iff $yA = 0$

But $y \cdot c_i = 0 \; \forall i$ iff $x \cdot \sum a_i c_i = \sum a_i x \cdot c_i = 0$

iff $x \cdot v = 0 \; \forall \; v \in V$

$\therefore$ $N(A^T) = V^\perp = Col(A)^\perp$

The picture is now justified:



- A takes Row(A) (1-1 & onto) Col(B).
  & it takes $\text{Row}(A)^\perp = N(A)$ to zero
- This tells how <u>every</u> $x \in \mathbb{R}^n$ gets mapped over to $\mathbb{R}^n$ because every x can be uniquely written as

$$x = x_r + x_n \longrightarrow b$$

- No 2 elements of Row(A) can map to the same b (A is 1-1 on $\text{Row}(A) \to \text{Col}(A)$)

Ie.

$$Ax_r = b \ \& \ A\bar{x}_r = b$$

$$\Rightarrow A(x_r - \bar{x}_r) = 0$$

$$\Rightarrow x_r - \bar{x}_r \in \text{Row}(A)^{\perp} = K(A)$$

$\therefore$

$$x_r = \bar{x}_r + \underbrace{(x_r - \bar{x}_r)} = x_r + \underbrace{0}$$

$\exists!$ way to write $x = v + v^{\perp}$!

$\therefore \ x_r = \bar{x}_r$ ✓

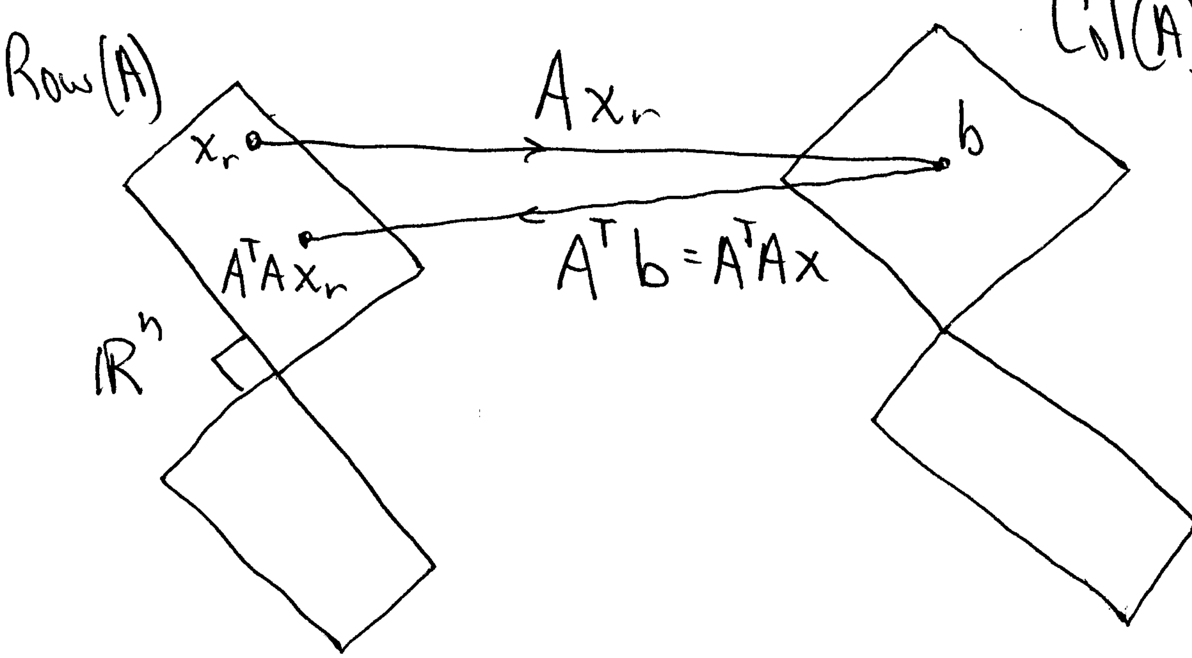
- Every $b \in \text{Col}(A)$ is "hit" by some x_r (onto)

Ie. $b \in \text{Col}(A) \Rightarrow b = \sum_{i=1}^{n} x_i c_i = Ax$ ✓

Note: That z=v+w for v in V and w in V^{perp}, just use the ON basis and dot with e_i to get unique representation in terms of ON basis. That means there is a unique way to write a vector in terms of v and its Orthogonal complement.

- **Note**: By symmetry, $A^T: Col(A) \to Row(A)$ but $A^T \neq A^{-1}$ in general:

Picture A^TA:



Here: if $x_r = A^TAx_r$ then $A^TA = id$ &

A^T = left inverse of A on $Col(A)$

Conclude: A^TA is a "bijection" from $Row(A) \to Row(A)$