Gramm-Schmidt Orthoganalization –

- A vector space has no inner product $\langle v_1, v_2 \rangle$ and <u>extra</u> structure for vector space
- $\mathbb{R}^n$ has a natural inner product $x, y \in \mathbb{R}^n$

$$\langle x, y \rangle = \sum x_i, y_i = \|x\|\|y\| \cos\theta = x \cdot y$$
$$= x^T \cdot y$$

<u>Defn</u>: A basis $\{q_1, \dots, q_n\}$ for $\mathbb{R}^n$ is orthonormal if $q_i \cdot q_j = \delta_{ij}$

I.e. $q_i \cdot q_j = 0$ if $i \neq j$

$\|e_i\| = 1 = q_i \cdot q_j$

When you have an O.N. basis, it is easy to find the comp's of b wrt basis:

• Let $\{q_1, \dots, q_n\}$ be on. basis, $b \in \mathbb{R}^n$

$$b = b_1 q_1 + \cdots + b_n q_n$$

Q: Formula for b_i?

Ans: Dot both sides with q_i^T

$$\underset{\substack{\uparrow \\ \text{matrix} \\ \text{mult}}}{q_i^T \cdot b} = b_1 \underbrace{\cancel{q_i^T \cdot q_1}}_{0} \cdots b_i \underbrace{q_i^T \cdot q_i}_{1} + \cdots + b_n \underbrace{\cancel{q_i^T q}}_{0}$$

$$= b_i$$

Conclude: $b_i = \underset{\substack{\uparrow \\ \text{matrix} \\ \text{mult}}}{q_i^T \cdot b} = \underset{\substack{\uparrow \\ \text{dot} \\ \text{product}}}{q_i \cdot b}$

Ie. $\mathrm{Proj}_{q_i} b = \dfrac{q_i \cdot b}{\|q_i\|^2} q_i = (q_i \cdot b) q_i$

Thm: $\{q_1, \dots, q_n\}$ ON. $\Rightarrow$

$$\boxed{b = \sum_{i=1}^{n} (q_i^T \cdot b)\, q_i}$$

Ex Orthogonal matrices — $\{q_1, \dots, q_n\}$ O.N.

Let $Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$

Defn: an orthogonal matrix is an $n \times n$ matrix whose colmns are an ON basis.

Claim: $Q^{-1} = Q^T$ iff Q is orthogonal

($\Rightarrow$) Assume Q ON. Then

$$Q^T Q = \begin{bmatrix} — q_1 — \\ \vdots \\ — q_n — \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Since Q is invertible = full rank, Q^T must also be right inverse - or

$$QQ^T = A \Rightarrow \underbrace{Q^TQ}\,Q^T = Q^TA$$
$$\Rightarrow I\,Q^T = Q^TA \Rightarrow A = id$$

($\Leftarrow$) Assume $Q^{-1} = Q^T$. Then

$$Q^TQ = \begin{bmatrix} - q_1 - \\ \vdots \\ - q_n - \end{bmatrix}\begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$\Rightarrow q_i \cdot q_j = \delta_{ij} \Rightarrow$ orthogonal ✓

~~• Note: To solve $Qx = b$ write~~

$$\cancel{Q^TQx = Q^Tb \Leftrightarrow x = Q^Tb = \begin{bmatrix} - q_1 - \\ \vdots \\ - q_n - \end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}$$

$$\cancel{x = \begin{bmatrix} q_1^T \cdot b \\ \vdots \\ q_n^T \cdot b \end{bmatrix}}$$ ~~as before~~ ✓

I.e. $Qx = b \Leftrightarrow \cancel{q_1x_1 + \cdots + q_nx_n = b}$ $\quad x_i = q_i^T \cdot b$

<u>Cor</u>: If the colms of Q are O.N., then so are the rows:

Pf. $Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & r_1 & - \\ & \vdots & \\ - & r_n & - \end{bmatrix}$

Q ON. $\Leftrightarrow$ $Q\, Q^T = \mathrm{id}$

$$\Leftrightarrow \begin{bmatrix} - & r_1 & - \\ & \vdots & \\ - & r_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$\Rightarrow$ rows are O.N.

• <u>Note</u> To solve $Qx = b$, write $Q^T Q x = Q^T b$

$$\Leftrightarrow \quad x = Q^T b = \begin{bmatrix} - & q_1 & - \\ & \vdots & \\ - & q_n & - \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{pmatrix} q_1^T b \\ \vdots \\ q_n^T b \end{pmatrix}$$

<u>So</u> $b = x_1 q_1 + \cdots + x_n q_n$, $x_i = q_i^T \cdot b$

"$= Ax$"

as before ✓

<u>Thm</u>: If Q is orthogonal, then Q viewed as a linear transformation preserves lengths ~~and~~ angles $\Leftrightarrow$ inner products

Pf. $$\langle Qx, Qy\rangle = (Qx) \underset{\uparrow \text{dot}}{\cdot} (Qy) = (Qx)^T \cdot Qy$$

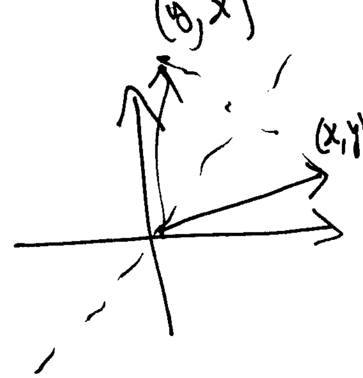
$$= x^T \underbrace{Q^T Q}_{I} y = x^T \cdot y = x \underset{\uparrow \text{dot}}{\cdot} y = \langle x, y\rangle \quad \checkmark$$

<u>Conclude</u>: The orthogonal transformations represent the "rotations & reflections" in $\mathbb{R}^n$

Ex: Rotation thru θ:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow \text{colms ON} \checkmark$$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad P\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$



reflects about $y=x$ ON. ✓

A permutation matrix is ON.

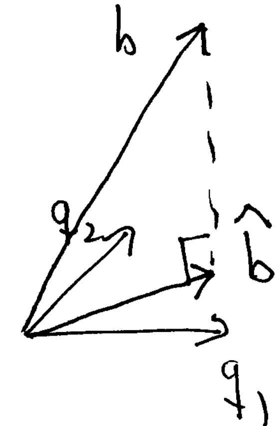
$P_\pi v = (v_{\pi_1}, \dots, v_{\pi_n})$ permute entries according to $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$

$$\Leftrightarrow \quad P_\pi = \begin{bmatrix} - e_{\pi_1} - \\ \vdots \\ - e_{\pi_n} - \end{bmatrix}$$ "identity matrix with rows permuted by π"

$\Rightarrow$ ON ✓

Ex: Given $\{q_1, \dots, q_n\}$ ON in $\mathbb{R}^m$ $m > n$, find the least squares approx $\hat{b}$ of $b \in \mathbb{R}^m$

We call: $\hat{b} = \mathrm{Proj}_{q_1 \cdots q_n} b$



Soln: $A_{m\times n} = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}_{m\times n}$

rank $n \Rightarrow (A^TA)^{-1}$ exists

$$\hat{b} = A(A^TA)^{-1}A^Tb$$

$$A^TA = \underset{m\times m}{\begin{bmatrix} - & q_1 & - \\ & \vdots & \\ - & q_n & - \end{bmatrix}} \underset{m\times n}{\begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}} = \underset{n\times n}{I}$$

$$\therefore \hat{b} = AA^Tb = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & q_1 & - \\ & \vdots & \\ - & q_n & - \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} q_1 \cdot b \\ \vdots \\ q_n \cdot b \end{bmatrix} = \sum_{i=1}^{n} (q_i \cdot b)\, q_i \in \mathbb{R}^m$$

$m > n$

Conclude:

$$\mathrm{Proj}_{q_1 \cdots q_n} b = (q_1^T b) q_1 + \cdots + (q_n^T b) q_n$$

As you would expect: eg if $m = n$, $\{q_1 \cdots q_n\}$ basis for $\mathbb{R}^n$, then

$$b = \underbrace{(q_1^T \cdot b) q_1 + \cdots + (q_k^T \cdot b) q_k}_{\text{gives projection onto the span of 1st } k \text{ } q\text{'s}} + \underbrace{\cdots + (q_n^T \cdot b) q_n}_{\text{because this is } \perp \text{ to 1st } k \text{ terms}}$$