Section 5.1

Eigenvalues / Eigenvectors (A $n \times n$)

- **Defn**: λ is an eigenvalue of A if $\exists\, x \in \mathbb{R}^n$, $x \neq 0$ s.t.

$$Ax = \lambda x$$

we call (λ, x) an eigen-pair of A

- **Defn**: For given eval λ, $\{x : Ax = \lambda\}$ is called the eigenspace of λ.

Thm: every eigenspace is a vector space

Pf. Need only show closed under $\cdot$ & $+$:

$$Ax_1 = \lambda x_1 \ \& \ Ax_2 = \lambda x_2 \Rightarrow A(c_1 x_1 + c_2 x_2) = c_1 A x_1 + c_2 A x_2$$
$$= c_1 \lambda x_1 + c_2 \lambda x_2$$
$$= \lambda (c_1 x_1 + c_2 x_2) \checkmark$$

- **Ex**: $N(A)$ = eigenspace for $\lambda = 0$ if A is not invertible

- Importance: Eigenpairs provide solutions of Linear ODE'S (homogeneous):

Consider:

$$\frac{dx}{dt} = 2x + y$$

$$\frac{dy}{dt} = x - y$$

$$\Leftrightarrow \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

ivp: Find $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ st $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

Two Coupled Linear Equations

- More generally: consider ivp

$$\frac{dx(t)}{dt} = Ax(t), \; x(0) = x_0$$

A system of n coupled linear 1st order ordinary diff eqn's ODE's

<u>Thm</u>: if (λ, x) is an eigenpair for A,
then

$$x(t) = x_0 e^{\lambda t} \quad \text{solves} \quad \frac{dx(t)}{dt} = A\,x(t)$$

constant vector $= x(0)$

Pf.

$$\frac{d}{dt}\left\{x_0 e^{\lambda t}\right\} = \lim_{\Delta t \to 0} x_0 \left[\frac{e^{\lambda(t+\Delta t)} - e^{\lambda t}}{\Delta t}\right]$$

$$= x_0 \frac{d}{dt} e^{\lambda t} = x_0 \lambda e^{\lambda t}$$

$$A\left\{\underset{\text{vector}}{x_0}\,\underset{\text{scalar}}{e^{\lambda t}}\right\} = e^{\lambda t} A x_0 = e^{\lambda t} \lambda x_0 \quad \checkmark$$

Q: How to find all eigenvalue —

__Thm__: λ is an eigenvalue of A iff $\det|A-\lambda I|=0$

Pf. ($\Rightarrow$) if λ is an eval then $Ax=\lambda x$ some $x\neq 0$

so $Ax-\lambda x=(A-\lambda I)x=0 \Rightarrow A-\lambda I$ is

__not invertible__ $= \det|A-\lambda I|=0$

($\Leftarrow$) if $\det|A-\lambda I|=0$ then $N(A-\lambda I)\neq\emptyset \Rightarrow$

$\exists$ nonzero $x\in N(A-\lambda I) \Rightarrow (A-\lambda I)x=0$

$\Rightarrow Ax=\lambda x$ ✓

__Turns out__: $\det(A-\lambda I)=$ "a polynomial of degree n in λ with coeff's determined by comp's of A" - __characteristic polynomial__ of A

* The roots of this polynomial are the evals of A!

Example: Find the characteristic polynomial & evals of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \det|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 1$$

$$= \lambda^2 - 3\lambda + 2 - 1 = \lambda^2 - 3\lambda + 1$$

$$\Rightarrow \lambda = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det|A-\lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc)$$

$$= \lambda^2 - Tr(A)\lambda + \det A = c(\lambda)$$

$$\lambda = \frac{Tr(A) \pm \sqrt{(Tr A)^2 - 4\det A}}{2} \Rightarrow \boxed{\begin{array}{l} \lambda_1 + \lambda_2 = \text{Trace} \\ \lambda_1 \cdot \lambda_2 = \text{Det} \end{array}}$$

Note: evals are complex when $(Tr A)^2 - 4 \det A < 0$!

Solve the system: $\frac{dx(t)}{dt} = A x(t)$, $x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A = \begin{bmatrix} +1 & +2 \\ 1 & 0 \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2) \quad \lambda = -1, 2$$

$$0 = \begin{bmatrix} 1-(-1) & +2 \\ 1 & 0-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1 + x_2 = 0$$

$$r_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$0 = \begin{bmatrix} 1-2 & 2 \\ 1 & 0-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad -x_1 + 2x_2 = 0$$

$$r_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$c_1 + 2c_2 = 1$
$-c_1 + c_2 = 2$
$3c_2 = 3 \Rightarrow c_2 = 1$
$c_1 = -1$

$$\Rightarrow x(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}, \quad x(0) = \underset{-1}{c_1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \underset{1}{c_2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \underbrace{e^{-t}}_{\substack{t \to \infty \\ \downarrow \\ 0}} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \underbrace{e^{2t}}_{\substack{t \to \infty \\ \downarrow \\ \infty}}$$

"negative eigenvalues correspond to stable modes, pos evals to unstable modes"

Eg is λ_1 & λ_2 both neg, the

$$x(t) = c_1 \begin{bmatrix} 1 \\ R_1 \\ 1 \end{bmatrix} \underbrace{e^{\lambda_1 t}}_{\substack{\downarrow \\ 0}} + c_2 \begin{bmatrix} 1 \\ R_2 \\ 1 \end{bmatrix} \underbrace{e^{\lambda_2 t}}_{\substack{\downarrow \\ 0}} \xrightarrow{t \to \infty} 0$$

$$\cancel{\xrightarrow{t \to \infty} c_1 \begin{bmatrix} 1 \\ R_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ R_2 \\ 1 \end{bmatrix} = x(0)}$$

$x(t) = 0$ is stable under perturbation!

"$x(t) = 0$ is a rest pt..."

Ex: Find char polynomial for A:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\det|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 3 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(-1-\lambda)$$

$$\lambda = 2, 1, -1$$

"Det of U or L is product of diag

Ex: $\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- $\det(A - \lambda I) = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1 \quad \lambda = \pm i$

- e-vectors: $[A - iI] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
  $(\lambda = i)$

$r_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\Leftrightarrow -ix_1 + x_2 = 0$

$x_2 = ix_1$

$[A - (-i)I] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow \quad ix_1 + x_2 = 0$
$(\lambda = -i)$

$x_1 = 1, x_2 = -i$

$r_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$\hat{x}_1(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos t + i \sin t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\hat{x}_2(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it} = \begin{pmatrix} 1 \\ -i \end{pmatrix} (\cos t - i \sin t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} - i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

"real & imaginary parts give two real solns"

$\therefore \quad x_1(t) = \hat{x}_1(t) + \hat{x}_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$

$x_2(t) = \frac{1}{2i}\left(\hat{x}_1(t) - \hat{x}_2(t)\right) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

$x(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

$x(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ ✓

Ex: $\frac{dx}{dt} = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad x(0) = x_0$

Soln: $|A - \lambda I| = \begin{vmatrix} \lambda_0 - \lambda & 1 \\ 0 & \lambda_0 - \lambda \end{vmatrix} = (\lambda_0 - \lambda)^2 - 0 = (\lambda - \lambda_0)^2$

$\lambda = \lambda_0$

"double root"

$$\begin{bmatrix} \lambda_0 - \lambda_0 & 1 \\ 0 & \lambda_0 - \lambda_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_2 = 0$$

$$R_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

One Soln: $x_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda_0 t}$: $x_2(t) = \begin{pmatrix} t e^{\lambda_0 t} \\ e^{\lambda_0 t} \end{pmatrix}$

Need another to ~~solve~~ meet initial condt.

Another one: $\begin{pmatrix} t e^{\lambda_0 t} \\ e^{\lambda_0 t} \end{pmatrix}' = \begin{pmatrix} e^{\lambda_0 t} + \lambda_0 t e^{\lambda_0 t} \\ \lambda_0 e^{\lambda_0 t} \end{pmatrix} = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix} \begin{bmatrix} t e^{t} \\ e^{t} \end{bmatrix}$ ✓

"Can reduce general resonant case to this one!