Eigenvalues / Eigenvectors (A $n \times n$)

- **Defn**: λ is an eigenvalue of A if $\exists\, x \in \mathbb{R}^n$, $x \neq 0$ s.t.

$$Ax = \lambda x$$

we call (λ, x) an eigen-pair of A

- **Defn**: For given eval λ, $\{x : Ax = \lambda\}$ is called the eigenspace of λ.

Thm: every eigenspace is a vector space

Pf. Need only show closed under $\cdot$ & $+$:

$$Ax_1 = \lambda x_1 \ \& \ Ax_2 = \lambda x_2 \Rightarrow A(c_1 x_1 + c_2 x_2) = c_1 A x_1 + c_2 A x$$
$$= c_1 \lambda x_1 + c_2 \lambda x_2$$
$$= \lambda (c_1 x_1 + c_2 x_2) \checkmark$$

- **Ex**: $N(A)$ = eigenspace for $\lambda = 0$ if A is not invertible

- Importance: Eigenpairs provide solutions of linear ODE's (homogeneous):

Consider:

$$\frac{dx}{dt} = 2x + y$$

$$\frac{dy}{dt} = x - y$$

$$\iff \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

ivp: Find $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ st $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

Two Coupled Linear Equations

- More generally: consider ivp

$$\frac{dx(t)}{dt} = Ax(t), \quad x(0) = x_0$$

A system of n coupled linear 1st order ordinary diff eqn's ODE's

<u>Thm</u>: if (λ, x) is an eigenpair for A, then

$$x(t) = x_0 e^{\lambda t} \quad \text{solves} \quad \frac{dx(t)}{dt} = A\,x(t)$$

constant vector $= x(0)$

Pf.

$$\frac{d}{dt}\left\{x_0 e^{\lambda t}\right\} = \lim_{\Delta t \to 0} x_0 \left[\frac{e^{\lambda(t+\Delta t)} - e^{\lambda t}}{\Delta t}\right]$$

$$= x_0 \frac{d}{dt} e^{\lambda t} = x_0 \lambda e^{\lambda t}$$

$$A\{x_0 e^{\lambda t}\} = e^{\lambda t} A x_0 = e^{\lambda t} \lambda x_0 \quad \checkmark$$

vector scalar

Q: How to find all eigenvalue —

Thm: λ is an eigenvalue of A iff $\det|A-\lambda I|=0$

Pf. ($\Rightarrow$) if λ is an eval then $Ax=\lambda x$ some $x\neq 0$

so $Ax-\lambda x=(A-\lambda I)x=0 \Rightarrow A-\lambda I$ is not invertible $= \det|A-\lambda I|=0$

($\Leftarrow$) if $\det|A-\lambda I|=0$ then $N(A-\lambda I)\neq \emptyset \Rightarrow$

$\exists$ nonzero $x\in N(A-\lambda I) \Rightarrow (A-\lambda I)x=0$

$\Rightarrow Ax=\lambda x$ ✓

Turns out: $\det(A-\lambda I)=$ "a polynomial of degree n in λ with coeff's determined by comp's of A" - characteristic polynomial of A

* The roots of this polynomial are the evals of A"

<u>Example</u>: Find the characteristic polynomial & evals of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \det|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 1$$

$$= \lambda^2 - 3\lambda + 2 - 1 = \lambda^2 - 3\lambda + 1$$

$$\Rightarrow \lambda = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det|A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc)$$

$$= \lambda^2 - \mathrm{Tr}(A)\lambda + \det A = c(\lambda)$$

$$\lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{(\mathrm{Tr}A)^2 - 4\det A}}{2} \Rightarrow \boxed{\begin{aligned} \lambda_1 + \lambda_2 &= \text{Trace} \\ \lambda_1 \cdot \lambda_2 &= \text{Det} \end{aligned}}$$

<u>Note</u>: evals are <u>complex</u> when $(\mathrm{Tr}A)^2 - 4\det A < 0$!

Review

Start (6a)
Friday

$$\frac{dx}{dt} = A\,x(t) \qquad A_{n\times n}$$

Eg

$$\frac{dx}{dt} = 2x - 3y + z$$

$$\frac{dy}{dt} = x - y$$

$$\frac{dz}{dt} = y + z$$

$$\Leftrightarrow \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\frac{d}{dt}x(t) = A \qquad x$$

$$\frac{d}{dt}x(t) = \dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix}$$

Note: $x = 0$ solves the problem

- (λ, r) an eigenpair for $A \Rightarrow c\,r\,e^{\lambda t} = x(t)$ solves $\dot{x} = Ax$

const vector scalar

- Conclude: if $r_1, \ldots, r_n$ is a basis of evects, then we can solve $\dot{x} = Ax$, $x(0) = x_0 \in \mathbb{R}^n$

Ie. $x(t) = c_1 r_1 e^{\lambda_1 t} + \cdots + c_n r_n e^{\lambda_n t}$

@ $t = 0$: $x(0) = c_1 r_1 + \cdots + c_n r_n = x_0$ Basis $\Rightarrow$
$\exists!$ const's $c_1, \ldots, c_n$ that do it. $\rightsquigarrow$ Example

Solve the system: $\frac{dx(t)}{dt} = Ax(t)$, $x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A = \begin{bmatrix} +1 & +2 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2) \qquad \lambda = -1, 2$$

evals

$$0 = \begin{bmatrix} 1-(-1) & +2 \\ 1 & 0-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1 + x_2 = 0$$

e-vector

$$r_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$0 = \begin{bmatrix} 1-2 & 2 \\ 1 & 0-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad -x_1 + 2x_2 = 0$$

eigenvector

$$r_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$c_1 + 2c_2 = 1$
$-c_1 + c_2 = 2$
$3c_2 = 3 \Rightarrow c_2 = 1$
$c_1 = -1$

$$\Rightarrow x(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}, \quad x(0) = \underbrace{c_1}_{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \underbrace{c_2}_{1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x(t) = \underbrace{c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}}_{\xrightarrow{t \to \infty} 0} + \underbrace{c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}}_{\xrightarrow{t \to \infty} \infty}$$

"negative eigenvalues correspond to $\underline{\text{stable}}$ modes, pos evals to unstable modes"

Eg if λ_1 & λ_2 both neg, then

$$x(t) = \underbrace{c_1 \begin{bmatrix} 1 \\ R_1 \\ 1 \end{bmatrix} e^{\lambda_1 t}}_{\to 0} + \underbrace{c_2 \begin{bmatrix} 1 \\ R_2 \\ 1 \end{bmatrix} e^{\lambda_2 t}}_{\to 0} \xrightarrow{t \to \infty} 0$$

$$\cancel{\xrightarrow{t \to \infty} c_1 \begin{bmatrix} 1 \\ R_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ R_2 \\ 1 \end{bmatrix} = x(0)}$$

$x(t) = 0$ is stable under perturbation!

"$x(t) = 0$ is a rest pt..."

In general: if $(r_1, \dots, r_n)$ basis & (λ_i, r_i) eigenpairs of A, then

$$x(t) = c_1 r_1 e^{\lambda_1 t} + \dots + c_n r_n e^{\lambda_n t}$$

Conclude: if $\lambda_i < 0$, then $x(t) \xrightarrow{t \to \infty} 0$

Ie, $e^{\lambda_i t} \left(\sim e^{-2t} \right) \xrightarrow{t \to \infty} 0$

"0 is a stable rest point"

Ex: Dynamical system: $\dot{x} = F(x)$

Nonlinear F: Eg, $\boxed{\begin{array}{l} \dot{x} = x^2 - \sin xy \\ \dot{y} = e^{xy} + y^2 \end{array}}$

Assume x_0 a rest point: $F(x_0) = 0$

$x = 0$ solves the problem:

Taylor expand: $F(x) = \underbrace{F(x_0)}_{0} + \overset{\nabla F}{A}(x - x_0) + B(x-x_0)^2$

constant coeff matrix

nonlin

- General (nonlinear) dynamical system

$$\dot{y} = F(y) \quad y = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad F = \begin{pmatrix} F_1(y_1 \cdots y_n) \\ \vdots \\ F_n(y_1 \cdots y_n) \end{pmatrix}$$

Eg: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} xy - 12 \\ x^2 + y^2 - 25 \end{pmatrix}$

- y_0 a <u>rest point</u> if $F(y_0) = 0$.
- Then expand by Taylor Series about y_0:

$$F(y) = \underbrace{\cancel{F(y_0)}}_{0} + \underbrace{\nabla F}_{A_{n\times n}}(y - y_0) + \underbrace{G}_{\text{nonlinear fn}}|y - y_0|^2$$

Since $|y-y_0|^2 \ll |y-y_0|$ when $y \approx y_0$, solutions near y_0 solve $\approx$

$$\dot{y} = A(y - y_0) \Leftrightarrow \dot{(y - y_0)} = A(y - y_0)$$

$$\dot{x} = Ax \quad (x = y - y_0)$$

<u>Thm</u>: if (λ_i, r_i) eigenpairs, $\{r_1, \dots, r_n\}$ basis, then rest point is stable because $x(t) \xrightarrow{t \to \infty} 0$!

Ex: Find char polynomial for A:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\det|A-\lambda I| = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 3 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(-1-\lambda)$$

$$\lambda = 2, 1, -1$$

"Det of U or L is product of diag

Ex: $\dfrac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- $\det(A - \lambda I) = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1 \quad \lambda = \pm i$

- e-vectors: $[A - iI] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
  $(\lambda = i)$

$r_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\quad \Leftrightarrow \; -ix_1 + x_2 = 0$

$x_2 = i x_1$

$[A - (-i)I] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Leftrightarrow \; ix_1 + x_2 = 0$
$(\lambda = -i)$

$x_1 = 1, \; x_2 = -i$

$r_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\hat{x}_1(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos t + i \sin t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

$\hat{x}_2(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it} = \begin{pmatrix} 1 \\ -i \end{pmatrix} (\cos t - i \sin t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} - i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

"real & imaginary parts give two <u>real</u> solns"

$$\therefore \quad x_1(t) = \hat{x}_1(t) + \hat{x}_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

$$x_2(t) = \frac{1}{2}\left(\hat{x}_1(t) - \hat{x}_2(t)\right) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$x(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$x(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \checkmark$$

Eigenvalues / Eigenvectors

Skip §5.3

16

$\dot{x} = Ax \qquad x = (x_1, \ldots, x_n) = x(t)$

$x(0) = x_0$

- (λ, x) eigenpair $Ax = \lambda x \Rightarrow xe^{\lambda t}$ solves $\dot{x} = Ax$. Soln space is vector space $\Rightarrow$

$cxe^{\lambda t}$ is 1-d vector space of soln's.

arb const | e-vector | scalar fn that gives time dynamics

- If you have basis of evectors, then you can meet any i-condt.

$(\lambda_1, x_1) \ldots (\lambda_n, x_n)$ e-prs, $\{x_1, \ldots, x_n\}$ basis

$x(t) = c_1 x_1 e^{\lambda_1 t} + \cdots + c_n x_n e^{\lambda_n t}$

$x(0) = c_1 x_1 + \cdots + c_n x_n = x_0$ for $!$ $c_1, \ldots, c_n$

- Basis of e-vectors <u>most</u> important case but there are degenerate cases $\sim A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

__Thm__: (Most Important Case) if $\underset{n\times n}{A}$ has ~~real and~~ distinct evals, then A has a basis of eigenvectors

Pf. Assume $\{r_1 \cdots r_n\}$ linearly indept $\Rightarrow$ can solve for one in terms of others.

$$r_k = \sum_{i\neq k} c_i r_i \quad \text{not all } c_i \text{ are zero}$$

mult by λ_k: $\lambda_k r_k = \sum_{i\neq k} c_i \lambda_k r_i$

operate by A: $\lambda_k r_k = \sum_{i\neq k} c_i \lambda_i r_i$

Subtract: $0 = \sum_{i\neq k} c_i(\lambda_i - \lambda_k) r_i$ not all c_i are zero

Thus $\{r_i\}_{i\neq k}$ are linearly dept. By __same__ __arg__. some $n-2$ of the r_i are lin dep, then $n-3$, ... some one $\{r_e\}$ linearly dept ※

(allows for complex evals!)

$\boxed{2}$ Exponential.

Scalar problem: $\frac{dx}{dt} = \lambda x \Rightarrow x(t) = e^{\lambda t} x_0$

Matrix problem: $\frac{dx}{dt} = Ax \Rightarrow x(t) = e^{At} x_0$

$$e^{\lambda t} = 1 + \lambda t + \frac{(\lambda t)^2}{2} + \cdots + \frac{(\lambda t)^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}$$

$$e^{At} = I + At + \frac{A^2 t^2}{2} + \cdots + \frac{A^n t^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

defines ~~complex~~ matrix exponential.

$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}}$$

"check" $\frac{d}{dt} e^{At} = \frac{d}{dt}\left(1 + At + \cdots + \frac{A^n t^n}{n!} + \cdots\right)$

$$= \frac{d}{dt} \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{d}{dt}\left(\frac{A^n}{n!} t^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{A^n (n-1) t^{n-1}}{n!} = A \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

Assume A has basis $\{r_1, \ldots, r_n\}$ e-vect's.

Consider: $B = \begin{bmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{bmatrix}$ rank $= n \Rightarrow B^{-1}$ exists

$$B^{-1} r_k = e_k = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow k\text{th slot}$$

Now: $B^{-1}AB = B^{-1}A \begin{bmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{bmatrix} = B^{-1} \begin{bmatrix} | & & | \\ Ar_1 & \cdots & Ar_n \\ | & & | \end{bmatrix}$

$$= B^{-1} \begin{bmatrix} | & & | \\ \lambda_1 r_1 & \cdots & \lambda_n r_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1 B^{-1} r_1 & \cdots & \lambda_n B^{-1} r_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ \lambda_1 e_1 & \cdots & \lambda_n e_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

"Matrix whose colms are e-vectors diagonalizes A"

<u>Defn</u>: We say $\hat{A} \sim A$ if $\exists B$ st $\hat{A} = B^{-1}AB$
(B invertible $n \times n$)

Thm: (λ, r) e-pair of A iff $(\lambda, B^{-1}r)$ e-pr of $B^{-1}AB$

[A & $B^{-1}AB$ have <u>same</u> evals]

Pf. (λ, r) e-pr of A iff $(A-\lambda I)r = 0$

Mult on left B^{-1}: $B^{-1}Ar - \lambda B^{-1}r = 0$

$$B^{-1}ABB^{-1}r - \lambda B^{-1}r = 0$$

$$(B^{-1}AB - \lambda I)\underbrace{B^{-1}r}_{} = 0$$

eval evect

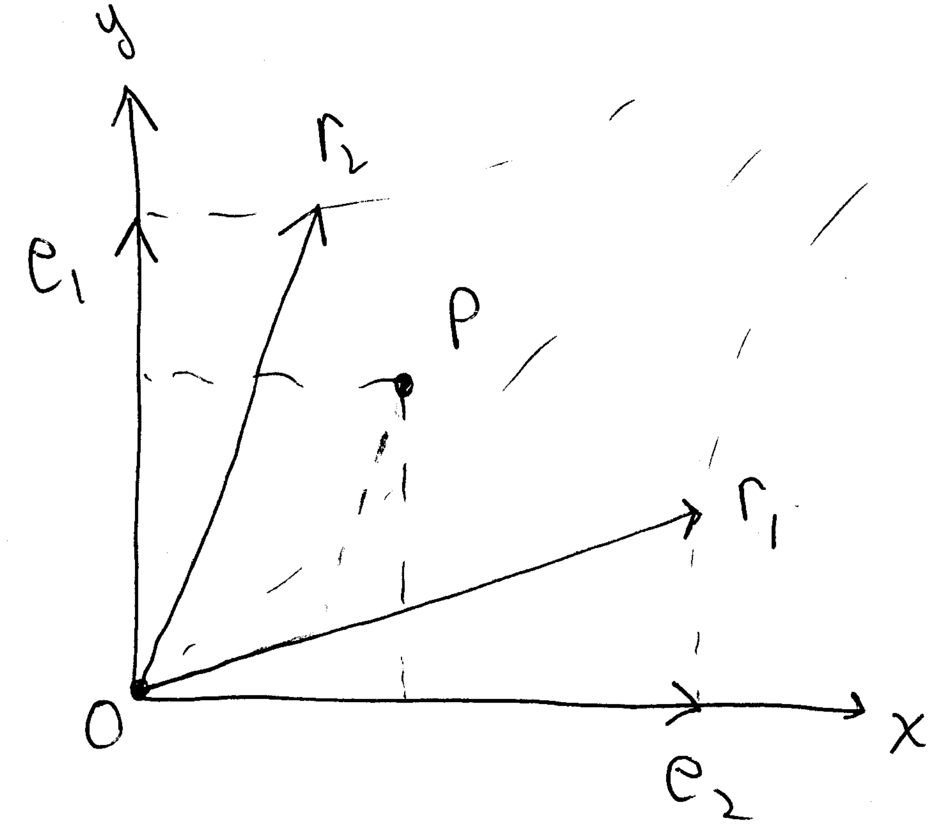
of $B^{-1}AB$ ✓

- $B^{-1}AB$ corresponds to a change of basis when viewing A as a linear transformation

I.e. $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad x \longmapsto Ax$

View x & Ax as components of vectors (in space) relative to standard basis $e_1, \dots, e_n$.

- That is, view vectors as real fixed things in space & components as measuring them wrt an arbitrary basis



$B = \begin{bmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{bmatrix}$ is <u>new</u> basis (components of the r's wrt the e's)

$$x = (x_1, \ldots, x_n)^{tr} \equiv x_1 e_1 + \cdots + x_n e_n$$

Q: $x_1 e_1 + \cdots + x_n e_n \overset{?}{=} \bar{x}_1 r_1 + \cdots + \bar{x}_n r_n$

$$= \begin{bmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix} = B\bar{x}$$

Thus: $x = B\bar{x}$ gives "comp's of a vector wrt r_i's to comp's wrt e_i's"

So: $\boxed{x = B\bar{x}, \ \bar{x} = B^{-1}x}$

$A : x \longrightarrow Ax$ (describes the mapping wrt $e_1, \ldots, e_n$)

$\underbrace{B^{-1}\underbrace{A\underbrace{B\bar{x}}_{x}}_{Ax}}_{\text{comp's of } Ax \text{ wrt } r_i\text{'s}}$ $\quad B^{-1}AB : \bar{x} \longrightarrow B^{-1}AB\bar{x}$

(describes the mapping wrt $r_1, \ldots, r_n$)

Assume A has basis $\{r_1, \dots, r_n\}$ of evectors

$$B^{-1}AB = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = D \text{ diagonal}$$

$$\frac{dx}{dt} = Dx \iff \frac{dx_1}{dt} = \lambda_1 x_1 \iff x_1 = e^{\lambda_1 t} x_1^0$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{dx_n}{dt} = \lambda_n x_n \iff x_n = e^{\lambda_n t} x_n^0$$

$$\therefore \text{ soln: } x(t) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

<u>But</u>: $$\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{(\lambda_1 t)^n}{n!} & & 0 \\ & \ddots & \\ 0 & & \sum_{n=0}^{\infty} \frac{(\lambda_n t)^n}{n!} \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \begin{bmatrix} \lambda_1^n t^n & & 0 \\ & \ddots & \\ 0 & & \lambda_n^n t^n \end{bmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_n^n \end{bmatrix} t^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} D^n t^n = e^{Dt} \text{ as defined } \checkmark$$

• **Now**: Consider $\frac{dx}{dt} = Ax$

Mult by B^{-1}: $$\frac{d}{dt}\underbrace{B^{-1}x}_{\bar{x}} = B^{-1}Ax = B^{-1}AB\underbrace{B^{-1}x}_{\bar{x}}$$

$$\frac{d}{dt}\bar{x} = D\bar{x} \Rightarrow \bar{x} = e^{Dt}\bar{x}_0$$

Now: $x = B\bar{x} \Rightarrow x = B\bar{x} = Be^{Dt}B^{-1}x_0$

$$Be^{Dt}B^{-1} = B\left(\sum_{n=0}^{\infty}\frac{D^n t^n}{n!}\right)B^{-1} = \sum_{n=0}^{\infty}\frac{BD^nB^{-1}t^n}{n!}$$

$$(BDB^{-1})^n = BDB^{-1}BDB^{-1}\cdots BDB^{-1} = BD^nB^{-1}$$

$$= \sum_{n=0}^{\infty}\frac{(BDB^{-1})^n t^n}{n!} = \sum_{n=0}^{\infty}\frac{A^n t^n}{n!} = e^{At}$$

as defined (Confirming the correctness!)

Thm: $e^{As}e^{At} = e^{A(s+t)} \Rightarrow e^{-At} = e^{At}$

$e^0 = I$, $e^{B^{-1}AB} = B^{-1}e^AB$

Not $e^{A+B} = e^Ae^B$ unless $AB = BA$