§5.2 Diagonalization

Basic Thms on eigenvalues / eigenvectors

Thm: If $(\lambda_1, r_1) \cdots (\lambda_k, r_k)$ eigenpairs from <u>different</u> evals, then $r_1 \cdots r_k$ are indep.

Pf. (induction) $n=1$ $Ar_1 = \lambda_1 r_1$ $r_1 \neq 0 \Rightarrow r_1$ "indept"

Assume $\lambda_1 \cdots \lambda_k$ distinct & $r_1 \cdots r_{k-1}$ indept. We show $r_1 \cdots r_k$ indept. Assume

$$0 = \sum_{i=1}^{k} c_i r_i = \sum_{i=1}^{k-1} c_i r_i + c_k r_k \quad (c_k \neq 0)$$

mult by A: $0 = \sum_{i=1}^{k} c_i \lambda_i r_i = \sum_{i=1}^{k-1} c_i \lambda_i r_i + c_k \lambda_k r_k$

mult 1st by λ_k: $0 = \sum_{i=1}^{k} c_i \lambda_k r_i = \sum_{i=1}^{k-1} c_i \lambda_k r_i + c_k \lambda_k r_k$

$$\sum_{i=1}^{k-1} c_i \underbrace{(\lambda_i - \lambda_k)}_{\text{not zero}} r_i = 0$$

$\Rightarrow c_i = 0$ $i = 1 \cdots k-1$ $\Rightarrow c_k = 0$ #

<u>Cor</u>: If A has n distinct evals then A has a basis of eigenvectors (Best case!)

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<u>Defn</u>: $B \sim A$ if $\exists$ nonsingular S such that

$$B = S^{-1} A S$$

<u>Thm</u>: if $B \sim A$ then $C_B(\lambda) = C_A(\lambda) \Rightarrow$ same evals

Pf. $S^{-1}(A - \lambda I) S = S^{-1} A S - \lambda I$

$$\therefore\ C_B(\lambda) = \det\left| S^{-1} A S - \lambda I \right| = \det\left| S^{-1}(A - \lambda I) S \right|$$

$$= \cancel{\det S^{-1}} \det|A - \lambda I| \cancel{\det S} = C_A(\lambda)$$

<u>Thm</u>: $Ax = \lambda x \Leftrightarrow By = \lambda y \qquad y = S^{-1} x$

Pf. $Ax = \lambda x$ iff $ASy = \lambda Sy$ iff $S^{-1} A S y = \lambda S^{-1} S y = \lambda y$

iff $By = \lambda y$ ✓

<u>Thm</u>: If A has a basis of eigenvectors $\{r_1, \dots, r_n\}$, then $A \sim \mathrm{diag}\{\lambda_1 \cdots \lambda_n\}$

Pf. Choose $S = \begin{bmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{bmatrix}$

We know $S^{-1}S = \begin{bmatrix} - & s_1 & - \\ & \vdots & \\ - & s_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

↑

"rows of S^{-1} satis $s_i \cdot r_j = \delta_{ij}$"

<u>Thus</u>: $S^{-1}AS = \begin{bmatrix} - & s_1 & - \\ & \vdots & \\ - & s_n & - \end{bmatrix} A \begin{bmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{bmatrix}$

$= \begin{bmatrix} - & s_1 & - \\ & \vdots & \\ - & s_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ \lambda_1 r_1 & \cdots & \lambda_n r_n \\ | & & | \end{bmatrix}$

$= (s_i \lambda_j r_j)_{ij} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ ✓

Ex: $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ no basis of eigenvectors

$\Rightarrow$ Not $\sim$ diag matrix!

Turns out: $A^T = A \Rightarrow$ o.n. basis of e.vectors

Thm: The eigenvalues of A^k are $\lambda_1^k \cdots \lambda_n^k$.

Moreover, if $\{r_1 \cdots r_n\}$ basis of evects for A then $S = \begin{bmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{bmatrix}$ diagonalizes A^k.

Pf. $A^k x_i = \underbrace{A \cdots A}_{k\text{-times}} x = \underbrace{A \cdots A}_{k-1 \text{ times}} \lambda_i x = \cdots \lambda_i^k x$ ✓

Also: $\Lambda = S^{-1} A S \Rightarrow \Lambda^k = (S^{-1} A S)^k$

$\Lambda^k = (S^{-1} A S) \cdots \underbrace{(S^{-1} A S)(S^{-1} A S)}_{S^{-1} A^2 S}$

$\vdots$

$= S^{-1} A^k S$ ✓

<u>Thm</u>: Assume A, B are diagonalizable. Then they diag with <u>same</u> S iff $AB = BA \equiv [A,B] = 0$

Pf. ($\Rightarrow$) $\Lambda_A = S^{-1}AS \quad \Lambda_B = S^{-1}BS$

$$\Rightarrow \Lambda_A \Lambda_B = S^{-1}ABS$$

$$\Lambda_B \Lambda_A = S^{-1}BAS$$

$$0 = S^{-1}(AB-BA)S \Rightarrow AB - BA = 0$$

($\Leftarrow$) assume $AB - BA = 0$ (For simplicity assume λ_i distinct)

Then $Ax_i = \lambda_i x_i \Rightarrow BAx_i = \lambda_i Bx_i \Rightarrow ABx_i = \lambda_i Bx_i$

$\Rightarrow Bx_i$ is a mult $x_i \Rightarrow x$ evect of B also

$\Rightarrow$ "same bassis of e-vectors"

<u>QM</u>: $[P, Q] = I \Rightarrow \|Px\| \|Qx\| \geq \frac{1}{2} \quad \forall \ x$ st $\|x\| = 1$

"cannot get Px Qx simult ~~in kernel of both~~ small!

$1 = x^T x = x^T [PQ - QP] x \underset{\uparrow \text{Schwarz}}{\leq} 2 \|Px\| \|Qx\|$