§5.3/5.4

(1)

(c) Applications:

(1) Difference Eqns

$$x_{k+1} = A x_k$$

what is limit $x_n \to \infty$?

(2) Differential Eqn's

$$\frac{dx}{dt} = Ax$$

Formula: $x(t) = x_0 e^{At}$

(2) Difference Eqn's - 2 appl's

Fibonacci Seqn's / Markov Chain

(1) $x_k = A x_{k-1} = A^2 x_{k-2} = \cdots = A^k x_0$

$\therefore x_k = A^k x_0 \longrightarrow \left(\lim_{k\to\infty} A^k\right) x_0$

Now if $A = S \Lambda S^{-1} \Rightarrow A^k = S \Lambda^k S^{-1}$

$= S \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} S^{-1} \Rightarrow A^k x_0 = \sum_{i=1}^{n} c_i x_{\lambda_i} \lambda_i^k$

$[x_{\lambda_1} \cdots x_{\lambda_n}]$

Thm (1) $x_k \to 0$ iff $|\lambda_j| < 1 \quad j = 1, \ldots, n$

① **Ex**: Fibonacci Seqv: $F_{k+1} = F_k + F_{k-1}$

$0, 1, 1, 2, 3, 5, 8, 13, 21 \ldots$ Find formula

- Write as matrix: $x_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$$x_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = A x_k$$

- Evals: $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1$

$$\lambda_1 = \frac{1+\sqrt{1+4}}{2} \qquad \lambda_2 = \frac{1-\sqrt{1+4}}{2}$$

$$\begin{bmatrix} 1-\frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \qquad a - \frac{1+\sqrt{5}}{2} b = 0$$

$b = 1, \ a = \lambda_1$

$$r_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$\therefore \quad x_k = c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \lambda_1^k + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \lambda_2^k$$

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$c_2 = -c_1, \quad c_1\lambda_1 - c_1\lambda_2 = 1$$

$$c_1 = \frac{1}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{5}}$$

$$x_k = \frac{1}{\sqrt{5}} \left\{ \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \lambda_1^k - \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \lambda_2^k \right\}$$

$$F_{k+1} = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^k \right\}$$

$$= \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} \right\} \quad \underline{\text{integer}} \; !!$$

$$-\frac{1.3}{2} < \frac{1-2.3}{2} < \frac{1-\sqrt{5}}{2} < \frac{1-2.2}{2} = -\frac{1.2}{2}$$

F_k = "nearest integer to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k$

⊠ Markov Chain – Wikopedia

- Assume "transition matrix"

$$P = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix} = P_{ij} \qquad \begin{bmatrix} \text{sunny} \\ \text{rainy} \end{bmatrix} \begin{bmatrix} S \\ r \end{bmatrix}$$

Ie. "Prob that a sunny day is followed by a sunny day is 90% ; Prob a rainy day is followed by a rainy day is 50%"

P_{ij} = Prob that if a given day is type j the next day will be type i

- Colms sum to $\underline{1}$
- Start $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $x_k = \begin{pmatrix} s_k \\ r_k \end{pmatrix}$

s_k = prob day-k is sunny

r_k = prob " " " rainy

Then: $x_1 = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .9 \\ .1 \end{bmatrix}$

$$x_2 = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix} \begin{bmatrix} .9 \\ .1 \end{bmatrix} = \begin{bmatrix} .81 + .05 \\ .09 + .05 \end{bmatrix}$$

$$\vdots$$

$$x_n = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Note: Since each colm sums to 1, the sum of the rows is (1,1). Since sum of rows of $-I$ is $(-1,-1)$, the sum of the rows of $(A-I)$ is $(0,0)$ $\Rightarrow$ $A-I$ has a nonzero kernel $\Rightarrow$ 1 is an eval!

$\therefore \det|A-\lambda I| = \begin{vmatrix} .9-\lambda & .5 \\ .1 & .5-\lambda \end{vmatrix} = (.9-\lambda)(.5-\lambda)-(.1)(.5)$

$\lambda^2-(.5+.9)\lambda + ((.9)(.5)-(.5)(.1)) = 0$

$(\lambda-1)(\lambda-d) = \lambda^2-(1+d)+d = 0$

$1+d = 1.4 \quad d = .4$

$\lambda_1 = 1 \quad \begin{bmatrix} .9-1 & .5 \\ .1 & .5-1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \quad \begin{matrix} (.1)a-(.5)b = 0 \\ b=1,\ a=5 \end{matrix}$

$r_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$\lambda_2 = .4 \quad \begin{bmatrix} .9-.4 & .5 \\ .1 & .5-.4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \quad \begin{matrix} (.1)a+(.1)b = 0 \\ -a=b=1 \end{matrix}$

$r_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$x_k = c_1\begin{bmatrix} +5 \\ 1 \end{bmatrix}\cdot 1^k + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}(.4)^k$

$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} +5 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad c_2 = -c_1 \Rightarrow 1 = c_1(+5+1) = +6c_1$

$c_1 = +\frac{1}{6}$

Conclude:

$$x_k = +\frac{1}{6}\begin{bmatrix} +5 \\ 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -1 \\ 1 \end{bmatrix} (.4)^k$$

$$\lim_{k \to \infty} x_k = \begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix}$$

$\Rightarrow$ "prob sunny at time $k \gg 1$ is $5/6$"