§5.4 e^{At}

Review:

- If $Ax = \lambda x$, then $xe^{\lambda t}$ solves

$$\frac{dx}{dt} = Ax \qquad (*)$$

- Eigenvectors from different evals are <u>indept</u>

- Evals solve $C_A(\lambda) = \det|A - \lambda I| \Rightarrow$ complex evals come in complex conj prs $\lambda_\pm = a \pm ib$

Then $(*)$ holds with

$$e^{\lambda_\pm t} = e^{at} e^{\pm ibt} = e^{at}\{\cos bt \pm i \sin bt\}$$

& real soln's can be extracted from two imaginary soln's

- Except in degenerate cases $\left(\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}\right)$ $\exists$ a <u>basis</u> of e-vectors. Then we can solve (*) [We restrict to this case!]

$$x(t) = \sum_{i=1}^{n} c_i x_i e^{\lambda_i t}$$

In case $\lambda_\pm = a \pm ib$, $x_- e^{\lambda_- t} = \bar{x}_- e^{\bar{\lambda}_- t}$

($\overline{a+ib} = a-ib$ ~~complex~~ complex conjugation: $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$)

Then

$$x_+ e^{(a+ib)t}, \quad x_- e^{(a-ib)t} = \bar{x}_+ e^{(a-ib)t}$$

are the two complex sol'ns, &

$$x_+ e^{(a+ib)t} = x_+ e^{at}(\cos bt + i \sin bt)$$
$$x_- e^{(a-ib)t} = \bar{x}_+ e^{at}(\cos bt - i \sin bt)$$

$$\Rightarrow \Big\{ \underbrace{(x_+ + \bar{x}_+)}_{\text{real}} e^{at}\cos bt, \ \underbrace{i(x_+ - \bar{x}_+)}_{\text{real}} e^{at} \sin bt \Big\} \text{ are 2 real sol'ns}$$

$$\Rightarrow \quad c_+ x_+ e^{\lambda_+ t} + c_- x_- e^{\lambda_- t} \quad \text{(complex)}$$

Can be replaced by the real pair

$$\hat{c}_+ (x_+ + \bar{x}_+) e^{at} \cos bt + \hat{c}_- (x_+ - \bar{x}_+) i e^{at} \sin bt$$

in the sum

$$x(t) = \sum_{i=1}^{n} c_i x_i e^{\lambda_i t}$$

$\Rightarrow$ real soln. To meet i-condts

$$x(0) = \sum_{i=1}^{n} c_i x_i$$

$(x_+ + \bar{x}_+), (x_+ - \bar{x}_+) i$ replacement for complex conj pairs. Still a real basis after replacement!

Easier to just keep complex soln's!

Thm: When $\exists$ basis of eigenvectors: $\exists$ nonsingular $S_{n\times n}$ s.t.

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

In fact:

$$S = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}$$

basis of e-vectors.

Pf.

$$\underbrace{\begin{bmatrix} — r_1 — \\ \vdots \\ — r_n — \end{bmatrix}}_{S^{-1}} A \underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{S} = \begin{bmatrix} — r_1 — \\ \vdots \\ — r_n — \end{bmatrix} \begin{bmatrix} | & & | \\ \lambda_1 x_1 & \cdots & \lambda_n x_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \checkmark$$

$r_i \cdot x_j = \delta_{ij}$

$S^{-1}S = id$

Ex: Show that if A has a basis of e-vectors, then A^n has same e-vector & evals are λ^n

Soln: $Ax = \lambda x \Rightarrow A^2 x = A\lambda x = \lambda^2 x$

$\ldots \Rightarrow A^n x = \lambda^n x$ ✓

Also: $A^n x = (S^{-1}\Lambda S)^n x = (S^{-1}\Lambda S S^{-1}\Lambda S \cdots S^{-1}\Lambda S)$

$= S^{-1}\Lambda^n S$

$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ ✓

☐ In fact: $\exists$ general soln to

$$\frac{dx}{dt} = Ax$$

$$x(0) = x_0$$

i-condition
$x \in \mathbb{R}^n$

Soln: $$x(t) = \underbrace{e^{At}}_{\substack{n \times n \\ \text{matrix}}} x$$

Defn: $$e^M = I + M + \frac{1}{2}M^2 + \cdots + \frac{1}{n!}M^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{M^n}{n!} \qquad M \equiv n \times n \text{ matrix}$$

$$e^{At} = I + At + \frac{(At)^2}{2} + \cdots + \frac{(At)^n}{n!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

<u>Thm</u>: e^{At} alway exists as $n \times n$ matrix

& $\frac{d}{dt} e^{At} = A e^{At}$

"Proof" $n!$ increases faster than A^n $\Rightarrow$ fast convergence (as matrices) $\Rightarrow$ can diff

TxT:

$$\frac{d}{dt} e^{At} = 0 + A + \frac{2At}{2} + \cdots + \underbrace{\frac{n(At)^{n-1} \cdot A}{n!}}_{A \frac{A^{n-1}}{(n-1)!} t^{n-1}} + \cdots$$

$$= A \cdot \left(\sum_{n=0}^{\infty} \frac{(At)^n}{n!} \right) \checkmark$$

<u>OR</u> $\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{n A^n t^{n-1}}{n!} = A \sum_{n=0}^{\infty} \frac{(At)^n}{n!} \checkmark$

(A complete proof needs to prove the series converges in the "matrix norm" fast enough so TxT diff justified)

<u>Thm</u>: $e^{As} \cdot e^{At} = e^{A(s+t)}$

Pf (T×T mult of series + fast convergence ≈ regular exponential)

<u>Cor</u>: $e^{-At} \cdot e^{At} = e^{A(-t+t)} = e^{0} = I$

$\therefore$ e^{At} always has an invers, & that is e^{-At} ✓

(All works fine with complex A!)

<u>Thm</u>: $e^{S^{-1}AS} = S^{-1} e^{A} S$

"Pf" $e^{S^{-1}AS} = \sum_{n=0}^{\infty} \frac{(S^{-1}AS)^n}{n!} = \sum_{n=0}^{\infty} \frac{S^{-1}A^n S}{n!} = S^{-1}\left(\sum_{n=0}^{\infty} \frac{A^n}{n!}\right) S$

<u>Thm</u>: $A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$

Pf. $e^{At} = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_n^n \end{bmatrix} = \begin{bmatrix} \sum \frac{\lambda_1^n}{n!} & & \\ & \ddots & \\ & & \sum \frac{\lambda_n^n}{n!} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}$

Now: Assume A has a basis of e-vectors $x_1, \dots, x_n$, $A = S\Lambda S^{-1}$ $(S^{-1}AS = \Lambda)$

Then: Soln $x(t) = e^{At}x_0$ has form

$$e^{At}x_0 = e^{(S\Lambda S^{-1})t} = S e^{\Lambda t} S^{-1} = S \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} S^{-1} x_0$$

$$= \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} S^{-1} x_0$$

$$= \begin{bmatrix} | & & | \\ x_1 e^{\lambda_1 t} & \cdots & x_n e^{\lambda_n t} \\ | & & | \end{bmatrix} \underbrace{S^{-1} x_0}_{= \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}$$

$$= c_1 x_1 e^{\lambda_1 t} + \cdots + c_n x_n e^{\lambda_n t} \checkmark$$

Recover same formula —

Stability: If $\lambda = a + ib$ then

$$e^{\lambda t} = e^{at} e^{ibt} = e^{at}(\cos bt + i \sin bt)$$

$$\xrightarrow{t \to \infty} 0 \quad \text{iff} \quad a < 0$$

Thm: The soln $x(t) = e^{At} x_0 \xrightarrow{t \to \infty} 0$

iff $Re\{\lambda_i\} < 0 \ \forall i$ ✓