5.4 $e^{st}$

- Review:
  - If $Ax = \lambda x$, then $xe^{\lambda t}$ solves
    \[
    \frac{dx}{dt} = Ax
    \] (*)

- Eigenvectors from different evals are independent.

- Eval solve $\mathcal{C}_n(\lambda) = \det(A - \lambda I) = 0$ complex evals come in complex conjugates $\lambda_\pm = a \pm ib$
  Then (*) holds with
  \[
  e^{\lambda t} = e^{a t} e^{\pm ibt} = e^{a t} \{\cos bt + i \sin bt\}
  \] 
  & real soln's can be extracted from two imaginary soln's.
Except in degenerate cases \( [\begin{array}{c} 1 \\ 0 \\ 1 \end{array}] \) \( \in \) a basis of e-vectors. Then we can solve (*) [We restrict to this case!]

\[
x(t) = \sum_{i=1}^{n} c_i x_i e^{\lambda_i t}
\]

In case \( \lambda = a \pm ib \), \( x e^{\lambda t} = \overline{x} e^{-\lambda t} \)

\[
(a + ib) = a - ib \text{ complex conjugate} \quad \overline{z_1 z_2} = \overline{z_1} \overline{z_2} \quad \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}
\]

Then

\[
x_+ e^{(a+ib)t}, \quad x_- e^{(a-ib)t}
\]

are the two complex solutions

\[
x_+ e^{(a+ib)t} = x_+ e^{at} (\cos bt + i \sin bt)
\]

\[
x_- e^{(a-ib)t} = \overline{x}_+ e^{at} (\cos bt - i \sin bt)
\]

\[
\left\{ \frac{(x_+ + \overline{x}_+) e^{at} \cos bt}{\text{real}}, \frac{i (x_+ - \overline{x}_+) e^{at} \sin bt}{\text{real}} \right\} \text{ are 2 real solutions}
\]
\[ c_+ x_+ e^{\lambda_+ t} + c_- x_- e^{\lambda_- t} \text{ (complex)} \]

Can be replaced by the real pair:

\[ c_+(x_++\bar{x}_+) e^{at} \cos bt + c_-(x_+-\bar{x}_+) i e^{at} \sin bt \]

in the sum

\[ x(t) = \sum_{i=1}^{n} c_i x_i e^{\lambda_i t} \]

\[ \Rightarrow \text{real soln. To meet i-conds} \]

\[ x(0) = \sum_{i=1}^{n} \bar{c}_i \bar{x}_i \]

\( (x_++\bar{x}_+) , (x_+-\bar{x}_+) i \) replaced for complex conj pairs. Still a real basis after replacment!

Easier to just keep complex soln's!
Thm: When $E$ basis of eigenvectors: $E$ nonsingular $S_{n \times n} \Rightarrow$

\[ S^{-1}AS = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix} \]

In fact:

\[ S = \begin{bmatrix} 1 & \cdots & 1 \\ \chi_1 & \cdots & \chi_n \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \]

basis of e-vectors.

Pf.

\[ \begin{bmatrix} -r_1 \\ \vdots \\ -r_n \end{bmatrix} A \begin{bmatrix} 1 \\ \chi_1 \\ \vdots \\ \chi_n \end{bmatrix} = \begin{bmatrix} -r_1 \\ \vdots \\ -r_n \end{bmatrix} \begin{bmatrix} 1 \\ \chi_1 \\ \vdots \\ \chi_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \]

\[ r_i \cdot \chi_j = \delta_{ij} \]

\[ S \cdot S = I \]
Ex: Show that if $A$ has a basis of eigenvectors, then $A^n$ has same eigenvectors $\lambda$.

**Soln:** $Ax = \lambda x \implies A^2 x = A(Ax) = \lambda^2 x$

$\vdots \implies A^n x = \lambda^n x$

Also: $A^n x = (S^{-1} \Lambda S)^n x = (S^{-1} \Lambda S \cdots S^{-1} \Lambda S)^n x = S^{-1} \Lambda^n S$

\[ \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix} \]
In fact: the general solution to
\[
\frac{dx}{dt} = Ax
\]
\[x(0) = x_0\]

**Solution:**
\[x(t) = e^{At} \quad \text{where} \quad A \in \mathbb{R}^{n \times n} \text{ matrix}\]

**Definition:**
\[e^M = I + M + \frac{1}{2!} M^2 + \cdots + \frac{1}{n!} M^n + \cdots\]
\[= \sum_{n=0}^{\infty} \frac{M^n}{n!} \quad M \in \mathbb{R}^{n \times n} \text{ matrix}\]

\[e^{At} = I + At + \frac{(At)^2}{2} + \cdots + \frac{(At)^n}{n!} + \cdots\]
\[= \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} \]
Thm: \( e^{At} \) always exists as an \( n \times n \) matrix.

\[
\frac{d}{dt} e^{At} = A e^{At}
\]

"Proof": \( n! \) increases faster than \( A^n \) \( \Rightarrow \) fast convergence (as matrices) \( \Rightarrow \) can differentiate.

\[
\frac{d}{dt} e^{At} = 0 + A + \frac{2A^2t}{2} + \ldots + \frac{n (A^t)^{n-1}}{n!} + \ldots
\]

\[
= A \left( \sum_{n=0}^{\infty} \frac{(At)^n}{n!} \right) = A \sum_{n=0}^{\infty} \frac{A^n t^n}{(n-1)!}
\]

Or

\[
\frac{d}{dt} e^{At} = \sum_{n=0}^{\infty} \frac{A^{n+1}t^n}{n!} = \sum_{n=0}^{\infty} \frac{nA^n t^{n-1}}{n!} \cdot A \sum_{n=0}^{\infty} \frac{(At)^n}{n!}
\]

(A complete proof needs to prove the series converges in the "matrix norm" fast enough so TXT diff \( A \) justified.)
Thm: \( e^{At} \cdot e^{At} = e^{A(s+t)} \)

pf (1x1 mult of series + fast convergence => regular exponential)

Cor: \( e^{-At} \cdot e^{At} = e^{A(-t+t)} = e^0 = I \)

\( e^{At} \) always has an inverse, \( B \) that is \( e^{-At} \)

(All works fine with complex \( A \)!

Thm: \( e^SAS = S^{-1}e^A S \)

pf: \( e^SAS = \sum_{n=0}^{\infty} \frac{(S^{-1}AS)^n}{n!} = \sum_{n=0}^{\infty} \frac{S^{-1}AS^n S}{n!} = S^{-1} \sum_{n=0}^{\infty} \frac{A^n S}{n!} S \)

Thm: \( A = \begin{bmatrix} x_1 & 0 \\ 0 & x_n \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{x_1 t} & 0 \\ 0 & e^{x_n t} \end{bmatrix} \)

pf: \( e = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} x_1^n & 0 \\ 0 & x_n^n \end{bmatrix} = \begin{bmatrix} e^{x_1 t} & 0 \\ 0 & e^{x_n t} \end{bmatrix} \)
Now: Assume $A$ has a basis of e-vectors $x_1, \ldots, x_n$, $A = S \Lambda S^{-1}$ ($S^T AS = \Lambda$)

Then: Soln $x(t) = e^{At}x_0$ has form

$e^{At}x_0 = e^{(S^{-1} \Lambda S)t} = S e^{\Lambda t} S^{-1} = S \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_n t} \end{bmatrix} S^{-1} x_0$

$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_n t} \end{bmatrix} S^{-1} x_0$

$= \begin{bmatrix} x_1 e^{\lambda_1 t} & \cdots & x_n e^{\lambda_n t} \\ 1 & \cdots & 1 \end{bmatrix} S^{-1} x_0 \sim \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$

$= c_1 x_1 e^{\lambda_1 t} + \cdots + c_n x_n e^{\lambda_n t}$

Recover same formula —
Stability: If \( \lambda = a + ib \) then
\[
e^{\lambda t} = e^{at} e^{ibt} = e^{at} (\cos bt + is \sin bt)
\]
\( t \to \infty \) if and only if \( a < 0 \)

Thm: The solution \( x(t) = e^{At} x_0 \) \( t \to \infty \) if and only if \( \text{Re} \{ \lambda_i \} < 0 \) for all \( i \).