§ 5.5 Symmetric / Hermitian Matrices

- **Defn**: a real valued matrix $A_{n\times n}$ is symmetric if $A^T = A$

- Let $\langle x, y\rangle = x \cdot y = x^T \cdot y$

$$\langle x,y\rangle = \underset{\text{dot}}{x \cdot y} = \underset{\text{matrix}}{x^T \cdot y}$$

Then $A^T = A$ iff $x^T A y = (x^T A y)^T$

$$= (y^T A^T x$$

$$= y^T A x$$

or said differently: $x \cdot (Ay) = y \cdot (Ax)$

or said differently: $\boxed{\langle x, Ay\rangle = \langle Ax, y\rangle}$

"A symmetric iff we can move A across inner product $\langle \, , \rangle$"

<u>Thm ①</u>: If A is symmetric, then A has real eigenvalues:

<u>Proof</u>: For complex vectors $x, y \in \mathbb{C}^n$, define complex inner product

$$\langle x, y \rangle = \bar{x} \cdot y \qquad \bar{x} = \text{complex conj of } x$$

Eg: $\overline{a+ib} = a - ib$, $x = \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix} \Rightarrow \bar{x} = \begin{pmatrix} a_1 - ib \\ a_2 - ib \end{pmatrix}$

etc. Then $(a+ib)(a-ib) = a^2 + b^2$ real & positive

$$x = \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix}, \; y = \begin{pmatrix} c_1 + id_1 \\ c_2 + id_2 \end{pmatrix}, \; \bar{x} \cdot y = (a_1 \; ib_1)(c_1 + id_1) + (a_2 + ib_2)(c_2 + id_2)$$

The bar in $\bar{x} \cdot y$ is <u>required</u> so that $|x| = \bar{x} \cdot x$ is real & positive *

<u>Then</u>: A real & symm $\Rightarrow \langle Ax, y \rangle = \langle x, Ay \rangle$ in the <u>complex</u> sense also: $(\overline{Ax}) \cdot y = \bar{x} \cdot Ay$

since $(A\bar{x}) \cdot y = \bar{x} \cdot Ay$ ✓

Now we show evals of A are real:

$Ax = \lambda x$ (complex sense) $\Rightarrow$

$$\langle Ax, x \rangle = \langle x, Ax \rangle$$

$$\langle \lambda x, x \rangle = \langle x, \lambda x \rangle$$

$$(\bar{\lambda}\,\bar{x}) \cdot x = \bar{x} \cdot (\lambda x)$$

$$\bar{\lambda}\, \cancel{|x|^2} = \lambda\, \cancel{|x|^2}$$

$\bar{\lambda} = \lambda$ iff λ is real ✓

<u>Thm 2</u>: A real & symmetric $\Rightarrow$ A has an ON. basis of eigenvectors

<u>Proof</u>: 1 Eigenvectors associated with different eigenspaces are $\perp$; I.e.

$Ax_1 = \lambda_1 x_1$ & $Ax_2 = \lambda_2 x_2$ $\Rightarrow$

$$\langle x_1, Ax_2 \rangle = \langle Ax_1, x_2 \rangle$$

$$\langle x_1, \lambda_2 x_2 \rangle = \langle \lambda_1 x_1, x_2 \rangle$$

$$\underbrace{\lambda_2}_{\text{real}} \underbrace{(x_1 \cdot x_2)}_{\text{dot prod}} = \underbrace{\lambda_1}_{\text{real}} \underbrace{(x_1 \cdot x_2)}_{\text{dot prod}}$$

Since $\lambda_1 \neq \lambda_2$, the only way this can hold is if $x_1 \cdot x_2 = 0$ ✓

<u>Thus</u>: if $\exists$ n distinct evals of $A = A^T$, then $\exists$ ON basis of e-vectors

2. If not all the evals of A are distinct, then still the evectors of A can be completed to an O.N. basis (Shur's Lemma §5.6) (we skip this pf)

Cor: $A = A^T \Rightarrow \exists$ matrix S st

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, \quad \lambda_i \text{ real}$$

$$S = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \quad \{x_1, \ldots, x_n\} \text{ ON basis.}$$

Since S is orthogonal, $S^{-1} = S^T \Rightarrow$

$$S^T A S = \Lambda \equiv \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

* This is the single most important result of the class *

☐ Hermitian matrices:

- In math physics, complex valued matrices A are important —

<u>Defn</u>: $A_{n\times n}$ (complex) is Hermitian Symmetric if

$$\langle Ax, y\rangle = \langle x, Ay\rangle \iff \bar{A}^T = A$$

$\forall$ complex x, y $\quad \langle x,y\rangle = \underset{\substack{\uparrow\\ \text{dot}}}{\bar{x}\cdot y} = \underset{\substack{\uparrow\\ \text{matrix}}}{\bar{x}^T y}$

<u>Defn</u>: $A^H \equiv \bar{A}^T$

<u>Thm</u>: If A is Hermitian, then evals of A are real, & $\exists$ o.n. basis of (complex) eigenvectors

Pf. (exactly same proof!)

Real evals: $\langle Ax, x\rangle = \langle x, Ax\rangle$

$$\langle \lambda x, x\rangle = \langle x, \lambda x\rangle$$

$$\bar{\lambda}\cancel{(\bar{x}\cdot x)} = \cancel{(\bar{x}\cdot x)}\lambda$$

$$\bar{\lambda} = \lambda \checkmark$$

$x_1 \perp x_2$: $\langle Ax_1, x_2\rangle = \langle x_1, Ax_2\rangle$

$$\langle \lambda_1 x_1, x_2\rangle = \langle x_1, \lambda_2 x_2\rangle$$

$$\lambda_1 \underbrace{\cancel{(\bar{x}_1\cdot x_2)}}_{0} = \lambda_2 \underbrace{\cancel{(\bar{x}_1\cdot x_2)}}_{0}$$

$\lambda_1 \neq \lambda_2 \Rightarrow \bar{x}_1\cdot x_2 = 0 \checkmark$

<u>That</u> x_i can be completed to ON-basis when A has repeated evals follows by same appl. of Shur's lemma ... (omit)

Thm: A Hermitian implies

$$U^{-1} A U = \Lambda$$

where U is unitary

Pf. Choose $U = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}$ (complex)

U unitary if colms of U are ON. in complex sense

Thm: U unitary $\Rightarrow$ ① $U^{-1} = \bar{U}^T = U^H$

and ② $|\lambda| = 1$ $\forall$ eval of U

Pf. ①

$$\underbrace{\begin{bmatrix} - \bar{x}_1 - \\ \vdots \\ - \bar{x}_n - \end{bmatrix}}_{U^H} \underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{U} = (\underbrace{\bar{x}_i \cdot x_j}_{\text{dot}}) = \delta_{ij}$$

① $\Rightarrow$ $\|Ux\|^2 = (\overline{Ux})^T Ux = \bar{x}^T \underbrace{\bar{U}^T U}_{\text{id}} x = \bar{x}^T x = |x|^2$

(length preserving)

For 2 Use: If $Ux = \lambda x$, then

$$\|\cancel{U}x\| = \|\lambda x\| = |\lambda| \, \|\cancel{x}\|$$

$$\|Ux\| = \|x\|$$

$$\Rightarrow \boxed{1 = |\lambda|}$$