Symmetric Matrices & SVD http://math.ucdavis.edu/class-accts (0)

e-vals / e-vectors   put in CRN for class 50106

(20)

(λ, x) e-pair for $\underset{n\times n}{A}$ iff $Ax = \lambda x$

- Only makes sense for square matrices
- λ eval iff $\det |A - \lambda I| = 0$.
- $\lambda = 0 \Rightarrow r \in \mathrm{Ker}(A)$. Thus A^{-1} exists iff A has no zero eigenvalue
- If A^{-1} exists, then A^{-1}

Ie: $Ax = \lambda x \Rightarrow A^{-1}Ax = \lambda A^{-1}x \quad A^{-1}x = \frac{1}{\lambda}x$

- $\underset{m\times n}{A}$ evals make sense only when $m = n$

$\underbrace{\underset{n\times m}{A^T}\,\underset{m\times n}{A}}_{n\times n}$ & $\underbrace{\underset{m\times n}{A}\,\underset{n\times m}{A^T}}_{m\times m}$ both symmetric square matrices of different sizes but...

They both have the same nonzero evals

$\underset{n\times n}{A^TA}\,\underset{n\times 1}{x} = \underset{n\times 1}{\lambda x} \Rightarrow \underset{m\times n}{A}\underbrace{\underset{n\times n}{A^TA}}x = \lambda \underset{m\times n}{A}\,\underset{n\times 1}{x} \Rightarrow AA^T(Ax) = \lambda(Ax)$

So $\lambda \neq 0$ eval of $A^TA \Rightarrow Ax \neq 0 \Rightarrow \lambda$ eval of AA^T

Same for $AA^Tx = \lambda x$

In fact we have:

(λ, x) e-pair for A^TA & $\lambda \neq 0 \Rightarrow (\lambda, Ax)$ e-pr for AA^T

<u>Cor</u>: The nonzero e-spaces of A^TA & AA^T have same dimension

Pf. $\{x_1, \ldots, x_k\}$ basis for λ-eigenspace of A^TA

$\Rightarrow \{Ax_1, \ldots, Ax_k\}$ are in λ-eigenspace of AA^T

& $c_1Ax_1 + \cdots + c_kAx_k = c_1\lambda x_1 + \cdots + c_k\lambda x_k = 0$

iff $c_1x_1 + \cdots + c_kx_k = 0$ so $\{Ax_1, \ldots, Ax_k\}$ indept.

<u>Thus</u>: dim λ-eigenspace of $A^TA \leq$ dim λ-eigenspace of AA^T

<u>Now</u> if A^T in place of A & it goes other way!

§ 5.5 Symmetric Matrices

- **Defn**: a real valued matrix $A_{n\times n}$ is **symmetric** if $A^T = A$

- Let $\langle x, y\rangle = x \cdot y = x^T \cdot y$

$$x \underset{\text{dot}}{\overset{\uparrow}{\cdot}} y \qquad x^T \underset{\text{matrix}}{\overset{\uparrow}{\cdot}} y$$

Then $A^T = A$ iff
$$x^T A y = (x^T A y)^T$$
$$= (y^T A^T x$$
$$= y^T A x$$

or said differently: $x \cdot (Ay) = y \cdot (Ax)$

or said differently: $\boxed{\langle x, Ay\rangle = \langle Ax, y\rangle}$

"A symmetric iff we can move A across inner product $\langle\, ,\rangle$"

<u>Thm ①</u>: If A is symmetric, then A has real eigenvalues:

<u>Proof</u>: For complex vectors $x, y \in \mathbb{C}^n$, define complex inner product

$$\langle x, y \rangle = \bar{x} \cdot y \qquad \bar{x} = \text{complex conj of } x$$

Eg: $\overline{a+ib} = a-ib$, $x = \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix} \Rightarrow \bar{x} = \begin{pmatrix} a_1 - ib \\ a_2 - ib \end{pmatrix}$

etc. Then $(a+ib)(a-ib) = a^2+b^2$ real & positive

$x = \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix}$, $y = \begin{pmatrix} c_1 + id_1 \\ c_2 + id_2 \end{pmatrix}$, $\bar{x} \cdot y = (a_1 - ib_1)(c_1 + id_1)$
$+ (a_2 - ib_2)(c_2 + id_2)$

* The bar in $\bar{x} \cdot y$ is <u>required</u> so that $|x| = \bar{x} \cdot x$ is real & positive *

<u>Then</u>: A real & symm $\Rightarrow \langle Ax, y \rangle = \langle x, Ay \rangle$ in the <u>complex</u> sense also: $(\overline{Ax}) \cdot y = \bar{x} \cdot Ay$

since $(A\bar{x}) \cdot y = \bar{x} \cdot Ay$ ✓

Now we show evals of A are real:

$Ax = \lambda x$ (complex sense) $\Rightarrow$

$$\langle Ax, x\rangle = \langle x, Ax\rangle$$

$$\langle \lambda x, x\rangle = \langle x, \lambda x\rangle$$

$$(\bar{\lambda}\,\bar{x})\cdot x = \bar{x}\cdot(\lambda x)$$

$$\bar{\lambda}\,\cancel{|x|^2} = \lambda\,\cancel{|x|^2}$$

$\bar{\lambda} = \lambda$ iff λ is real ✓

Thm 2: A real & symmetric $\Rightarrow$ A has an ON. basis of eigenvectors

Proof: 1 Eigenvectors associated with different eigenspaces are $\perp$; I.e.

$Ax_1 = \lambda_1 x_1$ & $Ax_2 = \lambda_2 x_2 \Rightarrow$

$$\langle x_1, Ax_2 \rangle = \langle Ax_1, x_2 \rangle$$

$$\langle x_1, \lambda_2 x_2 \rangle = \langle \lambda_1 x_1, x_2 \rangle$$

$$\underbrace{\lambda_2}_{\text{real}} \underbrace{(x_1 \cdot x_2)}_{\text{dot prod}} = \underbrace{\lambda_1}_{\text{real}} \underbrace{(x_1 \cdot x_2)}_{\text{dot prod}}$$

Since $\lambda_1 \neq \lambda_2$, the only way this can hold is if $x_1 \cdot x_2 = 0$ ✓

Thus: if $\exists$ n distinct evals of $A = A^T$, then $\exists$ ON basis of e-vectors

② If not all the evals of A are distinct, then still the evectors of A can be completed to an o.n. basis (Shur's Lemma §5.6) (we skip this pf)

Cor: $A = A^T \Rightarrow \exists$ matrix S s.t.

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, \quad \lambda_i \text{ real}$$

$$S = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \quad \{x_1, \ldots, x_n\} \text{ on basis.}$$

Since S is orthogonal, $S^{-1} = S^T \Rightarrow$

$$S^TAS = \Lambda \equiv \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

* This is the single most important result of the class *

Cor: $\underset{m\times n}{A} \Rightarrow A^TA$ & AA^T both have real evals & o.n. basis of eigenvectors.

We also know they have same nonzero e-vals.

Thm: A nonzero evals of A^TA & AA^T are positive (We say A^TA & AA^T positive semi-definite)

Proof (*) $A^TA = V^{-1}DV \quad V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

$\{v_i\}$ o.n. basis of eigenvectors

V orthogonal $\Rightarrow V^{-1} = V^T$ & $V = \begin{bmatrix} - & r_1 & - \\ & \vdots & \\ - & r_n & - \end{bmatrix}$ o.n.

• $Vr_i = \begin{bmatrix} - & r_1 & - \\ & \vdots & \\ - & r_n & - \end{bmatrix}\begin{bmatrix} | \\ r_i \\ | \end{bmatrix} = e_i$

(*) $\Rightarrow v_i^TA^TAv_i = \underbrace{v_i^TV^T}_{e_i^T}\begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ 0 & & \sigma_r & \\ & & & \ddots \ \sigma_n \end{bmatrix}\underbrace{Vv_i}_{e^+} = \sigma_i$

$\|Av\|^2 \geq 0$ ✓

☒ We have: II SVD Cont from (20)

- Thm: $\underset{n\times n}{A^T} = \underset{n\times n}{A} \Rightarrow$ real evals & O.N. basis of e-vects.

In particular — $A = VDV^T$ $\quad D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

- A^TA & AA^T symmetric $\Rightarrow$ $\quad V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ ON basis

- $\underset{m\times n}{A}\underset{n\times m}{A^T} = \underset{m\times m}{U}\underset{m\times m}{D_1}\underset{m\times m}{U^T} \quad A^TA = \underset{n\times n}{V}\underset{n\times n}{D_2}\underset{n\times n}{V^T}$

$U = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix}, V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ ON. bases

- (λ, v) nonzero epair of $A^TA \Rightarrow (\lambda, Av)$ e-pr of AA^T

& same nonzero evals & same dimension of e-spaces

$$\Rightarrow D_1 = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}_{m\times m} \quad D_2 = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}_{n\times n}$$

Same evalues — could be repeated if e-space has dim > 1.

- <u>Thm</u>: $\lambda_i = \sigma_i^2 > 0$ $\left(\begin{array}{l}\text{nonzero evals of } A^TA \\ \text{\& } AA^T \text{ are \underline{positive}}\end{array}\right)$

(Return to this) Order so $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_r^2 \geq 0$

- Now assume $\{v_1, \ldots, v_n\}$ ON basis of e-vects

$$\|Av_i\|^2 = \langle Av_i, Av_i \rangle = v_i^T \underbrace{A^TAv_i}_{\sigma_i^2 v_i} = \sigma^2$$

$\therefore$ $\boxed{u_i = \frac{1}{\sigma_i} Av_i}_{*}$ is correspondg evect of AA^T

<u>Check</u>: $\langle u_i, u_j \rangle = 0$ if $\sigma_i^2 \neq \sigma_j^2$.

$$\sigma_i^2 = \sigma_j^2 \Rightarrow \langle u_i, u_j \rangle = \left\langle \frac{1}{\sigma} Av_i, \frac{1}{\sigma} Av_j \right\rangle$$

$$= \frac{1}{\sigma^2} \langle Av_i, Av_j \rangle = \frac{1}{\sigma^2} v_i^T A^T A v_j$$

$$= \frac{1}{\sigma^2} \cdot \sigma^2 \, v_i \cdot v_j = 0 \quad v_i \perp v_j$$

$\therefore$ (*) <u>define</u> ON. basis of u's in terms of v's.

② Thm (SVD)

$$A_{m\times n} = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ 0 & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \, 0 \end{bmatrix} \begin{bmatrix} — & v_1 & — \\ & \vdots & \\ — & v_n & — \end{bmatrix}$$

Check on basis —

$\boxed{A v_j = \sigma_j u_j}$ (construction of u_j from v's)

$$U \Sigma V^T v_j = U \sigma_i e_i = \sigma_j u_j \quad \checkmark$$

Both agree on basis, so both are equal —

Thm: A^TA & AA^T are both pos semi-defi with the same nonzero evals $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_r^2$

To prove this we need:

Lemma: $A_{n\times n}$ symmetric. Then [A is pos semi-def] (Defn) iff

evals $\lambda_i \geq 0$

$$v^TAv \geq 0 \quad \forall v \in \mathbb{R}^n$$

Pf. ($\Rightarrow$) If evals of A satis $\lambda_i \geq 0$, then

$$A = SDS^T \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \lambda_i \geq 0$$

So

$$v^TAv = (v^TS)D(S^Tv) = (S^Tv)^TD\underbrace{(S^Tv)}_{w=(w_1,\dots,w_n)}$$

$$= w^TDw = \sum_{i=1}^n w_i^2\lambda_i \geq 0 \checkmark$$

<u>Note</u>: The SVD displays all of the fundamental vector spaces associated with A: $u_i = \frac{1}{\sigma_i} A v_i$

$$A_{m\times n} = \underbrace{\begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ 0 & & \sigma_r & \\ & & & 0 \ddots 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}}_{V^T}$$

$$A = \begin{bmatrix} | & & | & | & & | \\ u_1 & \cdots & u_r & u_{r+1} & \cdots & u_m \\ | & & | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \ddots 0 \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_r & - \\ - & v_{r+1} & - \\ & \vdots & \\ - & v_n & - \end{bmatrix} \begin{matrix} \} \text{Row}(A) \\ \oplus \\ \} \text{Ker}(A) = \text{Row}(A)^\perp \end{matrix}$$

$$= \begin{bmatrix} | & & | & | & & | \\ u_1 & \cdots & u_r & u_{r+1} & \cdots & u_m \\ | & & | & | & & | \end{bmatrix} \begin{bmatrix} - & \sigma_1 v_1 & - \\ & \vdots & \\ - & \sigma_r v_r & - \\ - & 0 & - \\ - & 0 & - \end{bmatrix} \} \text{Row}(A)$$

$$\underbrace{u_1 \cdots u_r}_{\text{Col}(A)} \oplus \underbrace{u_{r+1} \cdots u_m}_{\text{Ker}(A^T) = \text{Col}(A)^\perp}$$

$$\text{Ie; } \underset{m\times n\ n\times 1}{A x} = \begin{bmatrix} - & r_1 & - \\ & \vdots & \\ - & r_m & - \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 1$$

$$\text{Row}(A) \subset \text{Ker}(A)^\perp$$

$$U \Sigma V^T v_j = U \sigma_j e_j = \sigma_j u_j$$

$$\sum_{i=1}^r \sigma_i (u_i v_i^T) v_j = \sum_{i=1}^r \sigma_i \begin{pmatrix} | \\ u_i \\ | \end{pmatrix} (- v_i -) \begin{pmatrix} | \\ v_j \\ | \end{pmatrix} = \sum_{i=1}^r \sigma_i \begin{pmatrix} | \\ u_i \\ | \end{pmatrix} v_i \cdot v_j = \sigma_j v_j$$