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Introduction

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Blake Temple Applied Linear Alg
3148 MSB

General Relativity & Shock Waves
Experts in Appl Linear Alg — Saito/Strohmer

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http://www.math.ucdavis.edu/~temple/MAT167

- Find: syllabus + homework assignments + articles

Text: Gilbert Strang Linear Algebra with Applications

Grade will be determined by Midterm (100pts) and Final (200pts) together with homework assignments. Homework will be used to adjust grade by a Plus or Minus based on my judgement, and in cases where the Midterm and Final grades are highly inconsistent. Homework assignment is on the web, will be collected regularly as announced, and will be graded by the TA.

- Linear Algebra: "The study of matrices"
- Large Data sets usually stored as matrices of numebers ⇒ 1000x1000 matrices & higher ⇒ have to deal with nxn matrices

Problem: How to extract the info you want while ignoring the info you don't want —

- This Class - The mathematical theory that underlies applied linear algebra —

Topics:

1. Ch I: solving systems of equations by Gaussian Elimination —

Problem:

$$2u + v + w = 5$$
$$4u - 6v = -2$$
$$-2u + 7v + 2w = 9$$

Find (u, v, w)

Rewrite as matrix equation:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$(3\times3) \quad (3\times1) = (3\times1)$$

$(n\times m)$

rows $\uparrow$ $\quad$ $\uparrow$ colms

Abstractly

$$A\cdot x = b$$
$$(n\times n)(n\times1) = (n\times1)$$

<u>Problem</u>: 1. when does $Ax=b$ have a soln?

2. when is there no soln?

3. when is there a <u>unique</u> soln?

4. when $\exists$ more than one soln, how do you find all of them?

5. what is an efficient algorithm for the comuter to solve this problem?

6. How fast is the computer algorithm?

Ans: Gaussian Elimination –

Uses $\approx \frac{1}{3}n^3$ arithmatic operation to solve an $n \times n$ system

$\rightsquigarrow$ "pivots & factors"

$\rightsquigarrow$ Leads to the $LDU = A$ decomposition of matrices (Ch I, parts of II)

Least Squares:

Want: $y_k = Cx_k + D \quad k = 1, \dots, n$

Have: $\bar{y}_k = Cx_k + D \quad k = 1, \dots, n$

Minimize: $\bar{y}_k - y_k$ "Find C, D that minimize $\sum |\bar{y}_k - y_k|^2$"

Matrix:

$$\begin{bmatrix} D + C_1 X_1 \\ \vdots \\ D + C X_n \end{bmatrix} = \begin{bmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{bmatrix}$$

$n \times 2$ data matrix

output depends on C, D

$$A \begin{bmatrix} C \\ D \end{bmatrix} = \bar{\underset{\sim}{y}}$$

$$(n \times 2)(2 \times 1) = (n \times 1)$$

Min: $\|\underset{\sim}{y} - \bar{\underline{y}}\|^2$

Soln: Best (C, D) is: $\begin{bmatrix} C \\ D \end{bmatrix} = (A^T A)^{-1} A^T \underset{\sim}{y}$

To do this we need theory of

- Abstract Vector Spaces (Ch 2)
- Orthogonal Projection Matrices (Ch 3)

☒ Linear Differential / Difference Eqn's:

<u>Problem</u>: Find fn's $u(t), v(t), w(t)$ st:

$$\frac{du}{dt} = 2u + v + w$$

$$\frac{dv}{dt} = 4u - 6v \quad \Longleftrightarrow \quad \underset{3\times 1}{\begin{bmatrix} u \\ v \\ w \end{bmatrix}'} = \underset{3\times 3}{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}$$

$$\frac{dw}{dt} = -2u + 7v + 2w$$

$\rightsquigarrow$ Solve by finding the "eigen-solutions"

$\rightsquigarrow$ eigenvalues / eigenvectors Ch 5

Thm: A symmetric $\Rightarrow$ A has <u>real</u> evals & o.n. evectors

- Ch 6: Singular Value Decomposition:

Given large matrix A whose entries encode colors at each pixel of a computer generated picture –

Q: How do you pull out "the essential part of A that has the greatest influence on the picture" to reduce the complexity?

SVD: $A = U \Sigma V^T$
$n \times n$ $(n \times n)(n \times n)(n \times n)$

[Colms of U, V are ON unit vectors]

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \sigma_3 \cdots & \\ 0 & & & \sigma_n \; 0 \; 0 \; 0 \end{bmatrix}$$

$$A = \sigma_1 (u_1 v_1^T) + \cdots + \sigma_n (u_n v_n^T)$$

SVD: $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T + \sigma_{n+1}$

largest effect on A

unit vectors rank-1 matrix

Easy to compute as it involves only 2 vectors

next largest effect

$=0 \Rightarrow$ no effect

Conclusion: Singular Value Decomposition of matrix A:

- Pos Def Symmetric Matrix?
- FFT if time at end

$$A = \underset{(n\times n)}{U}\ \underset{(n\times n)}{\Sigma}\ \underset{(n\times n)}{V^T}$$

rows are ON unit vectors

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & 0 & \\ & & \ddots & & \\ & & & \sigma_n & \\ & 0 & & & 0 \\ & & & & & 0 \\ & & & & & & 0 \end{bmatrix}$$